On the speed in Wasserstein distances of heat distributions

Kazumasa Kuwada

(Tokyo Institute of Technology)

7th International Conference on Stochastic Analysis and its Applications

(Seoul National University) Aug. 6–11, 2014

1. Introduction

Speed in transportation cost

$(X_t)_{t\geq 0}$: stochastic process on M

$\Downarrow \Uparrow$

$(\mu_t)_{t\geq 0}$: curve in $\mathcal{P}(M)$, $\mu_t := \mathbb{P} \circ X_t^{-1}$

Speed in transportation cost

$(X_t)_{t \geq 0}$: stochastic process on M

$\Downarrow \Uparrow$

$(\mu_t)_{t \geq 0}$: curve in $\mathcal{P}(M)$, $\mu_t := \mathbb{P} \circ X_t^{-1}$

Q. $\mathcal{T}_c(\mu_t, \mu_s) \approx ? \quad (s \to t)$

$$\mathcal{T}_c(\mu, \nu) := \inf \left\{ \int_{M \times M} c \, d\pi \;\middle|\; \begin{array}{c} \pi\text{: coupling of} \\ \mu \text{ and } \nu \end{array} \right\}$$

(Optimal transportation cost for a cost function c)

Background

$\mu_t = \mathrm{e}^{t\Delta}\mu_0$: heat dist. on a met. meas. sp. (M, d, v)

$\Rightarrow$ "$\partial_t \mu_t = -\nabla \operatorname{Ent}_v(\mu_t)$" w.r.t. $W_2 = (\mathcal{T}_{d^2})^{1/2}$

[Jordan, Kinderlehrer & Otto '98]

[Ambrosio, Gigli & Savaré '05, ...]

Background

$\mu_t = e^{t\Delta}\mu_0$: heat dist. on a met. meas. sp. (M, d, v)

$\Rightarrow$ "$\partial_t \mu_t = -\nabla \operatorname{Ent}_v(\mu_t)$" w.r.t. $W_2 = (\mathcal{T}_{d^2})^{1/2}$

[Jordan, Kinderlehrer & Otto '98]

[Ambrosio, Gigli & Savaré '05, ...]

"$\partial_t \operatorname{Ent}_v(\mu_t) = -\langle \nabla \operatorname{Ent}_v, \partial_t \mu_t \rangle = -|\partial_t \mu_t|^2$"

Background

$\mu_t = e^{t\Delta}\mu_0$: heat dist. on a met. meas. sp. (M, d, v)

$\Rightarrow$ "$\partial_t \mu_t = -\nabla \operatorname{Ent}_v(\mu_t)$" w.r.t. $W_2 = (\mathcal{T}_{d^2})^{1/2}$

[Jordan, Kinderlehrer & Otto '98]

[Ambrosio, Gigli & Savaré '05, ...]

"$\partial_t \operatorname{Ent}_v(\mu_t) = -\langle \nabla \operatorname{Ent}_v, \partial_t \mu_t \rangle = -|\partial_t \mu_t|^2$"

$$\Downarrow$$

$$\overline{\lim_{s \downarrow t}} \left(\frac{W_2(\mu_s, \mu_t)}{s - t} \right)^2 = \int_M \frac{|\nabla \rho_t|^2}{\rho_t} dv =: I(\mu_t)$$

(Fisher information)

Background

"$\mathrm{Hess}\,\mathrm{Ent}_v \geq K$" ($\Leftrightarrow$ "$\mathrm{Ric} \geq K$") for $K \in \mathbb{R}$

$$\Downarrow$$

$$W_2(\mu_t^{(0)}, \mu_t^{(1)}) \leq \mathrm{e}^{-Kt} W_2(\mu_0^{(0)}, \mu_0^{(1)})$$

$$\Downarrow$$

$$W_2(\mu_t, \mu_{t+s}) \leq \mathrm{e}^{-Kt} W_2(\mu_0, \mu_s)$$

Background

"$\mathrm{Hess\,Ent}_v \geq K$" ($\Leftrightarrow$ "$\mathrm{Ric} \geq K$") for $K \in \mathbb{R}$

$$\Downarrow$$

$$W_2(\mu_t^{(0)}, \mu_t^{(1)}) \leq \mathrm{e}^{-Kt} W_2(\mu_0^{(0)}, \mu_0^{(1)})$$

$$\Downarrow$$

$$W_2(\mu_t, \mu_{t+s}) \leq \mathrm{e}^{-Kt} W_2(\mu_0, \mu_s)$$

$$\Downarrow$$

$$\boxed{I(\mu_t) \leq \mathrm{e}^{-2Kt} I(\mu_0)}$$

($\Rightarrow$ log Sobolev ineq. (when $K > 0$))

Questions

- What happens for other transportation costs?
- What happens when there is no gradient flow structure?

