The entropic curvature-dimension condition and Bochner's inequality

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(joint work with M. Erbar and K.-Th. Sturm (Univ. Bonn))

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1. Introduction

Purpose/History

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Unify the study of

$$\text{“}\mathbf{Ric} \geq \boldsymbol{K} \text{ and } \mathbf{dim} \leq \boldsymbol{N}\text{”}$$

in terms of

optimal transportation / heat distribution

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- Established for "$\mathbf{Ric} \geq \boldsymbol{K}$"

  Riem. mfd.: [von Renesse & Sturm '05]

  mm-sp.: [Ambrosio, Gigli & Savaré et al.]

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Purpose/History

The case $N = \infty$

- Γ_2-criterion (via $P_t = e^{t\Delta}$: [Bakry & Émery '84])
  $$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$$
- via Optimal transport:
  [Sturm '06, Lott & Villani, '09, Sturm & Bacher '10]
  in terms of the relative entropy Ent

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Key "fact"

$P_t^* \mu$: gradient curve of Ent

$$\left(\mathrm{Ent}(\mu) := \int_X \rho \log \rho \, d\mathfrak{m} \quad (\mu = \rho \mathfrak{m}) \right)$$

Purpose/History

The case $N < \infty$

- Bochner's inequality (cf. [Bakry & Ledoux '06])

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2$$

- via Optimal transport:

  [Sturm '06, Lott & Villani, '09, Sturm & Bacher '10]

  in terms of the Rényi entropy (NOT Ent)

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- via Optimal transport:

[Sturm '06, Lott & Villani, '09, Sturm & Bacher '10]

in terms of the Rényi entropy

★ Another approach via porous medium eq.

(gradient curve of the Rényi ent.)

[Ambrosio, Savaré & Mondino]

Goal

- Characterize "$\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$" by $\mathbf{Ent}$

- Find missing conditions

$$\text{connecting} \begin{cases} \text{optimal transport approach} \\ \qquad\quad \& \\ \boldsymbol{P_t} \text{ approach} \end{cases}$$

- Establish the equivalence

Goal

- Characterize "$\mathbf{Ric} \geq K$ & $\mathbf{dim} \leq N$" by $\mathbf{Ent}$

  $\Rightarrow$ (K, N)-convexity of $\mathbf{Ent}$

- Find missing conditions

  $$\text{connecting} \begin{cases} \text{optimal transport approach} \\ \qquad\& \\ P_t \text{ approach} \end{cases}$$

  $\Rightarrow$ (K, N)-evolution variational inequality

  Space-time W_2-control

- Establish the equivalence

Overview of applications

- Analysis/Geometry on non-smooth sp.'s
- Different viewpoints even on smooth sp.'s

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- Analysis/Geometry on non-smooth sp.'s
- Different viewpoints even on smooth sp.'s

- (Geometric) stability of analytic conditions
- Regulartity results on P_t
- (New) functional inequalities involving N & K (e.g. N-log Sobolev, N-Talagrand)
- Maximal diameter theorem [Ketterer]

Outline of the talk

1. **Introduction**

2. **Entropic curvature-dimension condition**

3. (K, N)**-evolution variational inequality**

4. **Connection with Bakry-Émery theory**

5. **Applications**

Framework

$(X, d, \mathfrak{m})$: Polish geodesic metric measure sp.,
$\mathfrak{m}$: loc. finite, σ-finite, $\operatorname{supp} \mathfrak{m} = X$,

Example

(X, g): complete Riem. mfd., $\partial X = \emptyset$,
d: Riem. dist., $\mathfrak{m} = e^{-V} \operatorname{vol}_g \quad (V : X \to \mathbb{R})$

Framework

L^2-Wasserstein distance W_2: For $\sigma_0, \sigma_1 \in \mathcal{P}(X)$,

$$\boxed{W_2(\sigma_0, \sigma_1) := \inf_{\pi} \|d\|_{L^2(\pi)}}$$

$$\pi \in \mathcal{P}(X^2), \begin{cases} \pi(A \times X) = \sigma_0(A), \\ \pi(X \times A) = \sigma_1(A) \end{cases}$$

$$\mathcal{P}_2(X) := \{\mu \in \mathcal{P}(X) \mid W_2(\delta_{x_0}, \mu) < \infty\}$$

