Wasserstein controls for heat distributions

Kazumasa Kuwada

(Tokyo Institute of Technology)

Stochastic Processes, Analysis and Mathematical Physics

In honor of Professor Masatoshi Fukushima's Sanju

(Kansai University)

Aug. 25-29, 2014

1. Introduction & Overview

What's Wasserstein control?

$\mu_t = \mathrm{e}^{t\mathcal{L}}\mu_0$: law at time t of diffusion process
on met. meas. sp. (M, d, v) generated by $\mathcal{L}$

$W_p(\mu, \nu) = \inf\{\|d\|_{L^p(\pi)} \mid \pi\text{: coupling of } \mu \text{ and } \nu\}$
(L^p-Wasserstein distance ($p \in [1, \infty]$))

<u>Wasserstein control</u>

A (Lipschitz type) upper bound of $W_p(\mu_t, \mu'_t)$
by $W_p(\mu_0, \mu'_0)$

($\mu_t = \mathrm{e}^{t\mathcal{L}}\mu_0$, $\mu'_t = \mathrm{e}^{t\mathcal{L}}\mu'_0$)

What's Wasserstein control?

$\mu_t = \mathrm{e}^{t\mathcal{L}}\mu_0$: law at time t of diffusion process
on met. meas. sp. (M, d, v) generated by $\mathcal{L}$

$W_p(\mu, \nu) = \inf\{\|d\|_{L^p(\pi)} \mid \pi$: coupling of μ and $\nu\}$
(L^p-Wasserstein distance ($p \in [1, \infty]$))

Space-time Wasserstein control

A (Lipschitz type) upper bound of $W_p(\mu_s, \mu'_t)$
by $W_p(\mu_0, \mu'_0)$ and "difference" between t & s
($\mu_t = \mathrm{e}^{t\mathcal{L}}\mu_0$, $\mu'_t = \mathrm{e}^{t\mathcal{L}}\mu'_0$)

Coupling by parallel transport

On a Riem. mfd M:

$({}^{\exists}X_t, {}^{\exists}X_t')$: coupling of BMs by parallel transp.

$$\Downarrow$$

$$\boxed{\mathrm{Ric} \geq K \Rightarrow W_p(\mu_t, \mu_t') \leq \mathrm{e}^{-Kt} W_p(\mu_0, \mu_0')}$$

$$({}^{\forall}p \in [1, \infty])$$

Similarly, on $\mathbb{R}^m$, $\mathrm{d}X_t = \mathrm{d}W_t + \nabla V(X_t)\mathrm{d}t$,

$\mathrm{Hess}\, V \geq K \Rightarrow$ the same bound

Coupling by parallel transport

On a Riem. mfd M:

$$\frac{\mathbf{d}}{\mathbf{d}t} d(\ X_t,\ X'_t\ ) \le - \int_{X_t}^{X'_t} \mathrm{Ric}$$

$$\Downarrow$$

$$\boxed{\mathrm{Ric} \ge K \Rightarrow W_p(\mu_t, \mu'_t) \le \mathrm{e}^{-Kt} W_p(\mu_0, \mu'_0)}$$

$$(\forall p \in [1, \infty])$$

Similarly, on $\mathbb{R}^m$, $\mathbf{d}X_t = \mathbf{d}W_t + \nabla V(X_t)\mathbf{d}t$,

$\mathrm{Hess}\, V \ge K \Rightarrow$ the same bound

Duality with gradient estimate

- $W_p(P_t^*\mu, P_t^*\mu') \le C W_p(\mu,\mu')$

  $\Updownarrow$ $\qquad$ $({}^\forall p \in [1,\infty])$

  $|\nabla P_t f| \le C P_t(|\nabla f|^{p_*})^{1/p_*}$

  [K. '10,'13 / Bakry, Gentil & Ledoux]

- $\mathrm{Ric} \ge K$ on a Riem. mfd

  $\Leftrightarrow \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle \ge K|\nabla f|^2$

  $\Leftrightarrow |\nabla P_t f| \le \mathrm{e}^{-Kt} P_t(|\nabla f|^{p_*})^{1/p_*}$

  $\Leftrightarrow W_p(P_t^*\mu, P_t^*\mu') \le \mathrm{e}^{-Kt} W_p(\mu,\mu')$

  [Bakry & Émery '84 / von Renesse & Sturm '05]

