

On the speed in transportation costs of heat distributions

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1. Introduction

Speed in transportation cost

$\partial_t \mu_t = \Delta \mu_t$: heat distribution

$\Rightarrow (\mu_t)_{t \geq 0}$: curve in $\mathcal{P}(M)$

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Q.1 $\mathcal{T}_c(\mu_t, \mu_s) \approx ? \quad (s \rightarrow t)$

$$\mathcal{T}_c(\mu, \nu) := \inf \left\{ \int_{M \times M} c \, d\pi \mid \begin{array}{l} \pi: \text{coupling of} \\ \mu \text{ and } \nu \end{array} \right\}$$

(Optimal transportation cost for a cost function c)

Q.2 Applications?

Background

On Q.1: Speed of gradient curve

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$\mu_t = e^{t\Delta} \mu_0$: heat dist. on a met. meas. sp. (M, d, ν)

$\Rightarrow$ “ $\partial_t \mu_t = -\nabla \text{Ent}_\nu(\mu_t)$ ” w.r.t. $W_2 = (\mathcal{T}_{d^2})^{1/2}$

[Jordan, Kinderlehrer & Otto '98]

[Ambrosio, Gigli & Savaré '05, ...]

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On Q.1: Speed of gradient curve

$\mu_t = e^{t\Delta} \mu_0$: heat dist. on a met. meas. sp. (M, d, v)

$\Rightarrow$ “ $\partial_t \mu_t = -\nabla \text{Ent}_v(\mu_t)$ ” w.r.t. $W_2 = (\mathcal{T}_{d^2})^{1/2}$

[Jordan, Kinderlehrer & Otto '98]

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$$“\partial_t \text{Ent}_v(\mu_t) = -\langle \nabla \text{Ent}_v, \partial_t \mu_t \rangle = -|\partial_t \mu_t|^2”$$

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$$“\partial_t \text{Ent}_\nu(\mu_t) = -\langle \nabla \text{Ent}_\nu, \partial_t \mu_t \rangle = -|\partial_t \mu_t|^2”$$

$$\overline{\lim}_{s \downarrow t} \left(\frac{W_2(\mu_s, \mu_t)}{s - t} \right)^2 \stackrel{\Downarrow}{=} \int_M \frac{|\nabla \rho_t|^2}{\rho_t} d\nu =: I(\mu_t)$$

(Fisher information)

Background

On Q.2: Monotonicity of transportation costs

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“ $\text{Hess Ent}_v \geq K$ ” ($\Leftrightarrow$ “ $\text{Ric} \geq K$ ”) for $K \in \mathbb{R}$

$\Downarrow$

$e^{Kt} W_2(\mu_t^{(0)}, \mu_t^{(1)}) \searrow \text{ in } t$

$\Downarrow$

$e^{Kt} W_2(\mu_t, \mu_{t+s}) \searrow \text{ in } t$

Background

On Q.2: Monotonicity of transportation costs

“Hess Ent_{*v*} ≥ *K*” (⇔ “Ric ≥ *K*”) for *K* ∈ ℝ

⇓

$$e^{Kt} W_2(\mu_t^{(0)}, \mu_t^{(1)}) \searrow \text{ in } t$$

⇓

$$e^{Kt} W_2(\mu_t, \mu_{t+s}) \searrow \text{ in } t$$

⇓

$$I(\mu_t) \leq e^{-2Kt} I(\mu_0)$$

(⇒ log Sobolev ineq. (when *K* > 0))

Remark on backgrounds

★ $\left. \begin{array}{l} \text{Q1} \\ \text{Q2} \end{array} \right\} \rightsquigarrow \text{grad. flow } \dot{\mu}_t = -\nabla U(\mu_t) \text{ on } \mathcal{P}(M)$
 [Ambrosio, Gigli & Savaré '05]

- Q1 $\rightsquigarrow$ Energy dissipation equality

$$-\frac{d}{dt}U(\mu_t) = \frac{1}{2}|\dot{\mu}_t|^2 + \frac{1}{2}|\nabla U|(\mu_t)^2$$

- Q2 $\rightsquigarrow$ (K -)Evolution variational inequality

$$\frac{1}{2}e^{-Kt}\frac{d}{dt}\left(e^{Kt}W_2(\mu_t, \nu)^2\right) \leq U(\nu) - U(\mu_t)$$

($\forall \nu \in \mathcal{P}_2(M)$)

Questions

- What happens for other trans. costs than $\mathcal{T}_{d^2}$?
- What happens when there is no gradient flow structure?

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- What happens when there is no gradient flow structure?
 - ↪ Heat distributions on (backward) Ricci flow

1. Introduction

2. Heat distributions on backward Ricci flow

3. Coupling methods (Thm 1 & 2)

4. Idea of the proof of Thm 3 & 5

5. Further problems

Framework

- $(M, g(t))$: cpl. Riem. mfd., $t \in [0, T]$
 $\partial_t g(t) = 2 \operatorname{Ric}_t$ (backward Ricci flow)
- $((X(t))_{t \geq 0}, (\mathbb{P}_x)_{x \in M})$: $g(t)$ -Brownian motion
 $\iff \Delta_{g(t)}$: generator
 $\mu_t = \mathbb{P}_{\mu_0} \circ X(t)^{-1}$: heat dist.
- v_t : $g(t)$ -volume meas., $\mu_t = \rho_t v_t$
 $\star \partial_t v_t = R_t v_t$ (R_t : $g(t)$ -scalar curv.)

Ass. $\sup_t |\operatorname{Rm}_t|_{g(t)} < \infty$ (Rm_t : $g(t)$ -curv. tensor)

$$\partial_t \mu_t \neq -\nabla \text{Ent}_{v_t}(\mu_t)$$

$$\text{Ent}_{v_t}(\mu_t) := \int_M \rho_t \log \rho_t \, dv_t = \int_M \log \rho_t \, d\mu_t$$

$$\star \partial_t \mu_t = \Delta_t \mu_t \text{ (weakly)}$$

$$\begin{aligned} \Rightarrow \partial_t \text{Ent}_{v_t}(\mu_t) &= - \int_M \left(\frac{|\nabla \rho_t|^2}{\rho_t^2} + R_t \right) d\mu_t \\ &=: -\mathcal{F}(\mu_t) \quad (\mathcal{F}\text{-functional}) \end{aligned}$$

$\Rightarrow$ No monotonicity of $\text{Ent}_{v_t}(\mu_t)$!

Monotonicity of transportation costs

Observation

When $g(t) \equiv g_0$, $\partial_t g(t) = 2 \operatorname{Ric}_t \Rightarrow \operatorname{Ric} \equiv 0$

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$\rightsquigarrow$ Monotonicity of transportation cost

★ $\mathcal{T}_{d_t^2}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$

- [McCann & Topping '10]: Opt. trans.
- [Arnaudon, Coulibaly & Thalmaier '09], [K. '12]:
Stochastic analysis (coupling of BMs)

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Stochastic analysis (coupling of BMs)

$\not\Rightarrow \mathcal{T}_{d_t^2}(\mu_t, \mu_{t+s}) \searrow$ (time-inhomogeneity)

Monotonicity of transportation costs

$$L_{\alpha}^{t,t'}(x,y) := \inf_{\substack{\gamma(t)=x, \\ \gamma(t')=y}} \left[\int_t^{t'} r^{\alpha/2} (|\dot{\gamma}(r)|_r^2 + R_r(\gamma(r))) dr \right]$$

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Theorem 1 ([Lott '09], [Amaba & K.]

$$\mathcal{T}_{L_0^{t,t+s}}(\mu_t, \mu_{t+s}) \searrow \text{ in } t$$

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Theorem 2 ([Topping '09], [K. & Philipowski '11])

$$\Xi_{\tau_0, \tau_1}(t) := (\sqrt{\tau_1 t} - \sqrt{\tau_0 t}) \mathcal{T}_{L_1^{\tau_0 t, \tau_1 t}}(\mu_{\tau_0 t}, \mu_{\tau_1 t}) - m(\sqrt{\tau_1 t} - \sqrt{\tau_0 t})^2$$