Outline of the talk

1. **Introduction**

2. **Heat distributions on backward Ricci flow**

3. **Idea of the proof**

4. **Further problems**

1. Introduction

2. Heat distributions on backward Ricci flow

3. Idea of the proof

4. Further problems

Framework

- $(M, g(t))$: cpl. Riem. mfds., $t \in [0, T]$

  $$\partial_t g(t) = 2\,\mathrm{Ric}_t \text{ (backward Ricci flow)}$$

- $((X(t))_{t \geq 0}, (\mathbb{P}_x)_{x \in M})$: $g(t)$-Brownian motion

  $\leftrightsquigarrow \Delta_{g(t)}$: generator

  $\mu_t = \mathbb{P}_{\mu_0} \circ X(t)^{-1}$: heat dist.

- v_t: $g(t)$-volume meas., $\mu_t = \rho_t v_t$

  $\bigstar\ \partial_t v_t = R_t v_t \quad$ (R_t: $g(t)$-scalar curv.)

<u>Ass.</u> $\sup_t |\mathrm{Rm}_t|_{g(t)} < \infty$ (Rm_t: $g(t)$-curv. tensor)

$\partial_t \mu_t \neq -\nabla \, \mathrm{Ent}_{v_t}(\mu_t)$

$$\mathrm{Ent}_{v_t}(\mu_t) := \int_M \rho_t \log \rho_t \, dv_t = \int_M \log \rho_t \, d\mu_t$$

★ $\partial_t \mu_t = \Delta_t \mu_t$ (weakly)

$$\Rightarrow \partial_t \, \mathrm{Ent}_{v_t}(\mu_t) = -\int_M \left(\frac{|\nabla \rho_t|^2}{\rho_t^2} + R_t \right) d\mu_t$$

$$=: -\mathcal{F}(\mu_t) \qquad (\mathcal{F}\text{-functional})$$

$\Rightarrow$ No monotonicity of $\mathrm{Ent}_{v_t}(\mu_t)$!

W_2-contraction

Observation

When $g(t) \equiv g_0$, $\partial_t g(t) = 2\,\mathrm{Ric}_t \Rightarrow \mathrm{Ric} \equiv 0$

W_2-contraction

Observation

When $g(t) \equiv g_0$, $\partial_t g(t) = 2\,\mathrm{Ric}_t \Rightarrow \mathrm{Ric} \equiv 0$

★ $\mathcal{T}_{d_t^2}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$ [McCann & Topping '10], [Arnaudon, Coulibaly & Thalmaier '09], [K. '12]

W_2-contraction

Observation

When $g(t) \equiv g_0$, $\partial_t g(t) = 2\,\mathrm{Ric}_t \Rightarrow \mathrm{Ric} \equiv 0$

★ $\mathcal{T}_{d_t^2}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$ [McCann & Topping '10], [Arnaudon, Coulibaly & Thalmaier '09], [K. '12]

$\not\Rightarrow$ $\mathcal{T}_{d_t^2}(\mu_t, \mu_{t+s}) \searrow$ (time-inhomogeneity)

W_2-contraction

Observation

When $g(t) \equiv g_0$, $\partial_t g(t) = 2\,\mathrm{Ric}_t \Rightarrow \mathrm{Ric} \equiv 0$

★ $\mathcal{T}_{d_t^2}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$ [McCann & Topping '10], [Arnaudon, Coulibaly & Thalmaier '09], [K. '12]

★ $\mathcal{T}_{L_0^{t,t+s}}(\mu_t, \mu_{t+s}) \searrow$ in t [Lott '09], [Amaba & K.]

$$\left(L_0^{t,t'}(x,y) := \inf_{\substack{\gamma(t)=x,\\ \gamma(t')=y}} \left[\int_t^{t'} \left(|\dot\gamma(r)|_r^2 + R_r \right) dr \right] \right)$$

(L_0-distance; introduced by Lott)

Monotonicity of $\mathcal{F}$

Theorem 1

Suppose $\mathrm{Ent}_{v_0}(\mu_0) < \infty$ *and* $\mathcal{F}(\mu_0) < \infty$

$$\Rightarrow \lim_{s \downarrow 0} \frac{\mathcal{T}_{L_0^{t,t+s}}(\mu_t, \mu_{t+s})}{s} = \mathcal{F}(\mu_t) \text{ a.e. } t \in [0, T]$$

Corollary 2

$\mathcal{F}(\mu_t) \searrow$

- Rem: $g(t) \equiv g$, $\mathrm{Ric} \geq 0 \Rightarrow I(\mu_t) \searrow$
- [Lott '09] when M: cpt.

  by Eulerian calculus (requires smoothness)