Duality with gradient estimate

- $W_p(P_t^* \mu, P_t^* \mu') \le C W_p(\mu, \mu')$

  $\Updownarrow$ $\qquad$ $({}^\forall p \in [1, \infty])$

  $|\nabla P_t f| \le C P_t(|\nabla f|^{p_*})^{1/p_*}$

  [K. '10,'13 / Bakry, Gentil & Ledoux]

- $\mathrm{Ric} \ge K$ on a Riem. mfd

  $$\Leftrightarrow \frac{1}{2}\Delta |\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \ge K |\nabla f|^2$$

  $$\Leftrightarrow |\nabla P_t f| \le \mathrm{e}^{-Kt} P_t(|\nabla f|^{p_*})^{1/p_*}$$

  $$\Leftrightarrow W_p(P_t^* \mu, P_t^* \mu') \le \mathrm{e}^{-Kt} W_p(\mu, \mu')$$

  [Bakry & Émery '84 / von Renesse & Sturm '05]

Duality with gradient estimate

- $W_p(P_t^* \mu, P_t^* \mu') \le C W_p(\mu, \mu')$

  $\Updownarrow$ $\qquad (^\forall p \in [1, \infty])$

  $|\nabla P_t f| \le C P_t(|\nabla f|^{p_*})^{1/p_*}$

  [K. '10,'13 / Bakry, Gentil & Ledoux]

- $\mathrm{Ric} \ge K$ on a Riem. mfd

  $\Leftrightarrow \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \ge K|\nabla f|^2$

  $\Leftrightarrow |\nabla P_t f| \le \mathrm{e}^{-Kt} P_t(|\nabla f|^{p_*})^{1/p_*}$

  $\Leftrightarrow W_p(P_t^* \mu, P_t^* \mu') \le \mathrm{e}^{-Kt} W_p(\mu, \mu')$

  [Bakry & Émery '84 / von Renesse & Sturm '05]

Duality with gradient estimate

- $W_p(P_t^* \mu, P_t^* \mu') \le C W_p(\mu, \mu')$

  $\Updownarrow$ $\quad$ $({}^\forall p \in [1, \infty])$

  $|\nabla P_t f| \le C P_t(|\nabla f|^{p_*})^{1/p_*}$

  [K. '10,'13 / Bakry, Gentil & Ledoux]

- $\mathrm{Ric} \ge K$ on a Riem. mfd

  $\Leftrightarrow \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \ge K|\nabla f|^2$

  $\Leftrightarrow |\nabla P_t f| \le \mathrm{e}^{-Kt} P_t(|\nabla f|^{p_*})^{1/p_*}$

  $\Leftrightarrow W_p(P_t^* \mu, P_t^* \mu') \le \mathrm{e}^{-Kt} W_p(\mu, \mu')$

  [Bakry & Émery '84 / von Renesse & Sturm '05]

Otto calculus

$$\nu_t = P_t^* \mu_0$$

$$\Leftrightarrow \boxed{\text{“}\partial_t \nu_t = -\nabla \operatorname{Ent}(\nu_t)\text{”}} \text{ on } (\mathcal{P}(M), W_2)$$

[Jordan, Kinderlehrer & Otto ’98, Otto ’01]

[Ambrosio, Gigli & Savaré ’05, ...]

Otto calculus

$\nu_t = P_t^* \mu_0$

$\Leftrightarrow$ "$\partial_t \nu_t = -\nabla \operatorname{Ent}(\nu_t)$" on $(\mathcal{P}(M), W_2)$

[Jordan, Kinderlehrer & Otto '98, Otto '01]

[Ambrosio, Gigli & Savaré '05, ...]

★ "$\operatorname{Hess} \operatorname{Ent} \geq K$" (w.r.t. W_2)

$\Rightarrow W_2(\nu_t, \nu_t') \leq e^{-Kt} W_2(\nu_0, \nu_0')$

★ "$\operatorname{Hess} \operatorname{Ent} \geq K$" $\Leftrightarrow \operatorname{Ric} \geq K$ on a Riem. mfd

[von Renesse & Sturm '05]

Further topics

- Coupling method

  $\Rightarrow$ Heat dist.'s on (backward) Ricci flows

  [Arnaudon, Koulibaly & Thalmaier '09 / K. '12]

  Monotonicity of $\mathcal{L}/\mathcal{L}_0$-transportation cost

  [K. & Philipowski '11 / Amaba & K.]