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Monotonicity of transportation costs

Comparison of results

- [Lott '09], [Topping '09]:
 - Optimal transportation
 - Ass: M : cpt.
- [K. & Amaba], [K. & Philipowski '11]
 - Stochastic analysis
 - Ass: $\text{Ric}_t \geq \exists K g(t) \ (\forall t)$

Monotonicity of transportation costs

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Recall:

$$\underline{\text{Ass.}} \quad \sup_t |\text{Rm}_t|_{g(t)} < \infty \quad (\text{Rm}_t: g(t)\text{-curv. tensor})$$

Monotonicity of $\mathcal{F}$

Theorem 3

Suppose $\text{Ent}_{v_0}(\mu_0) < \infty$ and $\mathcal{F}(\mu_0) < \infty$

$$\Rightarrow \lim_{s \downarrow 0} \frac{\mathcal{T}_{L_0^{t,t+s}}(\mu_t, \mu_{t+s})}{s} = \mathcal{F}(\mu_t) \text{ a.e. } t \in [0, T]$$

Corollary 4

$$\mathcal{F}(\mu_t) \searrow$$

- Rem: $g(t) \equiv g$, $\text{Ric} \geq 0 \Rightarrow I(\mu_t) \searrow$
- [Lott '09] when M : cpt.
by Eulerian calculus (requires smoothness)

Monotonicity of $\mathcal{W}$ -entropy

Theorem 5

Suppose $\text{Ent}_{v_0}(\mu_0) < \infty$ and $\mathcal{F}(\mu_0) < \infty$

$$\Rightarrow \lim_{s \downarrow 0} \frac{\mathcal{T}_{L_1^{t,t+s}}(\mu_t, \mu_{t+s})}{s} = \sqrt{t} \mathcal{F}(\mu_t) \text{ a.e. } t \in (0, T]$$

Corollary 6

$$t^2 \mathcal{F}(\mu_t) - \frac{mt}{2} \searrow. \text{ In particular, } \mathcal{W}(\mu_t) \searrow$$

$$\mathcal{W}(t) := t \mathcal{F}(\mu_t) - \text{Ent}(\mu_t) - \frac{m \log t}{2} + \text{const.}$$

- [Topping '09] when M : cpt.

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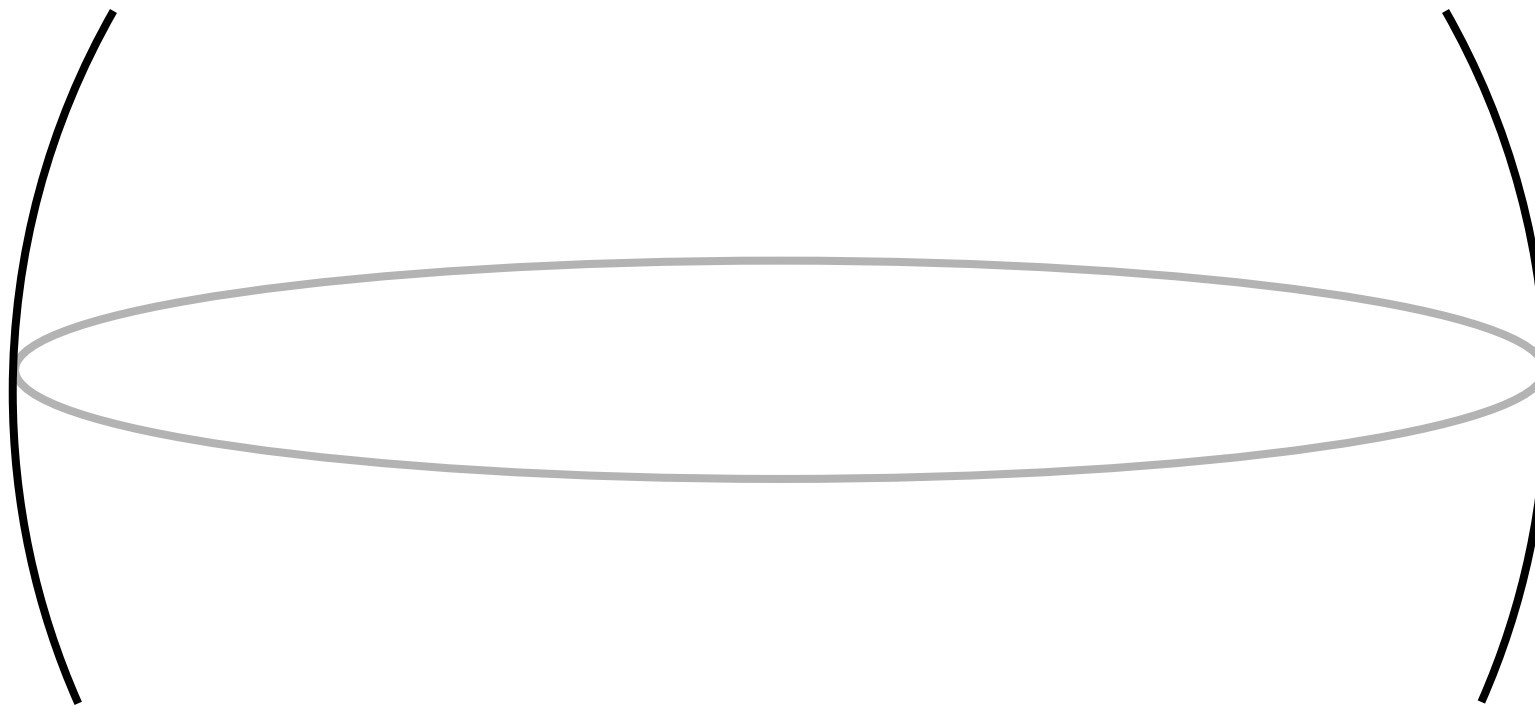
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$(X_0(t), X_1(t))$: coupling of BMs moving parallelly

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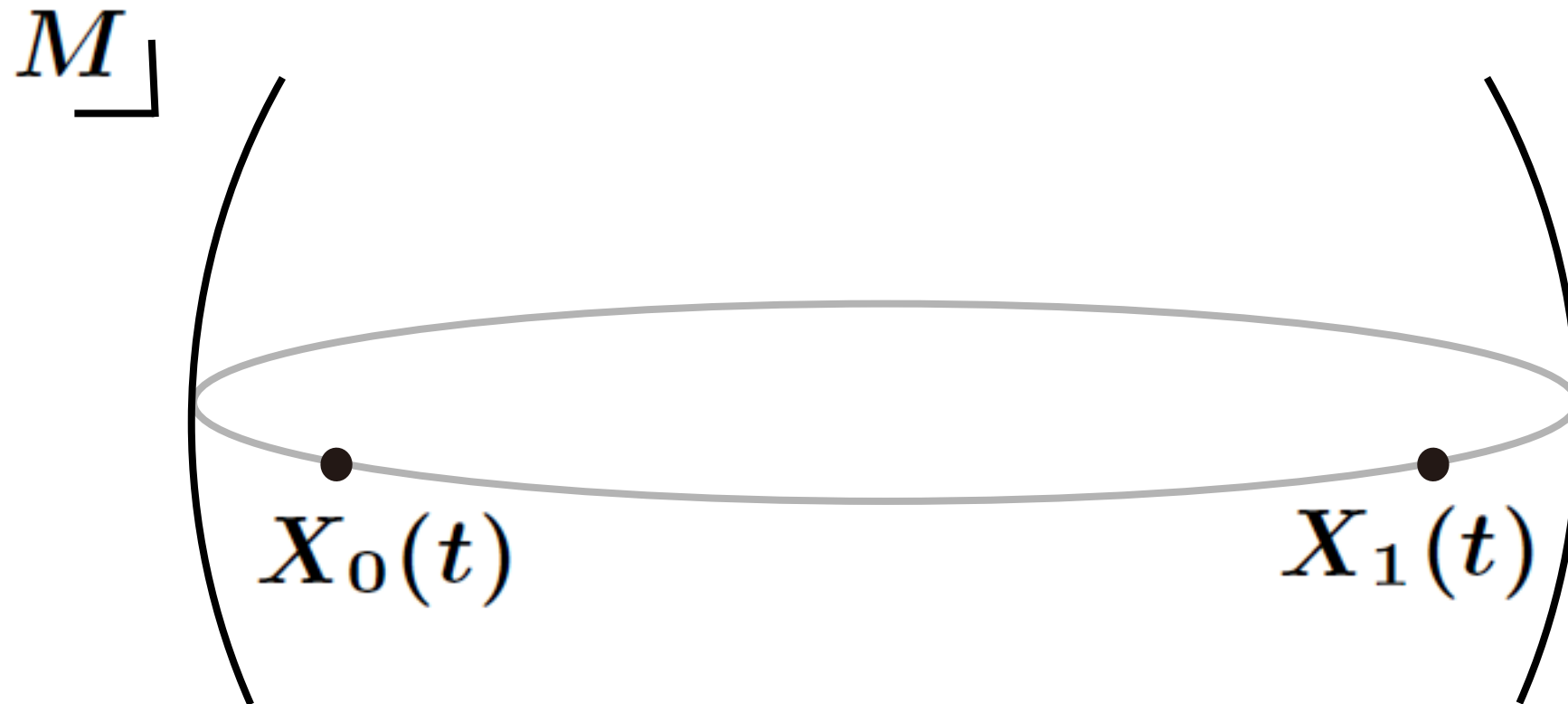
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$M_{\perp}$