Framework

L^2-Wasserstein distance W_2: For $\sigma_0, \sigma_1 \in \mathcal{P}(X)$,

$$W_2(\sigma_0, \sigma_1) := \inf_{\pi} \|d\|_{L^2(\pi)}$$

- $^\forall \sigma_0, \sigma_1 \in \mathcal{P}_2(X)$, $^\exists (\sigma_r)_{r \in [0,1]}$: W_2-geod.
- $^\exists \Gamma \in \mathcal{P}(\mathrm{Geo}(X))$ s.t.

$$\sigma_t(A) = \int_{\mathrm{Geo}(X)} 1_A(\gamma(t)) \Gamma(d\gamma),$$

$$W_2(\sigma_s, \sigma_t)^2 = \int_{\mathrm{Geo}(X)} d(\gamma(s), \gamma(t))^2 \Gamma(d\gamma)$$

($\mathrm{Geo}(X)$: sp. of const. speed geod.'s)

(K, N)-convexity

K-convexity of Ent: "$\mathrm{Hess}\,\mathrm{Ent} \geq K$" w.r.t. W_2

$(\Leftrightarrow \mathrm{Ric} \geq K)$

(K, N)-convexity

K-convexity of Ent: "Hess Ent $\geq K$"

$\forall \sigma_0, \sigma_1 \in \mathcal{P}_2(X)$, $\exists (\sigma_t)_{t\in[0,1]}$: W_2-geod. s.t.

$$\mathrm{Ent}(\sigma_t) \leq (1-t)\,\mathrm{Ent}(\sigma_0) + t\,\mathrm{Ent}(\sigma_1) - \frac{K}{2} t(1-t) W_2(\sigma_0, \sigma_1)^2$$

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★ $\varphi(t) :=$ (RHS) solves

$$\varphi''(t) = K W_2(\sigma_0, \sigma_1)^2,$$
$$\varphi(0) = \mathrm{Ent}(\sigma_0),$$
$$\varphi(1) = \mathrm{Ent}(\sigma_1)$$

(K, N)-convexity

(K, N)-convexity of Ent

$$\text{“}\mathrm{Hess}\,\mathrm{Ent} - \frac{1}{N} \nabla\, \mathrm{Ent}^{\otimes 2} \geq K\text{”}$$

(K, N)-convexity

(K, N)-convexity of Ent

$$\text{“}\mathrm{Hess}\,\mathrm{Ent} - \frac{1}{N}\nabla\,\mathrm{Ent}^{\otimes 2} \geq K\text{”}$$

$$\Updownarrow$$

$$\text{“}\mathrm{Hess}\,U_N \leq -\frac{K}{N}U_N\text{”},\ U_N := \exp\left(-\frac{1}{N}\,\mathrm{Ent}\right)$$

(K, N)-convexity

(K, N)-convexity of Ent

$$\text{“}\mathrm{Hess}\, U_N \le -\frac{K}{N} U_N\text{”},\ U_N := \exp\left(-\frac{1}{N}\,\mathrm{Ent}\right)$$

Entropic curvature-dimension cond. $\mathsf{CD}^e(K, N)$:

$^\forall \sigma_0, \sigma_1 \in \mathcal{P}_2(X)$, $^\exists (\sigma_t)_{t\in[0,1]}$: W_2-geod. s.t.

$$U_N(\sigma_t) \ge \psi(t),$$

where ψ solves $\psi''(t) = -\dfrac{K}{N} W_2(\sigma_0, \sigma_1)^2 \psi(t)$,

$$\psi(i) = U_N(\sigma_i)\ (i = 0, 1)$$

(K, N)-convexity

(K, N)-convexity of Ent

$$\text{"}\mathrm{Hess}\, U_N \le -\frac{K}{N} U_N\text{"}, \; U_N := \exp\left(-\frac{1}{N}\,\mathrm{Ent}\right)$$

strong $\mathsf{CD}^e(K, N)$:

$\forall \sigma_0, \sigma_1 \in \mathcal{P}_2(X)$, $\forall (\sigma_t)_{t\in[0,1]}$: W_2-geod.,

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Relation with known conditions

On cpl. Riem. mfd.,

Volume distortion est. for optimal transport

$$\Downarrow$$

Reduced curvature-dimension condition $\mathbf{CD}^*(K, N)$

[Bacher & Sturm '10]