- Duality

  $\Rightarrow$ W_p-controls for subelliptic diffusions (on Lie gr.)

  [Melcher '08 / Eldredge '10] & [K. '10]

Further topics

- Coupling method

  $\Rightarrow$ Heat dist.'s on (backward) Ricci flows

  [Arnaudon, Koulibaly & Thalmaier '09 / K. '12]

  Monotonicity of $\mathcal{L}/\mathcal{L}_0$-transportation cost

  [K. & Philipowski '11 / Amaba & K.]

- Duality

  $\Rightarrow$ $\boldsymbol{W_p}$-controls for subelliptic diffusions (on Lie gr.)

  [Melcher '08 / Eldredge '10] & [K. '10]

Further topics

- $\partial_t \mu_t = -\nabla \operatorname{Ent}(\mu_t)$ on $(\mathcal{P}(M), W_2)$

  † "$\operatorname{Hess}\operatorname{Ent} \geq K$" as a definition of "$\operatorname{Ric} \geq K$"

  [Sturm '06 / Lott & Villani '09]

  † Unifying "$\operatorname{Ric} \geq K$"

  on Riemannian metric measure sp.'s

  [Ambrosio, Gigli & Savaré '14 / AGS / AGS /

  Ambrosio, Gigli, Mondino & Rajala]

Further topics

- Extenstion to "$\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$" on mm sp.

  [Erbar, K. & Sturm]

† "$\mathrm{Hess\,Ent} - \frac{1}{N}\nabla\,\mathrm{Ent}^{\otimes 2} \geq K$"

† $\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{(\Delta f)^2}{N}$

† $|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt} P_t(|\nabla f|^2) - \frac{1 - \mathrm{e}^{-2Kt}}{NK}(\Delta P_t f)^2$

[Bakry & Ledoux '06]

† $W_2(P_s^* \mu, P_t^* \mu')^2$

$$\leq \mathrm{e}^{(-2Kt)\vee(-2Ks)} W_2(\mu, \mu')^2 + \left(\int_s^t \sqrt{\frac{NK}{\mathrm{e}^{2Kr} - 1}} \mathrm{d}r \right)^2$$

Further topics

- Extenstion to "$\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$" on mm sp. [Erbar, K. & Sturm]

† "$\mathbf{Hess\,Ent} - \frac{\mathbf{1}}{\boldsymbol{N}} \nabla \mathbf{Ent}^{\otimes \mathbf{2}} \geq \boldsymbol{K}$"

† $\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{(\Delta f)^2}{N}$

† $|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt} P_t(|\nabla f|^2) - \frac{1 - \mathrm{e}^{-2Kt}}{NK}(\Delta P_t f)^2$

[Bakry & Ledoux '06]

† $W_2(P_s^* \mu, P_t^* \mu')^2$

$$\leq \mathrm{e}^{(-2Kt) \vee (-2Ks)} W_2(\mu, \mu')^2 + \left(\int_s^t \sqrt{\frac{NK}{\mathrm{e}^{2Kr} - 1}} \mathrm{d}r \right)^2$$

Further topics

- Extenstion to "$\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$" on mm sp.

  [Erbar, K. & Sturm]

  † "$\mathbf{Hess\,Ent} - \frac{1}{N} \nabla \mathbf{Ent}^{\otimes 2} \geq K$"

  † $\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{(\Delta f)^2}{N}$

  † $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) - \frac{1 - e^{-2Kt}}{NK} (\Delta P_t f)^2$

  [Bakry & Ledoux '06]

  † $W_2(P_s^* \mu, P_t^* \mu')^2$

  $$\leq e^{(-2Kt)\vee(-2Ks)} W_2(\mu, \mu')^2 + \left(\int_s^t \sqrt{\frac{NK}{e^{2Kr} - 1}} \mathrm{d}r \right)^2$$

Further topics

- Extenstion to "$\mathrm{Ric} \geq K$ & $\dim \leq N$" on mm sp.