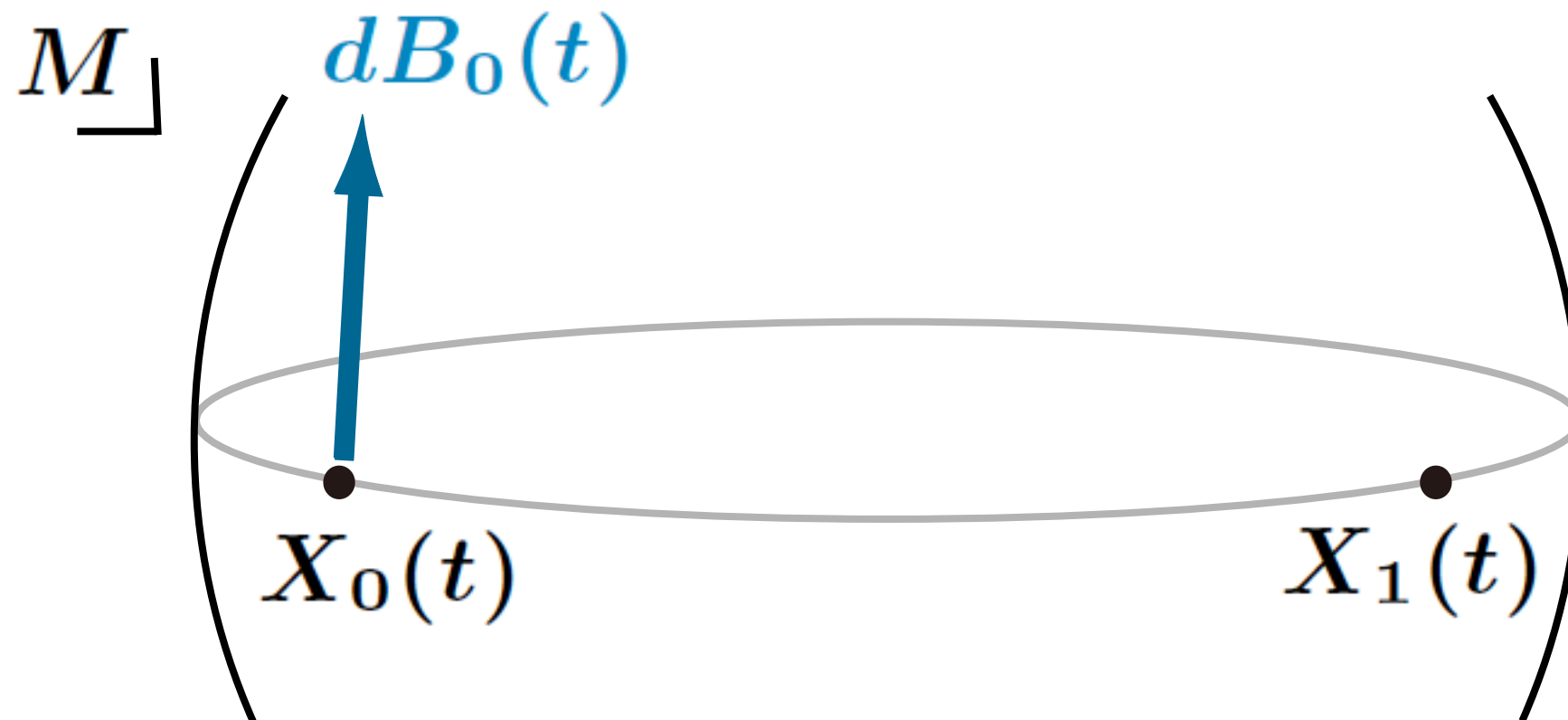
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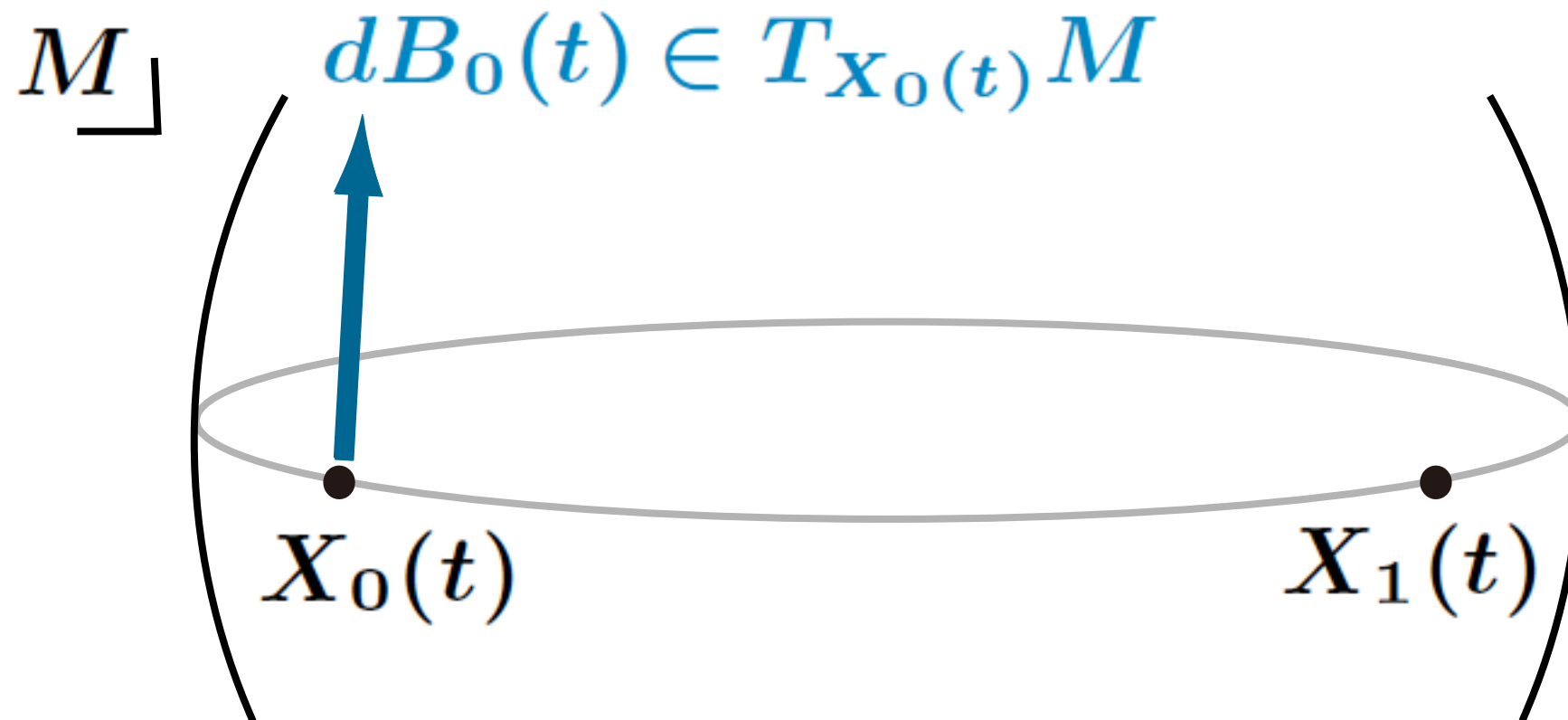
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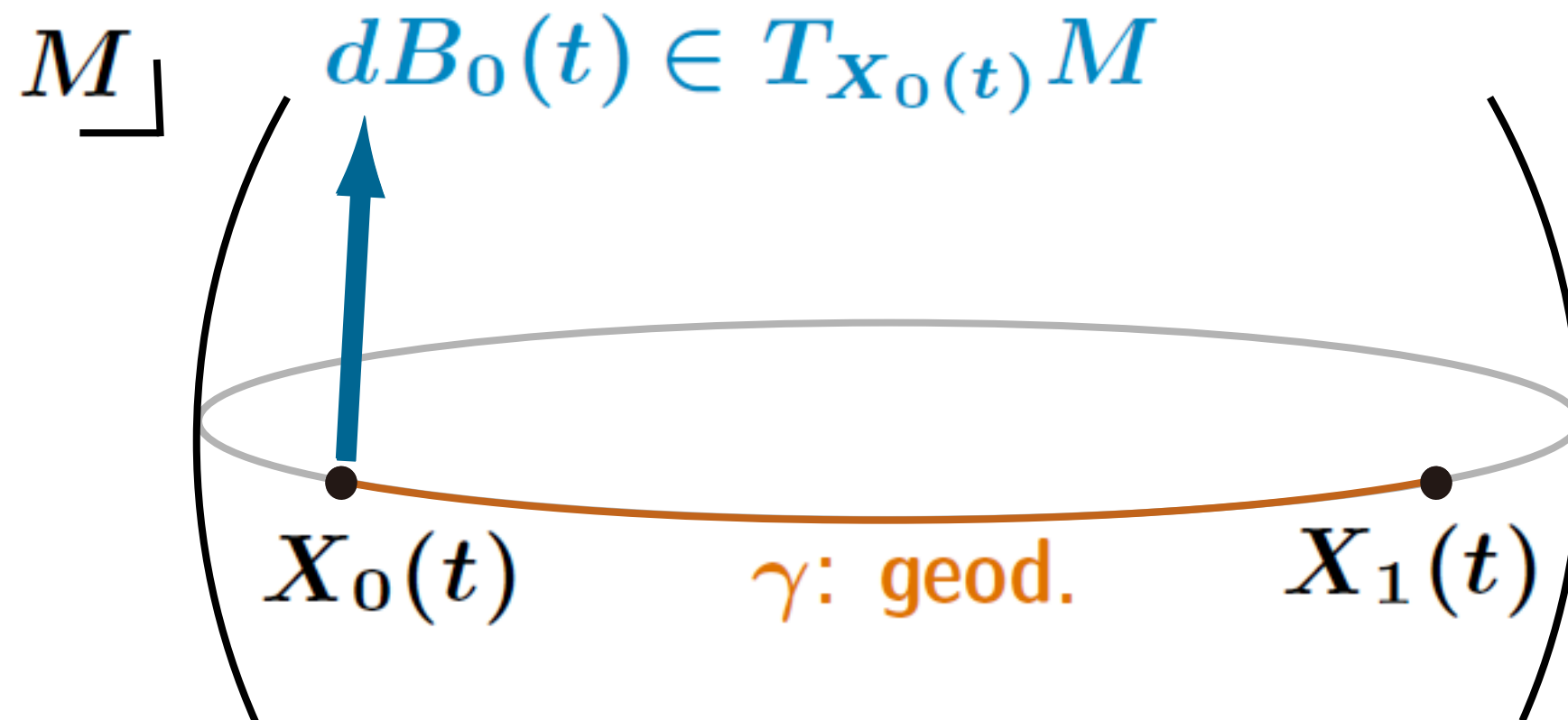
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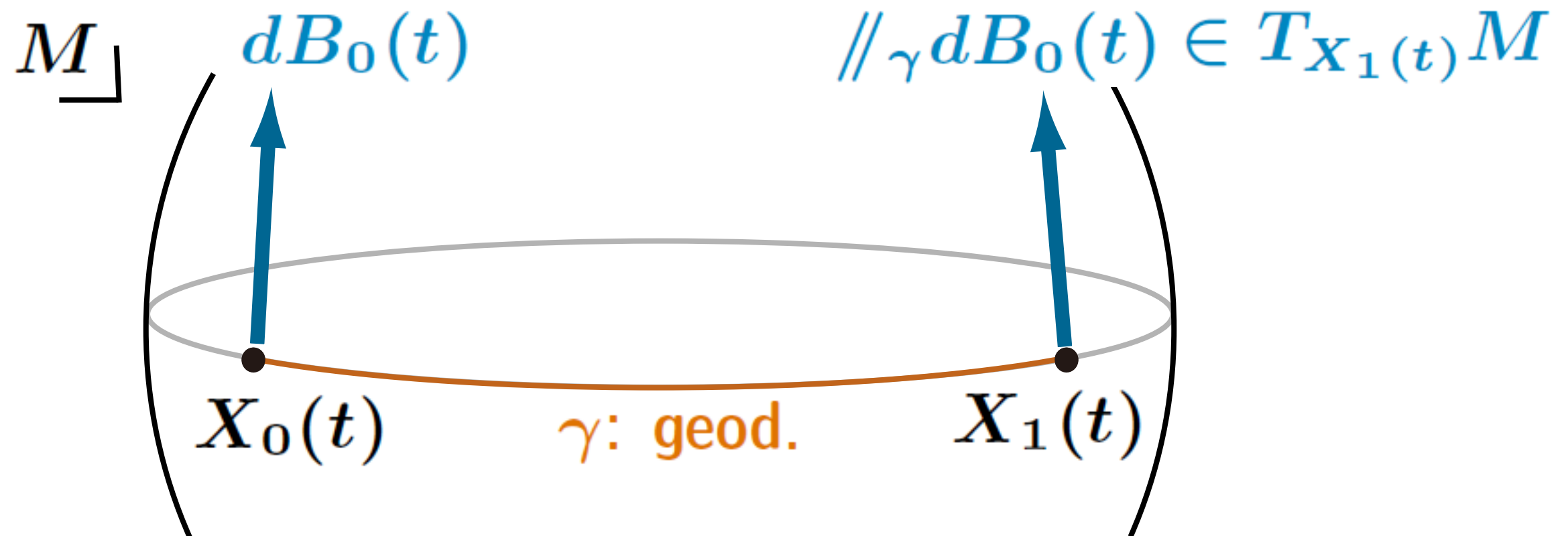