Relation with known conditions

$$\mathsf{CD}^*(K,N)$$

$\Downarrow$ geod.'s on X are non-branching

${}^\forall \sigma_0, \sigma_1 \in \mathcal{P}_2(X)$,
${}^\exists \Gamma \in \mathcal{P}(\mathrm{Geo}(X))$: lift of W_2-geod. s.t.

$$\rho_t(\gamma_t)^{-1/N} \geq \Psi(t; \gamma_0, \gamma_1)$$

for Γ-a.e. γ, where $\rho_t \mathfrak{m} = \sigma_t$ & Ψ solves

$$\Psi''(t) = -\frac{K}{N} d(\gamma_0, \gamma_1)^2 \Psi(t),$$

$$\Psi(i) = \rho_i(\gamma_i)^{-1/N} \ (i = 0, 1)$$

Relation with known conditions

$$\mathsf{CD}^*(K, N)$$

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$\Downarrow$

$$\rho_t(\gamma_t)^{-1/N} = \exp\left(-\frac{1}{N} \log \rho_t(\gamma_t)\right)$$

& Jensen's ineq.

CD$^e(K, N)$

Relation with known conditions

$$\rho_t(\gamma_t)^{-1/N} \geq \Psi(t; \gamma_0, \gamma_1)$$

for Γ-a.e. γ, where $\rho_t \mathfrak{m} = \sigma_t$

$\Updownarrow$ geod.'s on X are non-branching

$\mathsf{CD}^e(K, N)$

K-evolution variational ineq.

EVI$_K$ of Ent

$^\forall \mu_0$, $^\exists (\mu_t)_{t\geq 0}$: abs. conti. s.t. for $^\forall \nu$,

$$\frac{d}{dt}\left(\frac{W_2(\mu_t,\nu)^2}{2}\right) + \frac{K}{2}W_2(\mu_t,\nu)^2 + \mathrm{Ent}(\mu_t) \leq \mathrm{Ent}(\nu)$$

★ A variational formulation of "$\partial_t \mu_t = -\nabla \mathrm{Ent}(\mu_t)$"

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★ A variational formulation of "$\partial_t \mu_t = -\nabla \mathrm{Ent}(\mu_t)$"

$\because$) $^\forall$geod.$(\sigma_s)_{s\in[0,1]}$ with $\sigma_0 = \mu_t$, for $\nu = \sigma_s$,

$$-\langle \partial_t \mu_t, \dot{\sigma}_0 \rangle \leq \frac{\mathrm{Ent}(\sigma_s) - \mathrm{Ent}(\sigma_0)}{s} + o(1)$$

K-evolution variational ineq.

"$\mathrm{Hess\,Ent} \geq K \Rightarrow \mathsf{EVI}_K$"

$\because$) For $(\sigma_s)_{s\in[0,1]}$: W_2-geod. from μ_t to ν,

$$\langle \nabla \mathrm{Ent}(\sigma_0), \dot{\sigma}_0 \rangle \leq \mathrm{Ent}(\sigma_1) - \mathrm{Ent}(\sigma_0) - \frac{K}{2} W_2(\sigma_0, \sigma_1)^2$$

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By $\partial_t \mu_t = -\nabla \operatorname{Ent}(\mu_t)$ & "1st var. of W_2",

$$\langle \nabla \operatorname{Ent}(\mu_t), \dot{\sigma}_0 \rangle = -\langle \partial_t \mu_t, \dot{\sigma}_0 \rangle = \frac{1}{2} \frac{d}{dt} W_2(\mu_t, \nu)^2$$

EVI_K and P_t

$$P_t = e^{t\Delta} \leftrightarrow \text{Cheeger's } L^2\text{-energy}$$

$$\mathrm{Ch}(f) := \inf_{\substack{f_n:\ \text{Lip.} \\ f_n \to f \text{ in } L^2}} \liminf_{n} \int_X |\nabla f_n|^2 d\mathfrak{m}$$

$$= \int_X {}^{\exists}|\nabla f|_w^2 \, d\mathfrak{m}$$

($|\nabla f|_w$: minimal weak upper gradient)

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($|\nabla f|_w$: minimal weak upper gradient)

★ $(X, d, \mathfrak{m})$: infinitesimally Hilbertian

$\overset{\text{def}}{\Leftrightarrow}$ Ch: quadratic form

($\Leftrightarrow$ P_t: linear)

EVI_K and P_t

- $(\mu_t)_{t\geq 0}$ sol. to EVI_K of $\mathrm{Ent} \Rightarrow \mu_t = P_t^* \mu_0$
- $^\forall \mu_0$, $^\exists(\mu_t)_{t\geq 0}$ sol. to EVI_K of Ent & **(V)**

  $\Leftrightarrow$ Ent: strongly K-convex & infin. Hilb.