  [Erbar, K. & Sturm]

  † "$\mathrm{Hess\,Ent} - \frac{1}{N}\nabla\,\mathrm{Ent}^{\otimes 2} \geq K$"

  † $\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle \geq K|\nabla f|^2 + \frac{(\Delta f)^2}{N}$

  † $|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt} P_t(|\nabla f|^2) - \frac{1-\mathrm{e}^{-2Kt}}{NK}(\Delta P_t f)^2$

  [Bakry & Ledoux '06]

  † $W_2(P_s^* \mu, P_t^* \mu')^2$

  $$\leq \mathrm{e}^{(-2Kt)\vee(-2Ks)} W_2(\mu,\mu')^2 + \left(\int_s^t \sqrt{\frac{NK}{\mathrm{e}^{2Kr}-1}}\,\mathrm{d}r\right)^2$$

Further topics

- Extenstion to "$\mathbf{Ric} \geq K$ & $\mathbf{dim} \leq N$" on mm sp.

  [Erbar, K. & Sturm]

  † "$\mathbf{Hess\,Ent} - \frac{1}{N}\nabla\,\mathbf{Ent}^{\otimes 2} \geq K$"

  † $\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle \geq K|\nabla f|^2 + \frac{(\Delta f)^2}{N}$

  † $|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt} P_t(|\nabla f|^2) - \frac{1 - \mathrm{e}^{-2Kt}}{NK}(\Delta P_t f)^2$

  [Bakry & Ledoux '06]

  † $W_2(P_s^* \mu, P_t^* \mu')^2$

  $$\leq \mathrm{e}^{(-2Kt)\vee(-2Ks)} W_2(\mu, \mu')^2 + \left(\int_s^t \sqrt{\frac{NK}{\mathrm{e}^{2Kr} - 1}} \mathrm{d}r \right)^2$$

1. **Introduction & Overview**

2. **Duality in space-time estimates**

3. **W_p-space-time control on Riem. mfds**

1. Introduction & Overview

2. Duality in space-time estimates

3. W_p-space-time control on Riem. mfds

Framework

(M, d): Polish met. sp., d: geodesic metric

$P_t(x, \cdot) \in \mathcal{P}(M)$: semigroup of Markov kernels

<u>Ass.</u> $^\forall f \in C_b^{\mathsf{Lip}}(M)$, $^\forall x \in M$, $^\forall t > 0$

$$^\exists \mathcal{L} P_t f(x) = \lim_{s \to 0} \frac{P_{t+s} f(x) - P_t f(x)}{s}$$

Framework

(M, d): Polish met. sp., d: geodesic metric

$P_t(x, \cdot) \in \mathcal{P}(M)$: semigroup of Markov kernels

Ass. $\forall f \in C_b^{\mathsf{Lip}}(M)$, $\forall x \in M$, $\forall t > 0$

$$\exists \mathcal{L} P_t f(x) = \lim_{s \to 0} \frac{P_{t+s} f(x) - P_t f(x)}{s}$$

Hopf-Lax semigr.

$$Q_r f(x) := \inf_{y \in M} \left[f(y) + r \left(\frac{d(x, y)}{r} \right)^p \right]$$

$$(p \in (1, \infty))$$

Extended space-time duality

$a_t, b_t > 0$ conti., $p \geq \beta \geq 1$,

$J(du) = \dfrac{du}{b_u}$: loc. finite meas. on $[0, \infty)$

Theorem 1

The following are equivalent:

(i) $W_p(P_s^* \mu, P_t^* \mu')^\beta \leq \left(\fint_{[s,t]} a_u J(du) \right)^\beta W_p(\mu, \mu')^\beta + J([s,t])^\beta$

(ii) $|\nabla P_t f|^{\beta_*} \leq a_t^{\beta_*} \left(P_t(|\nabla f|^{p_*})^{\beta_*/p_*} - b_t^{\beta_*} |\mathcal{L} P_t f|^{\beta_*} \right)$
for $f = Q_\delta \tilde{f}$ $\quad (\delta > 0$ & $\tilde{f} \in C_b^{\mathrm{Lip}}(M))$

Extended space-time duality

$a_t, b_t > 0$ conti., $p \geq \beta \geq 1$,

$J(du) = \dfrac{du}{b_u}$: loc. finite meas. on $[0, \infty)$

Theorem 1

The following are equivalent:

(i) $W_p(P_s^* \mu, P_t^* \mu')^\beta \leq \left(\fint_{[s,t]} a_u J(du) \right)^\beta W_p(\mu, \mu')^\beta + J([s,t])^\beta$