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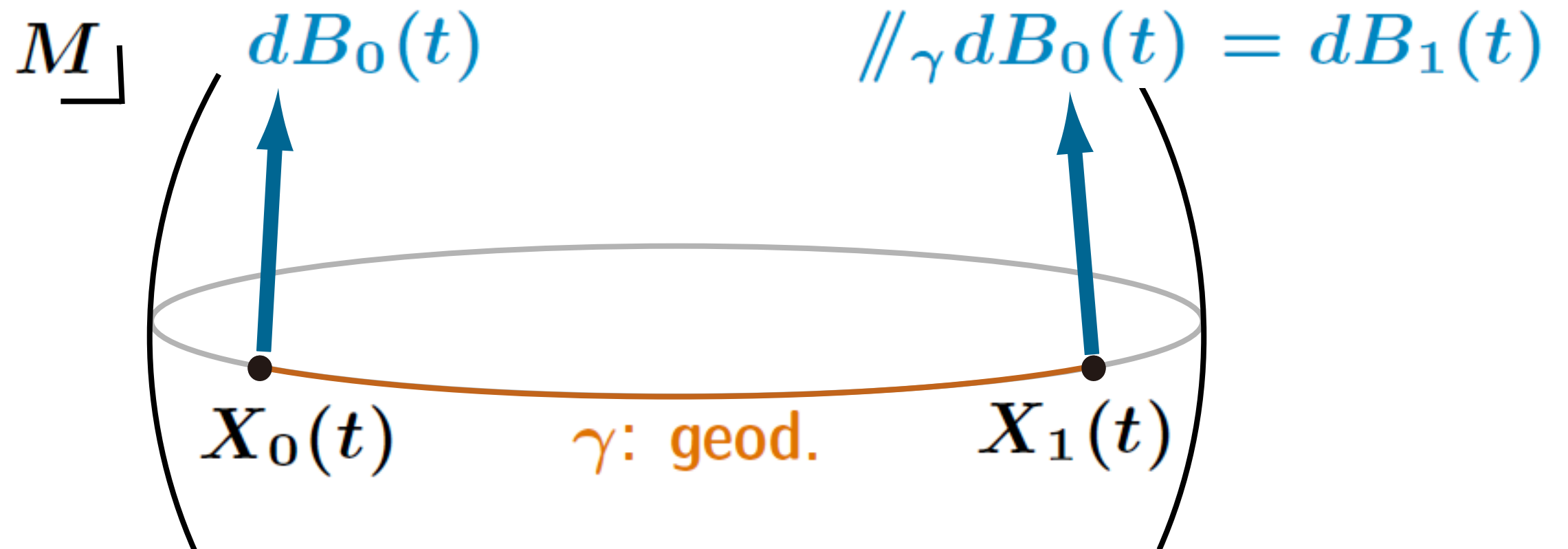
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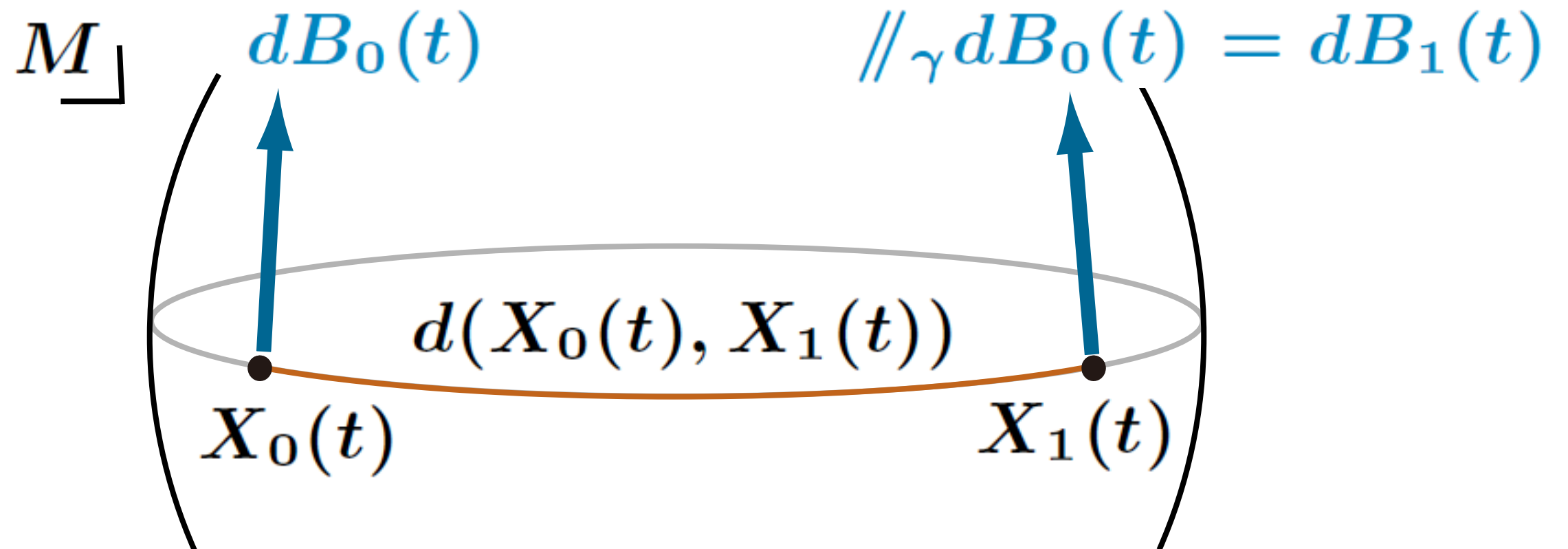
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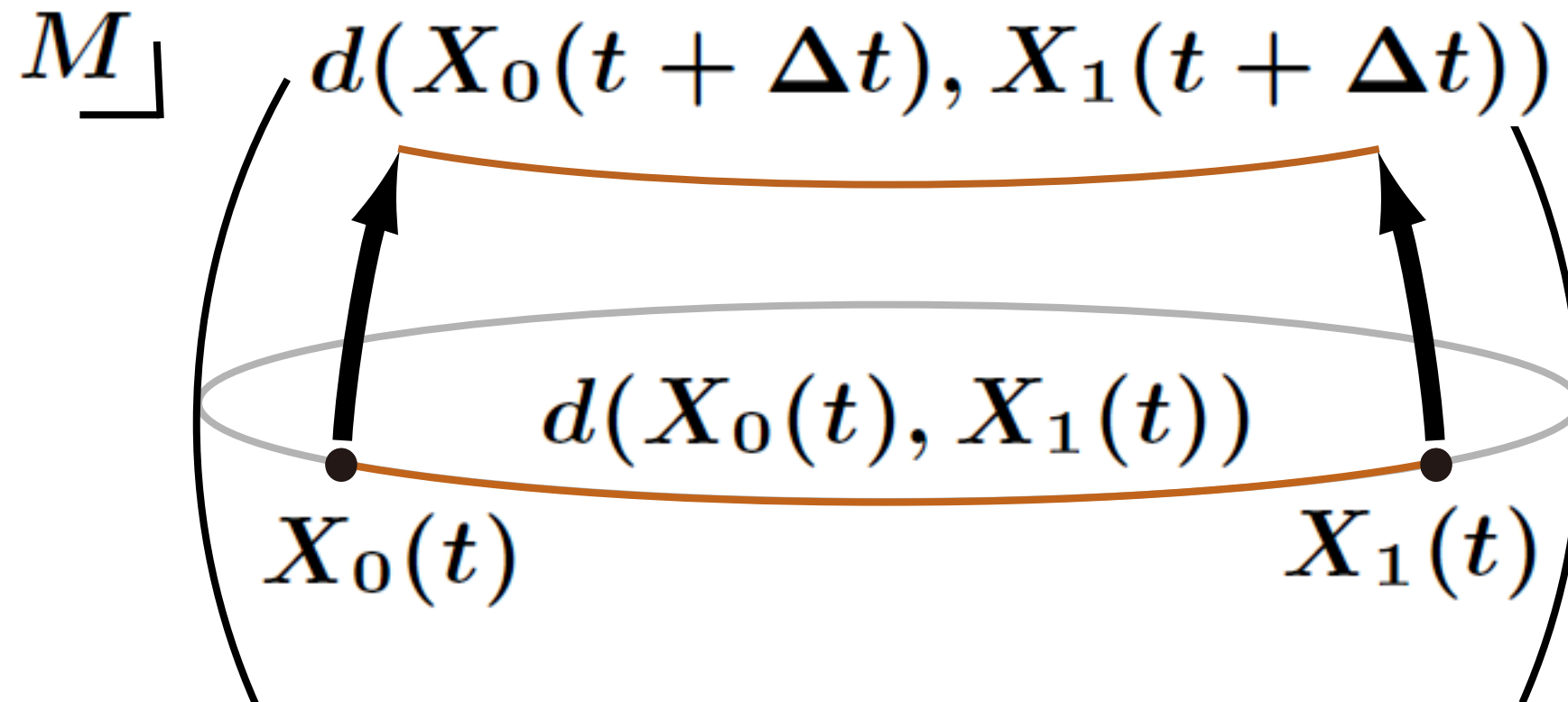
