On the speed in transportation costs of heat distributions

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The University of Warwick, Sept. 1–5, 2014

1. Introduction

Speed in transportation cost

$(\boldsymbol{X}_t)_{t\geq 0}$: stochastic process on M

$\Downarrow \Uparrow$

$(\boldsymbol{\mu}_t)_{t\geq 0}$: curve in $\mathcal{P}(M)$, $\boldsymbol{\mu}_t := \mathbb{P} \circ \boldsymbol{X}_t^{-1}$

Speed in transportation cost

$(X_t)_{t \geq 0}$: stochastic process on M

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$(\mu_t)_{t \geq 0}$: curve in $\mathcal{P}(M)$, $\mu_t := \mathbb{P} \circ X_t^{-1}$

Q. $\mathcal{T}_c(\mu_t, \mu_s) \approx ? \quad (s \to t)$

$$\mathcal{T}_c(\mu, \nu) := \inf \left\{ \int_{M \times M} c \, d\pi \;\middle|\; \begin{array}{c} \pi\text{: coupling of} \\ \mu \text{ and } \nu \end{array} \right\}$$

(Optimal transportation cost for a cost function c)

Background

$\boldsymbol{\mu_t = \mathrm{e}^{t\Delta}\mu_0}$: heat dist. on a met. meas. sp. $\boldsymbol{(M, d, v)}$

$\boldsymbol{\Rightarrow}$ "$\boldsymbol{\partial_t \mu_t = -\nabla \operatorname{Ent}_v(\mu_t)}$" w.r.t. $\boldsymbol{W_2 = (\mathcal{T}_{d^2})^{1/2}}$

[Jordan, Kinderlehrer & Otto '98]

[Ambrosio, Gigli & Savaré '05, . . .]

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$$\text{“}\partial_t \mathrm{Ent}_v(\mu_t) = -\langle \nabla \mathrm{Ent}_v, \partial_t \mu_t \rangle = -|\partial_t \mu_t|^2\text{”}$$

Background

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"$\partial_t \operatorname{Ent}_v(\mu_t) = -\langle \nabla \operatorname{Ent}_v, \partial_t \mu_t \rangle = -|\partial_t \mu_t|^2$"

$$\Downarrow$$

$$\overline{\lim_{s \downarrow t}} \left(\frac{W_2(\mu_s, \mu_t)}{s - t} \right)^2 = \int_M \frac{|\nabla \rho_t|^2}{\rho_t} dv =: I(\mu_t)$$

(Fisher information)

Background

"$\mathrm{Hess}\,\mathrm{Ent}_v \geq K$" ($\Leftrightarrow$ "$\mathrm{Ric} \geq K$") for $K \in \mathbb{R}$

$$\Downarrow$$

$$W_2(\mu_t^{(0)}, \mu_t^{(1)}) \leq \mathrm{e}^{-Kt} W_2(\mu_0^{(0)}, \mu_0^{(1)})$$

$$\Downarrow$$

$$W_2(\mu_t, \mu_{t+s}) \leq \mathrm{e}^{-Kt} W_2(\mu_0, \mu_s)$$

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$$\Downarrow$$

$$\boxed{I(\mu_t) \leq \mathrm{e}^{-2Kt} I(\mu_0)}$$

($\Rightarrow$ log Sobolev ineq. (when $K > 0$))

Questions

- What happens for other transportation costs?
- What happens when there is no gradient flow structure?

Outline of the talk

1. **Introduction**

2. **Heat distributions on backward Ricci flow**

3. **Idea of the proof**

4. **Further problems**

Framework

- $(M, g(t))$: cpl. Riem. mfds., $t \in [0, T]$

  $$\partial_t g(t) = 2\,\mathrm{Ric}_t \text{ (backward Ricci flow)}$$

- $((X(t))_{t \geq 0}, (\mathbb{P}_x)_{x \in M})$: $g(t)$-Brownian motion

  $\leftrightsquigarrow \Delta_{g(t)}$: generator

  $\mu_t = \mathbb{P}_{\mu_0} \circ X(t)^{-1}$: heat dist.