(ii) $|\nabla P_t f|^{\beta_*} \leq a_t^{\beta_*} \left(P_t(|\nabla f|^{p_*})^{\beta_*/p_*} - b_t^{\beta_*} |\mathcal{L} P_t f|^{\beta_*} \right)$

for $f = Q_\delta \tilde{f} \quad (\delta > 0 \;\&\; \tilde{f} \in C_b^{\mathsf{Lip}}(M))$

Rough idea of the proof

▷ **(i)** $\Rightarrow$ **(ii)**: Differentiation in space-time

$$P_s f(y) - P_t f(x) = \int_{M \times M} (f(z) - f(w)) \pi(dzdw)$$
$$\approx \int_{M \times M} |\nabla f|(w) d(z, w) \pi(dzdw)$$

for $^{\forall}\pi$: coupling of $P_s^* \delta_y$ and $P_t^* \delta_x$

Rough idea of the proof

▷ **(i) ⇒ (ii)**: Differentiation in space-time

$$P_s f(y) - P_t f(x) = \int_{M \times M} (f(z) - f(w)) \pi(dz dw)$$
$$\approx \int_{M \times M} \underline{|\nabla f|(w) d(z,w)} \pi(dz dw)$$

for ${}^\forall \pi$: coupling of $P_s^* \delta_y$ and $P_t^* \delta_x$

(LHS) $\to \underline{|\nabla P_t f|}(x) + \lambda \underline{\mathcal{L} P_t f}(x)$

$(y \to x$ & $s \to t$ with $\lambda(t-s) = d(x,y)$ $({}^\forall \lambda \in \mathbb{R}))$

Rough idea of the proof

▷ **(i)** $\Rightarrow$ **(ii)**: Differentiation in space-time

$$P_s f(y) - P_t f(x) = \int_{M \times M} (f(z) - f(w)) \pi(dzdw)$$
$$\approx \int_{M \times M} |\nabla f|(w) d(z,w) \pi(dzdw)$$

for $^\forall \pi$: coupling of $P_s^* \delta_y$ and $P_t^* \delta_x$

(LHS) $\to |\nabla P_t f|(x) + \lambda \mathcal{L} P_t f(x)$

($y \to x$ & $s \to t$ with $\lambda(t-s) = d(x,y)$ ($^\forall \lambda \in \mathbb{R}$))

Key Fact: $|\nabla Q_r f|$: u.s.c.

Rough idea of the proof

▷ **(ii) ⇒ (i)**: Integration in space-time

$(\mu_r)_{r\in[0,1]}$: W_p-geod. from μ to μ'

& consider W_p-speed of $P^*_{\xi(r)}\mu_r$ (ξ: parametrization)

$$\frac{W_p(P^*_{\xi(r)}\mu_s, P^*_{\xi(r+s)}\mu_{r+s})^p}{ps^p} \approx ???$$

Rough idea of the proof

▷ **(ii) ⇒ (i)**: Integration in space-time

$(\mu_r)_{r\in[0,1]}$: W_p-geod. from μ to μ'

& consider W_p-speed of $P^*_{\xi(r)}\mu_r$ (ξ: parametrization)

via the Kantorovich duality:

$$\frac{W_p(\mu,\mu')^p}{ps^{p-1}} = \sup_{\varphi\in C_b^{\mathrm{Lip}}(M)} \left[\int_M Q_s\varphi \, \mathbf{d}\mu' - \int_M \varphi \, \mathbf{d}\mu \right]$$

Rough idea of the proof

▷ **(ii) ⇒ (i)**: Integration in space-time

$(\mu_r)_{r\in[0,1]}$: W_p-geod. from μ to μ'

& consider W_p-speed of $P^*_{\xi(r)}\mu_r$ (ξ: parametrization)

via the Kantorovich duality:

$$\frac{W_p(P^*_{\xi(r)}\mu_s, P^*_{\xi(r+s)}\mu_{r+s})^p}{ps^p} \approx \partial_s \int P_{\xi(r+s)} Q_s \varphi \, d\mu_{r+s}$$

Rough idea of the proof

▷ **(ii) ⇒ (i)**: Integration in space-time

$(\mu_r)_{r\in[0,1]}$: W_p-geod. from μ to μ'
& consider W_p-speed of $P^*_{\xi(r)}\mu_r$ (ξ: parametrization)
via the Kantorovich duality:

$$\frac{W_p(P^*_{\xi(r)}\mu_s, P^*_{\xi(r+s)}\mu_{r+s})^p}{ps^p} \approx \partial_s \int P_{\xi(r+s)} Q_s\varphi \, d\mu_{r+s}$$