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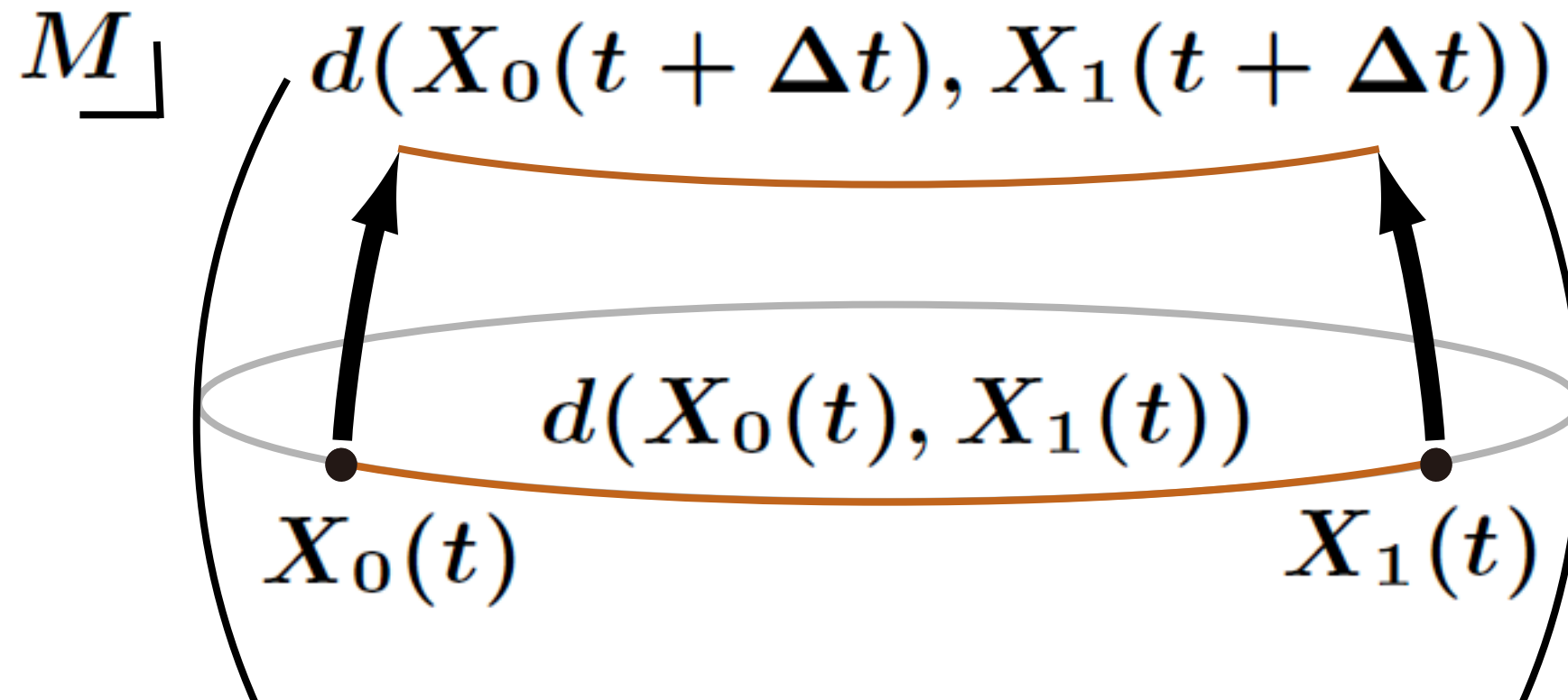
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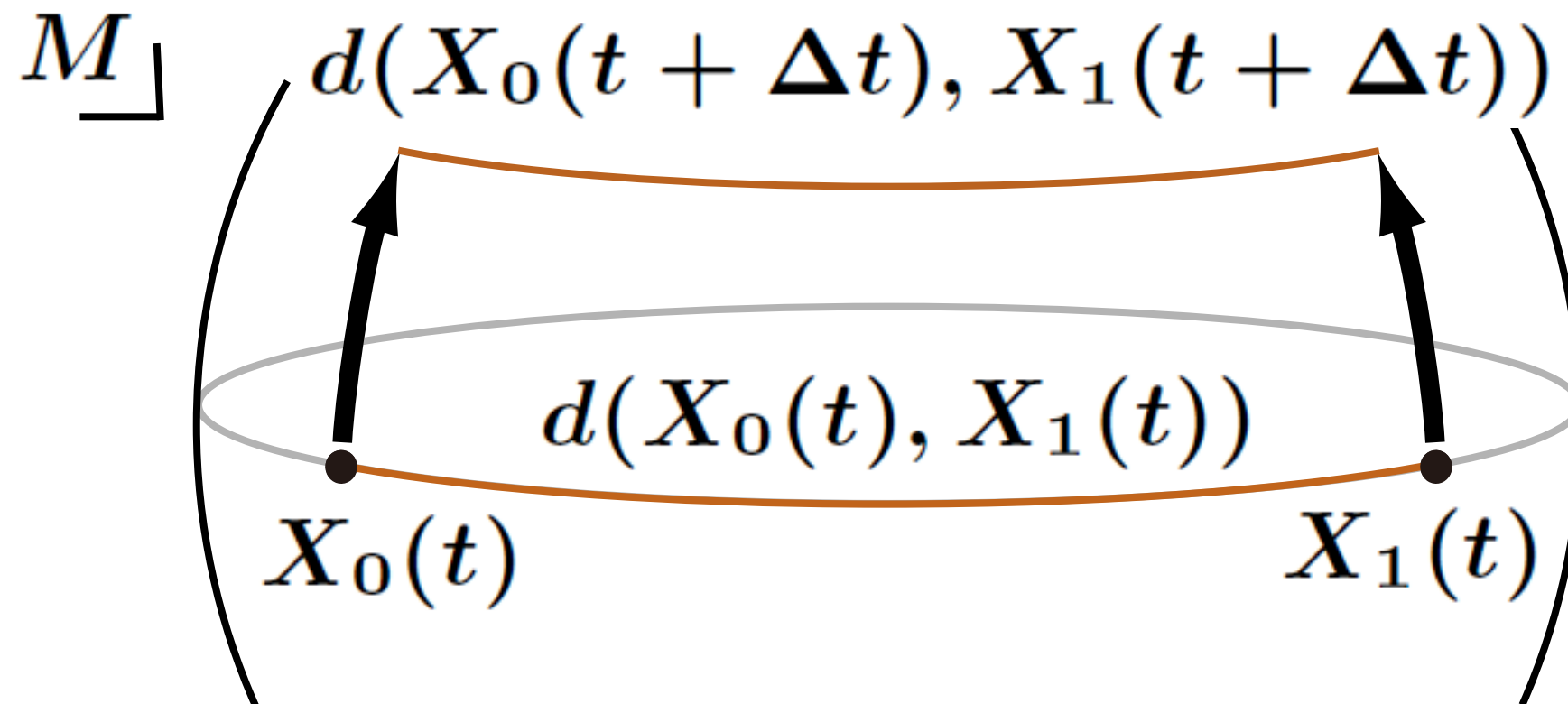
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- (mart. part of $d(X_0(t), X_1(t))$) = 0