- v_t: $g(t)$-volume meas., $\mu_t = \rho_t v_t$

  ★ $\partial_t v_t = R_t v_t$ (R_t: $g(t)$-scalar curv.)

Ass. $\sup_t |\mathrm{Rm}_t|_{g(t)} < \infty$ (Rm_t: $g(t)$-curv. tensor)

$\partial_t \mu_t \neq -\nabla \operatorname{Ent}_{v_t}(\mu_t)$

$$\operatorname{Ent}_{v_t}(\mu_t) := \int_M \rho_t \log \rho_t \, dv_t = \int_M \log \rho_t \, d\mu_t$$

★ $\partial_t \mu_t = \Delta_t \mu_t$ (weakly)

$$\Rightarrow \partial_t \operatorname{Ent}_{v_t}(\mu_t) = -\int_M \left(\frac{|\nabla \rho_t|^2}{\rho_t^2} + R_t \right) d\mu_t$$

$$=: -\mathcal{F}(\mu_t) \qquad (\mathcal{F}\text{-functional})$$

$\Rightarrow$ No monotonicity of $\operatorname{Ent}_{v_t}(\mu_t)$!

W_2-contraction

Observation

When $g(t) \equiv g_0$, $\partial_t g(t) = 2\,\mathrm{Ric}_t \Rightarrow \mathrm{Ric} \equiv 0$

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★ $\mathcal{T}_{d_t^2}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$ [McCann & Topping '10], [Arnaudon, Coulibaly & Thalmaier '09], [K. '12]

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$\not\Rightarrow$ $\mathcal{T}_{d_t^2}(\mu_t, \mu_{t+s}) \searrow$ (time-inhomogeneity)

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★ $\mathcal{T}_{d_t^2}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$ [McCann & Topping '10], [Arnaudon, Coulibaly & Thalmaier '09], [K. '12]

★ $\mathcal{T}_{L_0^{t,t+s}}(\mu_t, \mu_{t+s}) \searrow$ in t [Lott '09], [Amaba & K.]

$$L_0^{t,t'}(x,y) := \inf_{\substack{\gamma(t)=x,\\ \gamma(t')=y}} \left[\int_t^{t'} \left(|\dot\gamma(r)|_r^2 + R_r(\gamma(r)) \right) dr \right]$$

(L_0-distance; introduced by Lott)

Monotonicity of $\mathcal{F}$

Theorem 1

Suppose $\mathrm{Ent}_{v_0}(\mu_0) < \infty$ *and* $\mathcal{F}(\mu_0) < \infty$

$$\Rightarrow \lim_{s \downarrow 0} \frac{\mathcal{T}_{L_0^{t,t+s}}(\mu_t, \mu_{t+s})}{s} = \mathcal{F}(\mu_t) \text{ a.e. } t \in [0, T]$$

Corollary 2

$\mathcal{F}(\mu_t) \searrow$

- Rem: $g(t) \equiv g$, $\mathrm{Ric} \geq 0 \Rightarrow I(\mu_t) \searrow$
- [Lott '09] when M: cpt.

  by Eulerian calculus (requires smoothness)

$\mathcal{L}$-transportation cost

$$L^{t,t'}(x,y) := \inf_{\substack{\gamma(t)=x,\\ \gamma(t')=y}} \left[\int_t^{t'} \sqrt{r} \left(|\dot\gamma(r)|_r^2 + R_r \right) dr \right]$$

(L-distance; introduced by Perelman)

★ For $\tau_1 < \tau_2$ fixed,

$$\Xi_{\tau_1,\tau_2}(t) := \left(\sqrt{\tau_2 t} - \sqrt{\tau_1 t}\right) \mathcal{T}_{L^{\tau_1 t, \tau_2 t}}(\mu_{\tau_1 t}, \mu_{\tau_2 t}) - m(\sqrt{\tau_2 t} - \sqrt{\tau_1 t})^2$$

$\Rightarrow \Xi_{\tau_1,\tau_2}(t) \searrow$ [Topping '09], [K. & Philipowski '11]