Key fact: $\partial_r^+ Q_r\varphi + \dfrac{|\nabla Q_r\varphi|^{p_*}}{p_*} = 0$

Remarks

- **(i)** $\Leftrightarrow$ **(ii)** $\Leftrightarrow$ **(i)'**:

(i)' $W_p(P_s^*\mu, P_t^*\mu')^\beta \le A_{s,t}^\beta W_p(\mu,\mu')^\beta + B_{s,t}^\beta$

with $A_{t,t} = a_t$ & $\dfrac{B_{s,t}}{|t-s|} \to \dfrac{1}{b_t}$ as $s \to t$

($\Rightarrow$ **(i)'** enjoys self-improvement)

- If P_t is strong Feller,

  (ii) with $f \in C_b^{\mathrm{Lip}}(M)$ $\Leftrightarrow$ **(i)**

- The same holds for P_t defined on L^p-spaces (with a.e. umbiguity) under suitable assumptions

Remarks

- **(i)** $\Leftrightarrow$ **(ii)** $\Leftrightarrow$ **(i)'**:

(i)' $W_p(P_s^*\mu, P_t^*\mu')^\beta \le A_{s,t}^\beta W_p(\mu,\mu')^\beta + B_{s,t}^\beta$

with $A_{t,t} = a_t$ & $\dfrac{B_{s,t}}{|t-s|} \to \dfrac{1}{b_t}$ as $s \to t$

($\Rightarrow$ **(i)'** enjoys self-improvement)

- If P_t is strong Feller,

  (ii) with $f \in C_b^{\mathrm{Lip}}(M)$ $\Leftrightarrow$ **(i)**

- The same holds for P_t defined on L^p-spaces (with a.e. umbiguity) under suitable assumptions

Remarks

- **(i)** $\Leftrightarrow$ **(ii)** $\Leftrightarrow$ **(i)'**:

(i)' $W_p(P_s^*\mu, P_t^*\mu')^\beta \le A_{s,t}^\beta W_p(\mu,\mu')^\beta + B_{s,t}^\beta$

with $A_{t,t} = a_t$ & $\dfrac{B_{s,t}}{|t-s|} \to \dfrac{1}{b_t}$ as $s \to t$

($\Rightarrow$ **(i)'** enjoys self-improvement)

- If P_t is strong Feller,

  (ii) with $f \in C_b^{\mathsf{Lip}}(M)$ $\Leftrightarrow$ **(i)**

- The same holds for P_t defined on L^p-spaces (with a.e. umbiguity) under suitable assumptions

1. Introduction & Overview

2. Duality in space-time estimates

3. W_p-space-time control on Riem. mfds

Example of W_p-space-time control

M: cpl. Riem. mfd, $\dim M \le N$, $\mathrm{Ric} \le K$

Example of W_p-space-time control

M: cpl. Riem. mfd, $\dim M \le N$, $\mathrm{Ric} \le K$

$$p \ge 2,\ b_u = \sqrt{\frac{e^{2Ku} - 1}{(N + p - 2)K}},\ J(du) = \frac{du}{b_u}$$

Theorem 2

The following holds:

(a) $$W_p(P_s^* \mu, P_t^* \mu')^2 \le \left(\fint_{[s,t]} e^{-Kt} J(du) \right)^2 W_p(\mu, \mu')^2 + J([s,t])^2$$

(b) $$|\nabla P_t f|^2 \le e^{-Kt} \left(P_t(|\nabla f|^{p_*})^{2/p_*} - b_t^2 |\mathcal{L} P_t f|^2 \right)$$

for $f \in C_b^{\mathrm{Lip}}(M)$

Example of W_p-space-time control

M: cpl. Riem. mfd, $\dim M \le N$, $\mathrm{Ric} \le K$

$$p \ge 2,\ b_u = \sqrt{\frac{\mathrm{e}^{2Ku} - 1}{(N + p - 2)K}},\ J(du) = \frac{du}{b_u}$$

Theorem 2

The following holds:

(a) $$W_p(P_s^* \mu, P_t^* \mu')^2 \le \left(\fint_{[s,t]} \mathrm{e}^{-Kt} J(du) \right)^2 W_p(\mu, \mu')^2 + J([s,t])^2$$