  ($\mathsf{RCD}(K,\infty)$ cond.)

  [Ambrosio, Gigli & Savaré]
  [Ambrosio, Gigli, Mondino & Rajala]

EVI_K and P_t

- $(\mu_t)_{t\geq 0}$ sol. to EVI_K of Ent $\Rightarrow$ $\mu_t = P_t^* \mu_0$
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  [Ambrosio, Gigli & Savaré]
  [Ambrosio, Gigli, Mondino & Rajala]

The volume growth cond. **(V)**

$$\int_X \exp\left(-{}^\exists c\, d(x_0, x)^2\right) \mathfrak{m}(dx) < \infty$$

EVI$_{K,N}$ and Riemannian CD$^e(K,N)$

EVI$_{K,N}$ of Ent

$^\forall\mu_0$, $^\exists(\mu_t)_{t\ge 0}$: abs. conti. s.t. for $^\forall\nu$,

$$\frac{d}{dt}\mathfrak{s}_{K/N}{}^2\left(\frac{W_2(\mu_t,\nu)}{2}\right)+K\,\mathfrak{s}_{K/N}{}^2\left(\frac{W_2(\mu_t,\nu)}{2}\right)\le\frac{N}{2}\left(1-\frac{U_N(\nu)}{U_N(\mu_t)}\right)$$

$$\left(\mathfrak{s}_\kappa(r):=\frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}},\ U_N:=\exp\left(-\frac{1}{N}\,\mathrm{Ent}\right)\right)$$

$\mathsf{EVI}_{K,N}$ and Riemannian $\mathsf{CD}^e(K,N)$

Theorem 1 ($\mathsf{RCD}^*(K,N)$ cond.'ns)

For $K \in \mathbb{R}$ *and* $N > 0$, *TFAE:*

(i) *strong* $\mathsf{CD}^*(K,N)$ *& infin. Hilb.*

(ii) *strong* $\mathsf{CD}^e(K,N)$ *& infin. Hilb.*

(iii) $^\forall \mu_0$, $^\exists (\mu_t)_{t \ge 0}$: $\mathsf{EVI}_{K,N}$*-curve &* **(V)**

$$\textbf{(V)} \quad \int_X \exp\left(-^\exists c\, d(x_0,x)^2\right) \mathfrak{m}(dx) < \infty$$

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★ $\mathsf{RCD}^*(K,N) \Rightarrow \mathsf{RCD}(K,\infty)$

$\mathsf{EVI}_{K,N}$ and Riemannian $\mathsf{CD}^e(K,N)$

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★ $\mathsf{RCD}^*(K,N) \Rightarrow \mathsf{RCD}(K,\infty)$

$\Rightarrow$ geod.'s on X are essentially non-branching

[Rajala & Sturm]

Regularity results on RCD(K, ∞) sp.

[Ambrosio, Gigli, Savaré et al.]: **RCD**(K, ∞) yields

- The volume growth bound **(V)**
- $P_t f \in C_b^{\mathsf{Lip}}$ for $f \in L^\infty(\mathfrak{m})$
- $|\nabla f|_w \leq 1$ $\mathfrak{m}$-a.e. $\Rightarrow f$: 1-Lip.

Regularity results on RCD(K, ∞) sp.