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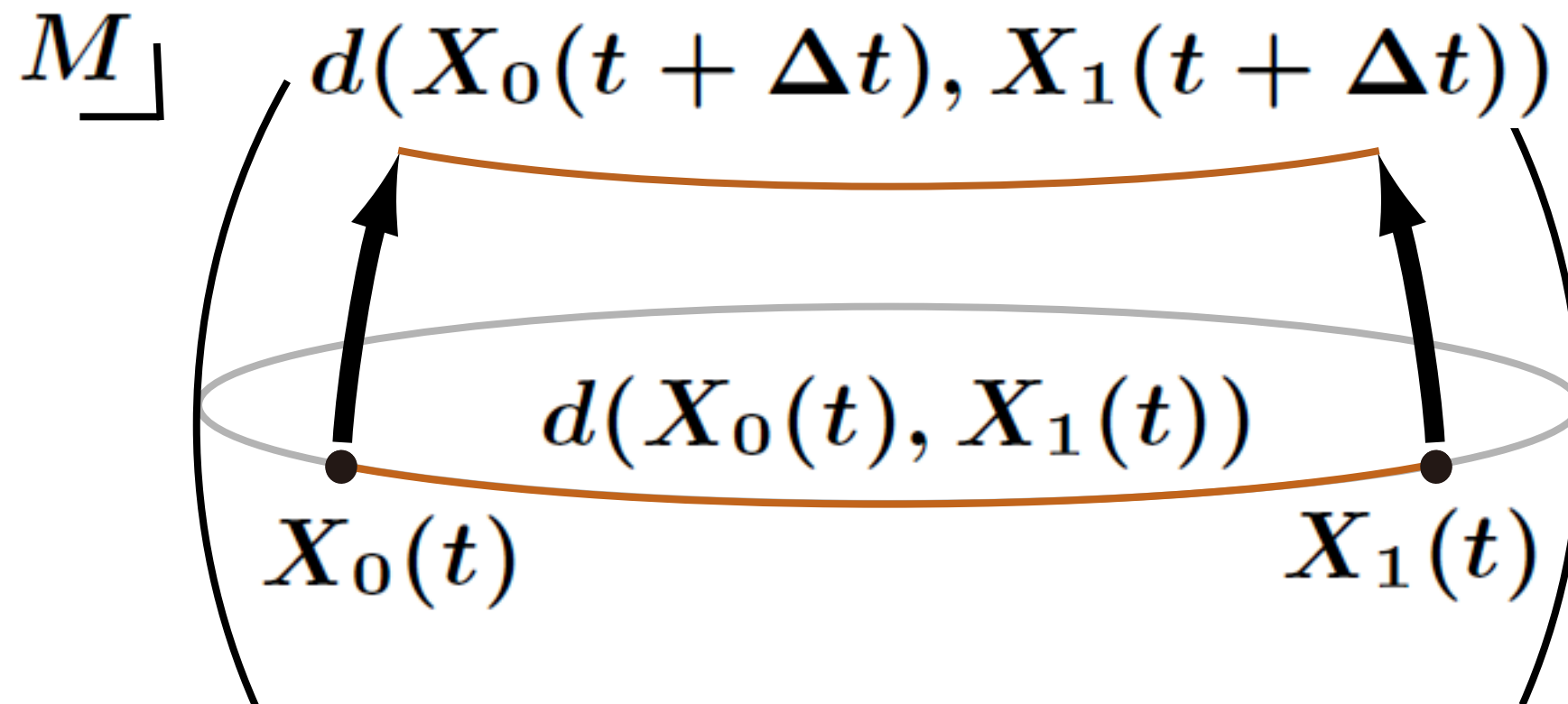
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- (“bdd. var.” part): Controlled by $\text{Ric} \geq K$