Monotonicity of $\mathcal{W}$-entropy

Theorem 3

Suppose $\mathrm{Ent}_{v_0}(\mu_0) < \infty$ *and* $\mathcal{F}(\mu_0) < \infty$

$$\Rightarrow \lim_{s \downarrow 0} \frac{\mathcal{T}_{L^{t,t+s}}(\mu_t, \mu_{t+s})}{s} = \sqrt{t}\mathcal{F}(\mu_t) \text{ a.e. } t \in (0, T]$$

Corollary 4

$$t^2 \mathcal{F}(\mu_t) - \frac{mt}{2} \searrow. \text{ In particular, } \mathcal{W}(\mu_t) \searrow$$

$$\mathcal{W}(t) := t\mathcal{F}(\mu_t) - \mathrm{Ent}(\mu_t) - m\left(1 + \frac{\log(4\pi t)}{2}\right)$$

- [Topping '09] when M: cpt. by optimal transport

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Kantorovich duality

Observation: W_2 in the case $g(t) \equiv g$

$$\frac{W_2(\mu_t, \mu_{t+s})^2}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s\varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$Q_s\varphi(x) := \inf_{y \in M} \left(\varphi(y) + \frac{d(y,x)^2}{2s} \right)$$

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(Hopf-Lax semigroup)

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(Hopf-Lax semigroup)

$$\star \ \partial_s Q_s\varphi + \frac{1}{2} |\nabla Q_s\varphi|^2 = 0 \quad \text{(Hamilton-Jacobi eq.)}$$

Upper bound

$$\frac{W_2(\mu_t, \mu_{t+s})^2}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s \varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$[\cdots] = \int_0^s \left(\partial_r \int_M Q_r \varphi \, d\mu_{t+r} \right) \, dr$$

$$\partial_r (\cdots) = \int \left(-\langle \nabla Q_r \varphi, \frac{\nabla \rho_{t+r}}{\rho_{t+r}} \rangle - \frac{1}{2} |\nabla Q_r \varphi|^2 \right) d\mu_{t+r}$$

$$\leq \frac{1}{2} I(\mu_{t+r})$$

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$$\overline{\lim_{s \downarrow t}} \frac{W_2(\mu_t, \mu_{t+s})^2}{s^2} \leq \overline{\lim_{s \downarrow t}} \frac{1}{s} \int_t^{t+s} I(\mu_r) dr = I(\mu_t)$$

Lower bound

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$$\text{“}{=}\text{”} \int \left(-\langle \nabla \varphi, \frac{\nabla \rho_t}{\rho_t} \rangle - \frac{1}{2} |\nabla \varphi|^2 \right) d\mu_t$$

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$$\Downarrow \quad \varphi = -\log\rho_t$$

$$\varliminf_{s\downarrow t} \frac{W_2(\mu_t,\mu_{t+s})^2}{s^2} \geq I(\mu_t)$$

Outline of the proof of Thm1

- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry

  No a priori integrability of $\nabla \rho_t$ & $\Delta_t \rho_t$

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† Show $\overline{\lim_{s \downarrow t}} \frac{\mathrm{Ent}_{v_s}(\mu_s) - \mathrm{Ent}_{v_t}(\mu_t)}{s - t} \leq \mathcal{F}(\mu_t)$

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- For W_p ($p \in [1, \infty)$) in time-homogeneous case?

† $\mathrm{Ric} \geq K \Rightarrow \mathrm{e}^{Kt} W_p(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$

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★ $\displaystyle \varlimsup_{s \downarrow 0} \left(\frac{W_p(\mu_t, \mu_{t+s})}{s} \right)^p \leq \int_M \frac{|\nabla \rho_t|^p}{\rho_t^{p-1}} dv$

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$\mathrm{Ric} \geq K > 0$, $v \in \mathcal{P}(M)$

$$\Rightarrow \boxed{W_p(\mu, v)^p \leq \frac{1}{K^p} \int_M \frac{|\nabla \rho|^p}{\rho^{p-1}} dv}$$

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- The solution to the p_*-heat equation

$$\partial_t u = \mathrm{div}(|\nabla u|^{p_* - 2} \nabla u)$$

seems to fit better with W_p. (work in progress)