Assumption 1 (cf. [Ambrosio, Gigli & Savaré])

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W_2-control

EVI$_K$-curve

$$\frac{d}{dt}\left(\frac{W_2(\mu_t,\nu)^2}{2}\right)+\frac{K}{2}W_2(\mu_t,\nu)^2 + \mathrm{Ent}(\mu_t) \le \mathrm{Ent}(\nu)$$

$\Downarrow$ $\nu = \nu_t$: another **EVI**$_K$-curve

For $\Xi(t) := W_2(P_t^*\mu, P_t^*\nu)$,

$$\Xi(t) \le e^{-Kt}\Xi(0)$$

Space-time W_2-control

EVI$_{K,N}$-curve

$$\frac{d}{dt}\mathfrak{s}^2_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right) + K\,\mathfrak{s}^2_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right) \le \frac{N}{2}\left(1 - \frac{U_N(\nu)}{U_N(\mu_t)}\right)$$

Space-time W_2-control

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$$\Downarrow$$

For $\hat{\Xi}(s,t) := W_2(P_s^*\mu, P_t^*\nu)$,

$$\mathfrak{s}^2_{K/N}\left(\frac{\hat{\Xi}(s,t)}{2}\right) \le e^{-K(s+t)}\mathfrak{s}^2_{K/N}\left(\frac{\hat{\Xi}(0,0)}{2}\right) + \frac{N}{2}\frac{1-e^{-K(s+t)}}{K(s+t)}(\sqrt{t}-\sqrt{s})^2$$

Space-time W_2-control

EVI$_{K,N}$-curve

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Space-time W_2-control

$$\mathrm{Hess\,Ent} \ge \frac{1}{N} \nabla\,\mathrm{Ent}^{\otimes 2} \Rightarrow$$ Sp.-t. W_2-control $(K=0)$

$\because)$ $(\sigma_r)_{r\in[0,1]}$: W_2-geod. from $P^*_{t_0 u}\mu$ to $P^*_{t_1 u}\nu$,

$\chi_r := \langle \nabla\,\mathrm{Ent}(\sigma_r), \dot{\sigma}_r \rangle$, $(t_r)_{r\in[0,1]}$: interpolation

$$\frac{\partial}{\partial u} \frac{\hat{\Xi}(t_0 u, t_1 u)^2}{2} = t_0 \chi_0 - t_1 \chi_1$$

$$= -\int_0^1 \frac{\partial}{\partial r}(t_r \chi_r) dr \le -\int_0^1 \dot{t}_r \chi_r + \frac{1}{N} t_r \chi_r^2 \, dr$$

$$\le \frac{N}{4} \int_0^1 \frac{\dot{t}_r^2}{t_r} \, dr$$

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$t_r := ((1-r)\sqrt{t_0} + r\sqrt{t_1})^2$

$$\frac{\partial}{\partial u}\frac{\hat{\Xi}(t_0 u, t_1 u)^2}{2} = t_0\chi_0 - t_1\chi_1$$

$$= -\int_0^1 \frac{\partial}{\partial r}(t_r\chi_r)dr \leq -\int_0^1 \dot{t}_r\chi_r + \frac{1}{N}t_r\chi_r^2\,dr$$

$$\leq \frac{N}{4}\int_0^1 \frac{\dot{t}_r^2}{t_r}\,dr$$

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$t_r := ((1-r)\sqrt{t_0} + r\sqrt{t_1})^2$

$$\frac{\partial}{\partial u}\frac{\hat{\Xi}(t_0 u, t_1 u)^2}{2} = t_0\chi_0 - t_1\chi_1$$

$$= -\int_0^1 \frac{\partial}{\partial r}(t_r\chi_r)dr \leq -\int_0^1 \dot{t}_r\chi_r + \frac{1}{N}t_r\chi_r^2\,dr$$

$$\leq \frac{N}{4}\int_0^1 \frac{\dot{t}_r^2}{t_r}\,dr = N(\sqrt{t_1} - \sqrt{t_0})^2$$

Bakry-Ledoux gradient estimate

Space-time W_2-control: $\hat{\Xi}(s,t) := W_2(P_s^*\nu, P_t^*\mu)$,

$$\mathfrak{s}_{K/N}^2\left(\frac{\hat{\Xi}(s,t)}{2}\right) \le e^{-K(s+t)}\mathfrak{s}_{K/N}^2\left(\frac{\hat{\Xi}(0,0)}{2}\right) + \frac{N}{2}\frac{1-e^{-K(s+t)}}{K(s+t)}(\sqrt{t}-\sqrt{s})^2$$

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$\Downarrow$ Derivative in space & time