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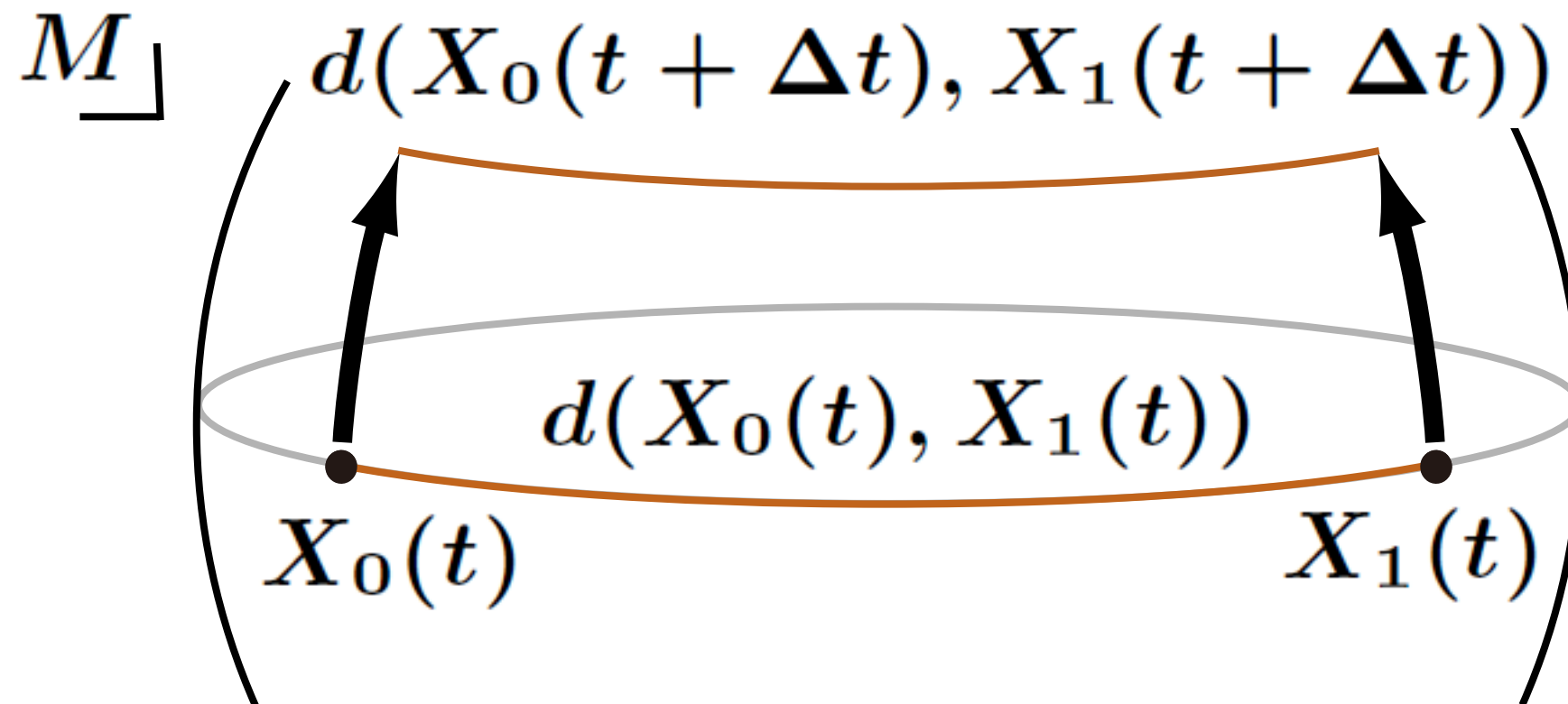
- (mart. part of $d(X_0(t), X_1(t))$) = 0
- (“bdd. var.” part): Controlled by $\text{Ric} \geq K$