Kantorovich duality

Observation: W_2 in the case $g(t) \equiv g$

$$\frac{W_2(\mu_t, \mu_{t+s})^2}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s\varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$Q_s\varphi(x) := \inf_{y \in M} \left(\varphi(y) + \frac{d(y,x)^2}{2s} \right)$$

Kantorovich duality

Observation: W_2 in the case $g(t) \equiv g$

$$\frac{W_2(\mu_t, \mu_{t+s})^2}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s\varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$Q_s\varphi(x) := \inf_{y \in M} \left(\varphi(y) + \frac{d(y,x)^2}{2s} \right)$$

$$= \inf_{\substack{\gamma(t)=y \\ \gamma(t+s)=x}} \left(\varphi(\gamma(t)) + \frac{1}{2} \int_t^{t+s} |\dot\gamma(r)|^2 \, dr \right)$$

(Hopf-Lax semigroup)

Kantorovich duality

Observation: W_2 in the case $g(t) \equiv g$

$$\frac{W_2(\mu_t, \mu_{t+s})^2}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s\varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$Q_s\varphi(x) := \inf_{y \in M} \left(\varphi(y) + \frac{d(y,x)^2}{2s} \right)$$

$$= \inf_{\substack{\gamma(t)=y \\ \gamma(t+s)=x}} \left(\varphi(\gamma(t)) + \frac{1}{2} \int_t^{t+s} |\dot{\gamma}(r)|^2 \, dr \right)$$

(Hopf-Lax semigroup)

$\bigstar$ $\partial_s Q_s\varphi + \frac{1}{2}|\nabla Q_s\varphi|^2 = 0$ (Hamilton-Jacobi eq.)

Upper bound

$$\frac{W_2(\mu_t, \mu_{t+s})^2}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s\varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$[\cdots] = \int_0^s \left(\partial_r \int_M Q_r\varphi \, d\mu_{t+r} \right) \, dr$$

$$\partial_r(\cdots) = \int \left(-\langle \nabla Q_r\varphi, \frac{\nabla \rho_{t+r}}{\rho_{t+r}} \rangle - \frac{1}{2} |\nabla Q_r\varphi|^2 \right) d\mu_{t+r}$$

$$\leq \frac{1}{2} I(\mu_{t+r})$$

Upper bound

$$\frac{W_2(\mu_t, \mu_{t+s})^2}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s \varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$[\cdots] = \int_0^s \left(\partial_r \int_M Q_r \varphi \, d\mu_{t+r} \right) dr$$

$$\partial_r(\cdots) = \int \left(-\langle \nabla Q_r \varphi, \frac{\nabla \rho_{t+r}}{\rho_{t+r}} \rangle - \frac{1}{2} |\nabla Q_r \varphi|^2 \right) d\mu_{t+r}$$

$$\leq \frac{1}{2} I(\mu_{t+r})$$

$$\overline{\lim_{s \downarrow t}} \, \frac{W_2(\mu_t, \mu_{t+s})^2}{s^2} \leq \overline{\lim_{s \downarrow t}} \, \frac{1}{s} \int_t^{t+s} I(\mu_r) dr = I(\mu_t)$$

Lower bound

$$\frac{W_2(\mu_t, \mu_{t+s})^2}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s \varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$\varliminf_{s \downarrow t} \frac{1}{s} \left[\int_M Q_s \varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$\text{“=”} \int \left(-\langle \nabla \varphi, \frac{\nabla \rho_t}{\rho_t} \rangle - \frac{1}{2} |\nabla \varphi|^2 \right) d\mu_t$$

Lower bound

$$\frac{W_2(\mu_t, \mu_{t+s})^2}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s \varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$\varlimsup_{s \downarrow t} \frac{1}{s} \left[\int_M Q_s \varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$\text{``=''} \int \left(-\langle \nabla \varphi, \frac{\nabla \rho_t}{\rho_t} \rangle - \frac{1}{2} |\nabla \varphi|^2 \right) d\mu_t$$

$$\Downarrow \boxed{\varphi = -\log \rho_t}$$

Lower bound

$$\frac{W_2(\mu_t, \mu_{t+s})^2}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s\varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$\lim_{s \downarrow t} \frac{1}{s} \left[\int_M Q_s\varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$\text{“}{=}\text{”} \int \left(-\langle \nabla\varphi, \frac{\nabla\rho_t}{\rho_t} \rangle - \frac{1}{2} |\nabla\varphi|^2 \right) d\mu_t$$