Bakry-Ledoux gradient est.: For $f \in W^{1,2}$, $\mathfrak{m}$-a.e.,

$$|\nabla P_t f|_w^2 \le e^{-2Kt} P_t(|\nabla f|_w^2) - \frac{2tC(t)}{N}|\Delta P_t f|^2,$$

$$C(t) = 1 + O(t) \ (t \to 0)$$

$\mathsf{RCD}^*(K,N) \Rightarrow$ Bakry-Ledoux

Theorem 2

$(X, d, \mathfrak{m})$: *infinitesimally Hilbertian*

For $K \in \mathbb{R}$ *&* $N > 0$, **(iii)** $\Rightarrow$ **(iv)** $\Rightarrow$ **(v)**

(iii) $^\forall \mu_0$, $^\exists$**EVI**$_{K,N}$*-curve* $(\mu_t)_t$ *&* **(V)**

(iv) *Ass. 1 & Space-time* W_2*-control*

(v) *Ass. 1 & Bakry-Ledoux gradient estimate*

Bakry-Ledoux $\Leftrightarrow$ Bochner

Theorem 3

$(X, d, \mathfrak{m})$: *infinitesimally Hilbertian*

For $K \in \mathbb{R}$ & $N > 0$, **(v) $\Leftrightarrow$ (vi)**

(v) $|\nabla P_t f|_w^2 \le e^{-2Kt} P_t(|\nabla f|_w^2) - \frac{2C(t)}{N} |\Delta P_t f|^2$

(vi) $^\forall f \in W^{1,2}$ *with* $\Delta f \in W^{1,2}$ &

$g \in D(\Delta) \cap L^\infty$ *with* $g \ge 0$ & $\Delta g \in L^\infty$

$$\int_X \left(\frac{1}{2} \Delta g |\nabla f|_w^2 - g \langle \nabla f, \nabla \Delta f \rangle \right) d\mathfrak{m} \ge \int_X g \left(K |\nabla f|_w^2 + \frac{1}{N} |\Delta f|^2 \right) d\mathfrak{m}$$

Bakry-Ledoux $\Leftrightarrow$ Bochner

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$(X, d, \mathfrak{m})$: *infinitesimally Hilbertian*

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Bakry-Ledoux $\Rightarrow$ RCD$^*(K, N)$

"Bochner $\Rightarrow$ **RCD**$^*(K, N)$" via Otto calculus:

$(\sigma_t)_{t \in [0,1]}$: W_2-geod.

$$\Rightarrow \begin{cases} \dot{\sigma}_t + \mathrm{div}_{\sigma_t}(\nabla \xi_t)\sigma_t = 0, \\ \partial_t \xi_t + \frac{1}{2}|\nabla \xi_t|^2 = 0 \end{cases}$$

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$\Downarrow$

$$\|\dot{\sigma}_t\|^2 = \|\nabla \xi_t\|^2_{L^2(\sigma_t)},$$

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$\Downarrow$

$$\|\dot{\sigma}_t\|^2 = \|\nabla\xi_t\|^2_{L^2(\sigma_t)}, \ \langle\nabla\,\mathrm{Ent}(\sigma_t), \dot{\sigma}_t\rangle = \int_X \Delta\xi_t\, d\sigma_t,$$

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$$\|\dot{\sigma}_t\|^2 = \|\nabla\xi_t\|^2_{L^2(\sigma_t)}, \ \langle\nabla \operatorname{Ent}(\sigma_t), \dot{\sigma}_t\rangle = \int_X \Delta\xi_t \, d\sigma_t,$$

$$\operatorname{Hess}\operatorname{Ent}(\dot{\sigma}_t, \dot{\sigma}_t) = \int_X \left(\frac{1}{2}\Delta|\nabla\xi_t|^2 - \langle\nabla\xi_t, \nabla\Delta\xi_t\rangle\right) d\sigma_t$$

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$$\Rightarrow \mathrm{Hess}\,\mathrm{Ent}(\dot{\sigma}_t,\dot{\sigma}_t) \geq K\int_X |\nabla\xi_t|^2 d\sigma_t + \frac{1}{N}\int_X (\Delta\xi_t)^2\,d\sigma_t \geq K\langle\dot{\sigma}_t,\dot{\sigma}_t\rangle + \frac{1}{N}\nabla\,\mathrm{Ent}^{\otimes 2}(\dot{\sigma}_t,\dot{\sigma}_t)$$