(b) $$|\nabla P_t f|^2 \le \mathrm{e}^{-Kt} \left(P_t(|\nabla f|^{p_*})^{2/p_*} - b_t^2 |\mathcal{L} P_t f|^2 \right)$$

for $f \in C_b^{\mathsf{Lip}}(M)$

Method of the proof

- Coupling by parallel transport of BMs
  with different speed
- **(i)'** $\Rightarrow$ **(ii)**: still rough
- **(ii)** $\Rightarrow$ **(c)** below $\Rightarrow$ **(b)** $\Rightarrow$ **(a)**

(c) $$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{(\mathcal{L} f)^2}{N + p - 2} + \frac{p-2}{4(p-1)} \frac{|\nabla|\nabla f|^2|^2}{|\nabla f|^2}$$

(Bakry-Émery theory)

Method of the proof

- Coupling by parallel transport of BMs
  with different speed ($\Rightarrow$ (Rough) est. of $\boldsymbol{W_p}$-speed)
- **(i)'** $\Rightarrow$ **(ii)**: still rough
- **(ii)** $\Rightarrow$ **(c)** below $\Rightarrow$ **(b)** $\Rightarrow$ **(a)**

(c) $$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{(\mathcal{L}f)^2}{N+p-2} + \frac{p-2}{4(p-1)} \frac{|\nabla|\nabla f|^2|^2}{|\nabla f|^2}$$

(Bakry-Émery theory)

Method of the proof

- Coupling by parallel transport of BMs with different speed ($\Rightarrow$ (Rough) est. of $\boldsymbol{W_p}$-speed)
- **(i)'** $\Rightarrow$ **(ii)**: still rough
- **(ii)** $\Rightarrow$ **(c)** below $\Rightarrow$ **(b)** $\Rightarrow$ **(a)**

(c) $$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{(\mathcal{L} f)^2}{N+p-2} + \frac{p-2}{4(p-1)} \frac{|\nabla |\nabla f|^2|^2}{|\nabla f|^2}$$

(Bakry-Émery theory)

Method of the proof

- Coupling by parallel transport of BMs
  with different speed ($\Rightarrow$ (Rough) est. of $\boldsymbol{W_p}$-speed)
- **(i)'** $\Rightarrow$ **(ii)**: still rough
- **(ii)** $\Rightarrow$ **(c)** below $\Rightarrow$ **(b)** $\Rightarrow$ **(a)**

(c) $$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{(\mathcal{L}f)^2}{N+p-2} + \frac{p-2}{4(p-1)} \frac{|\nabla|\nabla f|^2|^2}{|\nabla f|^2}$$

(Bakry-Émery theory)

Method of the proof

- Coupling by parallel transport of BMs
  with different speed ($\Rightarrow$ (Rough) est. of W_p-speed)
- **(i)'** $\Rightarrow$ **(ii)**: still rough
- **(ii)** $\Rightarrow$ **(c)** below $\Rightarrow$ **(b)** $\Rightarrow$ **(a)**

(c) $$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{(\mathcal{L}f)^2}{N + p - 2} + \frac{p-2}{4(p-1)} \frac{|\nabla|\nabla f|^2|^2}{|\nabla f|^2}$$

(Bakry-Émery theory)

Some remarks

- $\Delta \rightsquigarrow \mathcal{L} = \Delta + Z$ (nonsymm.) is possible
$$\left(\mathrm{Ric} \rightsquigarrow \mathrm{Ric} - \nabla Z - \frac{1}{N-m} Z \otimes Z\right)$$
- For $\mathrm{Ric} \geq K$ & $\dim \leq N \Rightarrow$ **(c)**,
no straightforward proof is known

Some remarks

- $\boldsymbol{\Delta} \rightsquigarrow \boldsymbol{\mathcal{L}} = \boldsymbol{\Delta} + \boldsymbol{Z}$ (nonsymm.) is possible
  $$\left(\mathbf{Ric} \rightsquigarrow \mathbf{Ric} - \boldsymbol{\nabla} \boldsymbol{Z} - \frac{\mathbf{1}}{\boldsymbol{N} - \boldsymbol{m}} \boldsymbol{Z} \otimes \boldsymbol{Z}\right)$$
- For $\boxed{\mathbf{Ric} \geq \boldsymbol{K} \ \& \ \mathbf{dim} \leq \boldsymbol{N} \Rightarrow \textbf{(c)}}$,
  no straightforward proof is known