$$\Downarrow \quad \boxed{\varphi = -\log \rho_t}$$

$$\lim_{s \downarrow t} \frac{W_2(\mu_t, \mu_{t+s})^2}{s^2} \geq I(\mu_t)$$

Outline of the proof of Thm1

- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry

  No a priori integrability of $\nabla \rho_t$ & $\Delta_t \rho_t$

Outline of the proof of Thm1

- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry

  No a priori integrability of $\nabla \rho_t$ & $\Delta_t \rho_t$

Outline of the proof of Thm1

- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry

  No a priori integrability of $\nabla\rho_t$ & $\Delta_t\rho_t$

Outline of the proof of Thm1

- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry

  No a priori integrability of $\nabla \rho_t$ & $\Delta_t \rho_t$

  $\Rightarrow$ Approximation

Outline of the proof of Thm1

- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry

  No a priori integrability of $\nabla \rho_t$ & $\Delta_t \rho_t$

  $\Rightarrow$ Approximation

† Use $^\exists$ of heat kernel & Gaussian (upper) bound

Outline of the proof of Thm1

- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry

  No a priori integrability of $\nabla \rho_t$ & $\Delta_t \rho_t$

  $\Rightarrow$ Approximation

† Use $\exists$ of heat kernel & Gaussian (upper) bound

† Show $$\mathrm{Ent}_{v_s}(\mu_s) - \mathrm{Ent}_{v_t}(\mu_t) \geq \int_s^t \mathcal{F}(\mu_r) dr$$

Outline of the proof of Thm1

- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry

  No a priori integrability of $\nabla \rho_t$ & $\Delta_t \rho_t$

  $\Rightarrow$ Approximation

† Use ∃ of heat kernel & Gaussian (upper) bound

† Show $\boxed{\mathrm{Ent}_{v_s}(\mu_s) - \mathrm{Ent}_{v_t}(\mu_t) \geq \int_s^t \mathcal{F}(\mu_r) dr}$

($\Rightarrow \mathcal{F}(\mu_r) < \infty$ a.e. r)

Outline of the proof of Thm1

- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry

  No a priori integrability of $\nabla \rho_t$ & $\Delta_t \rho_t$

  $\Rightarrow$ Approximation

† Use $^\exists$ of heat kernel & Gaussian (upper) bound

† Show $$\mathrm{Ent}_{v_s}(\mu_s) - \mathrm{Ent}_{v_t}(\mu_t) \geq \int_s^t \mathcal{F}(\mu_r)\,dr$$

($\Rightarrow \mathcal{F}(\mu_r) < \infty$ a.e. r)

† Show $$\overline{\lim_{s \downarrow t}} \frac{\mathrm{Ent}_{v_s}(\mu_s) - \mathrm{Ent}_{v_t}(\mu_t)}{s - t} \leq \mathcal{F}(\mu_t)$$

1. Introduction

2. Heat distributions on backward Ricci flow

3. Idea of the proof

4. Further problems

$\mathcal{T}_L/W_p$

- For Perelman's L-distance? (backward Ricci flow)

$\mathcal{T}_L/W_p$

- For Perelman's L-distance? (backward Ricci flow)

  † A monotonicity of transportation cost is known

  [Topping '09], [K. & Philipowski '11]

$\mathcal{T}_L / W_p$

- For Perelman's L-distance? (backward Ricci flow)

  † A monotonicity of transportation cost is known

  [Topping '09], [K. & Philipowski '11]

  $\Rightarrow$ Monotonicity of $\mathcal{W}$-functional

  ([Topping '09]; M: cpt.)

$\mathcal{T}_L/W_p$

- For W_p ($p \in [1, \infty)$) in time-homogeneous case?

† $\mathrm{Ric} \geq K \Rightarrow \mathrm{e}^{Kt} W_p(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$

$\mathcal{T}_L/W_p$

- For W_p ($p \in [1, \infty)$) in time-homogeneous case?

† $\mathrm{Ric} \geq K \Rightarrow \mathrm{e}^{Kt} W_p(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$

$$\bigstar \ \overline{\lim_{s \downarrow 0}} \left(\frac{W_p(\mu_t, \mu_{t+s})}{s} \right)^p \leq \int_M \frac{|\nabla \rho_t|^p}{\rho_t^{p-1}} dv$$

$\mathcal{T}_L/W_p$

- For W_p ($p \in [1,\infty)$) in time-homogeneous case?

  † $\mathrm{Ric} \ge K \Rightarrow \mathrm{e}^{Kt} W_p(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$

  $$\bigstar \ \varlimsup_{s\downarrow 0} \left(\frac{W_p(\mu_t, \mu_{t+s})}{s} \right)^p \le \int_M \frac{|\nabla \rho_t|^p}{\rho_t^{p-1}} dv$$

  $\Downarrow$

  $\mathrm{Ric} \ge K > 0$, $v \in \mathcal{P}(M)$

  $$\Rightarrow \boxed{W_p(\mu, v)^p \le \frac{1}{K^p} \int_M \frac{|\nabla \rho|^p}{\rho^{p-1}} dv}$$