Bakry-Ledoux $\Rightarrow$ RCD$^*(K, N)$

Theorem 4

$(X, d, \mathfrak{m})$: *infinitesimally Hilbertian & Ass. 1*

For $K \in \mathbb{R}$ *&* $N > 0$, **(v) $\Rightarrow$ (ii)**:

(v) *Bakry-Ledoux gradient estimate*

$$|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) - \frac{2C(t)}{N} |\Delta P_t f|^2$$

(ii) *Entropic curv.-dim.:*

$$\text{“} \operatorname{Hess} \operatorname{Ent} - \frac{1}{N} \operatorname{Ent}^{\otimes 2} \geq K \text{”}$$

Bakry-Ledoux $\Rightarrow$ RCD$^*(K, N)$

Idea: Action estimate (as in [Ambrosio, Gigli, Savaré])

$$\frac{W_2(\sigma_0, P^*_\tau \sigma_1)^2}{2} - \frac{1}{2}\int_0^1 |\dot\sigma_s|^2 e^{-2K\tau} ds \leq Nt(U_N(P_\tau \sigma_1) - U_N(\sigma_0)) \qquad (\spadesuit)$$

for $t \ll 1$, where $\tau = \tau_{s,t} : \partial_t \tau = s\, U_N(P^*_\tau \sigma_s)$, $\tau_{s,0} = 0$

$$\left(\Rightarrow \partial_t P^*_\tau \sigma_s = sN\nabla U_N(P^*_\tau \sigma_s)\right)$$

Ingredients of the proof

Kantorovich duality, approximations

Bakry-Ledoux $\Rightarrow$ RCD$^*(K,N)$

Idea: Action estimate (as in [Ambrosio, Gigli, Savaré])

$$\frac{W_2(\sigma_0, P_\tau^*\sigma_1)^2}{2} - \frac{1}{2}\int_0^1 |\dot\sigma_s|^2 e^{-2K\tau}\,ds \leq Nt(U_N(P_\tau\sigma_1) - U_N(\sigma_0)) \qquad (\spadesuit)$$

for $t \ll 1$, where $\tau = \tau_{s,t}$: $\partial_t\tau = s\,U_N(P_\tau^*\sigma_s)$, $\tau_{s,0} = 0$

$$\left(\Rightarrow \partial_t P_\tau^*\sigma_s = sN\nabla U_N(P_\tau^*\sigma_s)\right)$$

Ingredients of the proof

Kantorovich duality, approximations & detailed calc.

Bakry-Ledoux $\Rightarrow$ RCD$^*(K, N)$

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$$\frac{W_2(\sigma_0, P_\tau^*\sigma_1)^2}{2} - \frac{1}{2}\int_0^1 |\dot{\sigma}_s|^2 e^{-2K\tau} ds \leq Nt(U_N(P_\tau\sigma_1) - U_N(\sigma_0)) \qquad (\spadesuit)$$

for $t \ll 1$, where $\tau = \tau_{s,t} : \partial_t \tau = s\, U_N(P_\tau^* \sigma_s)$,

$\Downarrow$

For $(\sigma'_s)_{s\in[0,1]}$: W_2-geod.,
$(\spadesuit)$ for $(\sigma_0, \sigma_1) = (\sigma'_0, \sigma'_r)$ or (σ'_1, σ'_r)
& $t \to 0$

Bakry-Ledoux $\Rightarrow$ RCD$^*(K, N)$

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$$\frac{W_2(\sigma_0, P_\tau^*\sigma_1)^2}{2} - \frac{1}{2}\int_0^1 |\dot{\sigma}_s|^2 e^{-2K\tau} ds \leq Nt(U_N(P_\tau\sigma_1) - U_N(\sigma_0)) \quad (\spadesuit)$$

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$(\spadesuit)$ for $(\sigma_0, \sigma_1) = (\sigma'_0, \sigma'_r)$ or (σ'_1, σ'_r)
& $t \to 0$

$$U_N(\sigma'_r) - (1-r)U_N(\sigma'_0) - rU_N(\sigma'_1) \geq \frac{K}{N}\int_0^1 (s(1-r)) \wedge ((1-s)r)\, U_N(\sigma'_r) dr$$

Summary

Theorem 5

$K \in \mathbb{R}$, $N > 0$

(1) *TFAE*

- **(i)** $\mathbf{CD}^*(K, N)$ *& infin. Hilb.*
- **(ii)** $\mathbf{CD}^e(K, N)$ *& infin. Hilb.*
- **(iii)** $^\exists\mathbf{EVI}_{K,N}$*-curves &* **(V)**