$\Downarrow$

$$\left\langle \frac{\partial}{\partial t} d(X_0(t), X_1(t)) \right\rangle \leq -K d(X_0(t), X_1(t))$$

Time-homogeneous case (for $\mathcal{T}_{d^2}$)

$(X_0(t), X_1(t))$: coupling of BMs moving parallelly



$$\therefore \text{Ric} \geq K$$

$$\Rightarrow e^{pKt} \mathcal{T}_{d^p}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow \text{ in } t \quad (1 \leq \forall p < \infty)$$

Ricci flow case (for L_0/L_1)

- Properties of L_0
being analogous to the Riem. dist.
(geodesic (minizing curve), 1st & 2nd variation,)
(index lemma, cut locus, ...)
- Coupling of $dX_0(t)$ and $dX_1(t + s)$

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- Coupling of $dX_0(t)$ and $dX_1(t + s)$
by space-time parallel transport

For $\gamma : [s, t] \rightarrow M$ & V : vector field along γ ,

$$\nabla_{\dot{\gamma}(u)}^{g(u)} V(u) = -\frac{1}{2} \partial_u g(u)^\# V(u)$$

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by space-time parallel transport along L_0 -geodesic

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$$\nabla_{\dot{\gamma}(u)}^{g(u)} V(u) = -\frac{1}{2} \partial_u g(u)^\# V(u)$$

Ricci flow case (for L_0/L_1)

- Properties of L_1

being analogous to the Riem. dist.

(geodesic (minizing curve), 1st & 2nd variation,)
(index lemma, cut locus, ...)

- Coupling of $dX_0(\tau_0 t)$ and $dX_1(\tau_1 t)$

by space-time parallel transport along L_1 -geodesic
& scaling

For $\gamma : [s, t] \rightarrow M$ & V : vector field along γ ,

$$\nabla_{\dot{\gamma}(u)}^{g(u)} V(u) = -\frac{1}{2} \partial_u g(u)^\# V(u)$$

Ricci flow case (for L_0/L_1)

Technicalities

- Non-smoothness of L_0/L_1 at their cut loci
 $\Leftarrow$ Approximation by coupling of random walks
 (Differential ineq. $\rightsquigarrow$ Difference ineq.)
- Lack of a (global) upper bound of **Ric**
 $\Leftarrow$ Localization by stopping times

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Remark

- Many other approaches in time-homogeneous case
- A method in [Arnaudon, Coulibaly & Thalmaier '09]
does not seem to work

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Kantorovich duality

Observation: $\mathcal{T}_{d^2}$ under $g(t) \equiv g$

$$\frac{\mathcal{T}_{d^2}(\mu_t, \mu_{t+s})}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s \varphi d\mu_{t+s} - \int_M \varphi d\mu_t \right]$$

$$Q_s \varphi(x) := \inf_{y \in M} \left(\varphi(y) + \frac{d(y, x)^2}{2s} \right)$$

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$$\begin{aligned} Q_s \varphi(x) &:= \inf_{y \in M} \left(\varphi(y) + \frac{d(y, x)^2}{2s} \right) \\ &= \inf_{\substack{\gamma: [t, t+s] \rightarrow M \\ \gamma(t+s) = x}} \left(\varphi(\gamma(t)) + \frac{1}{2} \int_t^{t+s} |\dot{\gamma}(r)|^2 \, dr \right) \end{aligned}$$

(Hopf-Lax semigroup)

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(Hopf-Lax semigroup)

$$\star \quad \partial_s Q_s \varphi + \frac{1}{2} |\nabla Q_s \varphi|^2 = 0 \quad (\text{Hamilton-Jacobi eq.})$$

Upper bound

$$\frac{\mathcal{T}_{d^2}(\mu_t, \mu_{t+s})}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s \varphi d\mu_{t+s} - \int_M \varphi d\mu_t \right]$$

$$[\dots] = \int_0^s \left(\partial_r \int_M Q_r \varphi d\mu_{t+r} \right) dr$$

$$\begin{aligned} (\partial_r \dots) &= \int \left(- \left\langle \nabla Q_r \varphi, \frac{\nabla \rho_{t+r}}{\rho_{t+r}} \right\rangle - \frac{1}{2} |\nabla Q_r \varphi|^2 \right) d\mu_{t+r} \\ &\leq \frac{1}{2} I(\mu_{t+r}) \end{aligned}$$

Upper bound

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$$[\dots] = \int_0^s \left(\partial_r \int_M Q_r \varphi d\mu_{t+r} \right) dr$$

$$(\partial_r \dots) = \int \left(- \left\langle \nabla Q_r \varphi, \frac{\nabla \rho_{t+r}}{\rho_{t+r}} \right\rangle - \frac{1}{2} |\nabla Q_r \varphi|^2 \right) d\mu_{t+r}$$

$$\leq \frac{1}{2} I(\mu_{t+r})$$

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- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry
No a priori integrability of $\nabla \rho_t$ & $\Delta_t \rho_t$

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1. Introduction
2. Heat distributions on backward Ricci flow
3. Coupling methods (Thm 1 & 2)
4. Idea of the proof of Thm 3 & 5
- 5. Further problems**

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$\text{Ric} \geq K \Rightarrow$ Wang's Harnack inequality

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$$\mathcal{T}_{d^2}(\mu, \nu) \geq \int g_1 d\nu - \int g_0 d\mu$$

($\because$ Kantorovich duality)

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$$g_1 := c^{-1} \alpha \log P_s \rho_t, \quad g_0 := c^{-1} \log P_s(\rho_t^\alpha)$$

$$\begin{aligned} \lim_{s \downarrow 0} \frac{\mathcal{T}_{d^2}(\mu_t, \mu_{t+s})}{s^2} &\geq \lim_{s \downarrow 0} \frac{1}{s^2} \left(\int g_1 d\mu_{t+s} - \int g_0 d\mu_t \right) \\ &= \dots = 4(\alpha - 1)(2 - \alpha)I(\mu_t) \end{aligned}$$

$$\Downarrow \alpha = 3/2$$

$$\lim_{s \downarrow 0} \frac{\mathcal{T}_{d^2}(\mu_t, \mu_{t+s})}{s^2} \geq I(\mu_t)$$



$\mathcal{T}_{d^p}$ (or W_p)

- For $\mathcal{T}_{d^p}$ ($p \in [1, \infty)$) in time-homogeneous case?
 - † $\text{Ric} \geq K \Rightarrow e^{pKt} \mathcal{T}_{d^p}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$
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$\text{Ric} \geq K > 0, \mu_t \rightarrow v \in \mathcal{P}(M)$

$\Rightarrow \mathcal{T}_{d^p}(\mu, v) \leq \frac{1}{K^p} \int_M \frac{|\nabla \rho|^p}{\rho^{p-1}} dv$

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- $u_t \in L^1_+(v)$: The solution to the p_* -heat eq.

$$\partial_t u = \operatorname{div}(|\nabla u|^{p_*-2} \nabla u)$$

(or gradient flow of $\int |\nabla f|^{p_*} dv$)

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[Ambrosio, Gigli & Savaré, '12], [Kell]

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Q. Monotonicity of W_p on “Riemannian” spaces?

(False on Finsler mfd's with $p = 2$)

$\mathcal{W}$ -entropy

Facts

- Monotonicity of $\mathcal{W}$ -entropy

$$\mathcal{W}(t) := tI(\mu_t) - \text{Ent}(\mu_t) - \frac{N}{2} \log t + (\text{const.})$$

on N -dim. Riem. mfd's with $\text{Ric} \geq 0$

- Rigidity: $\mathcal{W}$ -entropy is constant iff $M \simeq \mathbb{R}^n$

[L. Ni '04], [X.-D. Li '11], ...

Q. (X.-D. Li) The same for $\text{RCD}^*(0, N)$ spaces?

(The proof on smooth spaces relies on differential calc.)

★ Monotonicity holds on $\text{RCD}^*(0, N)$ met. meas. sp.