(2) *Under* $(X, d, \mathfrak{m})$*: infin. Hilb. & Ass. 1,* *either* **(iv)–(vi)** *is also equiv. to* **(i)–(iii)**

- **(iv)** *Space-time* W_2*-control*
- **(v)** *Bakry-Ledoux gradient estimate*
- **(vi)** *Bochner inequality*

1. Introduction

2. Entropic curvature-dimension condition

3. (K, N)-evolution variational inequality

4. Connection with Bakry-Émery theory

5. Applications

Properties of $\mathbf{RCD}^*(K,N)$

- Stability under mGH (or Sturm's $\mathbb{D}$)-conv.
- Tensorization
- From local to global
- Measure contraction property $\mathbf{MCP}(K,N)$
  (via $\mathbf{CD}^*(K,N)$; [Cavalletti & Sturm])
  $\Rightarrow$ (sharp) Bishop-Gromov volume comparison
  $\Rightarrow$ (sharp) Bonnet-Myers diameter bound

Properties of RCD$^*(K, N)$

$\Downarrow$

– volume doubling property

– (local unif.) Poincaré ineq. [Rajala]

$\Rightarrow$ $^\exists$heat kernel, two-sided Gaussian bound

$\Rightarrow$ Ultracontractivity of P_t

- Lipschitz regularity of the heat kernel/eigenfn.'s
- Lichnerowicz bound of λ_1 [Ketterer]
- Li-Yau's ineq. [Garofalo & Mondino]

Properties of RCD$^*(K,N)$

- N-precision of f'nal ineq.'s
  – N-HWI ineq.

$$\frac{U_N(\mu_1)}{U_N(\mu_0)} \le \mathfrak{s}'_{K/N}(W_2) + \frac{1}{N}\mathfrak{s}_{K/N}(W_2)\sqrt{I(\mu_0)}$$

$$\left(I(\rho\mathfrak{m}) = 4\int_X |\nabla\sqrt{\rho}|_w^2 \, d\mathfrak{m}\text{: Fisher info.} \right)$$

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- N-precision of f'nal ineq.'s

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  $\Rightarrow$ N-log Sobolev ineq. $(K > 0)$

  $$KN\left(U_N(\mu)^{-2} - 1\right) \le I(\mu)$$

$$\left(I(\rho\mathfrak{m}) = 4\int_X |\nabla\sqrt{\rho}|_w^2 \, d\mathfrak{m}\text{: Fisher info.} \right)$$

Properties of $\mathsf{RCD}^*(K,N)$

- N-precision of f'nal ineq.'s

  – N-HWI ineq.

$$\frac{U_N(\mu_1)}{U_N(\mu_0)} \le \mathfrak{s}'_{K/N}(W_2) + \frac{1}{N}\mathfrak{s}_{K/N}(W_2)\sqrt{I(\mu_0)}$$

$\Rightarrow$ N-log Sobolev ineq. $(K > 0)$

$$KN\left(U_N(\mu)^{-2} - 1\right) \le I(\mu)$$

$\Rightarrow$ N-Talagrand ineq. $(K > 0)$

$$U_N(\mu) \le \mathfrak{s}'_{K/N}(W_2(\mu, \mathfrak{m}))$$

$\left(I(\rho\mathfrak{m}) = 4\int_X |\nabla\sqrt{\rho}|_w^2 d\mathfrak{m}\text{: Fisher info.}\right)$

Properties of $\mathsf{RCD}^*(K, N)$

Maximal diameter theorem [Ketterer]

$(X, d, \mathfrak{m})$: $\mathsf{RCD}^*(N, N+1)$ $(N \geq 0)$,

$\mathrm{diam}(X) = \pi$

$\Rightarrow (X, d, \mathfrak{m}) \underset{\text{isom}}{\simeq} N$-spherical susp. of $^\exists (X', d', \mathfrak{m}')$,

- $N \geq 1 \Rightarrow (X', d', \mathfrak{m}')$: $\mathsf{RCD}^*(N-1, N)$
- $N < 1 \Rightarrow \# X' = 1$ or
  $\# X' = 2$ with $\mathrm{diam}(X') = \pi$

(homeo under **MCP** [Ohta '07])