Notations and statements of the talk

Tokyo Probability Seminar (Apr. 14, 2014)

- Local Lipschitz constant: For $f : X \to \mathbb{R}$, $|\nabla f|(x) := \limsup_{y \to x} \dfrac{|f(y) - f(x)|}{d(y,x)}$.
- Cheeger's energy functional:
$$\begin{aligned}\mathsf{Ch}(f) :&= \inf \left\{ \liminf_{n\to\infty} \int_X |\nabla f_n|^2 \, dm \;\middle|\; \begin{array}{l} f_n : X \to \mathbb{R}\text{: Lipschitz} \\ f_n \to f \text{ in } L^2(m) \end{array} \right\} \\ &= \int_X |\nabla f|_w^2 \, dm.\end{aligned}$$
Note that the existence of such a function $|\nabla f|_w$ is non-trivial.
- L^2-Wasserstein distance: $\mathscr{P}_2(X)$: L^2-Wasserstein space, consisting of probability measures on X with finite second moments:
$$\mathscr{P}_2(X) := \left\{ \mu \in \mathscr{P}(X) \;\middle|\; \int_X d(x_0,x)^2 \mu(dx) < \infty \text{ for some } x_0 \in X \right\}.$$
W_2: L^2-Wasserstein distance:
$$W_2(\mu,\nu) := \inf \left\{ \|d\|_{L^2(\pi)} \;\middle|\; \begin{array}{l} \sigma \in \mathscr{P}(X \times X),\ {}^\forall A \in \mathcal{B}(X), \\ \sigma(A \times X) = \mu(A),\ \sigma(X \times A) = \nu(A) \end{array} \right\}.$$
Note that $(\mathscr{P}_2(X), W_2)$ is a Polish geodesic metric space if so is (X,d).
- Comparison functions: For $\kappa \in \mathbb{R}$ and $\kappa\theta^2 \le \pi^2$,
$$\mathfrak{s}_\kappa(\theta) := \frac{\sin(\sqrt{\kappa}\theta)}{\sqrt{\kappa}}, \quad \sigma_\kappa^{(t)}(\theta) := \frac{\mathfrak{s}_\kappa(t\theta)}{\mathfrak{s}_\kappa(\theta)}.$$
- (K,N)-convexity: (Y, d_Y): a metric space. $V : Y \to \mathbb{R}$ (K,N)-convex if for ${}^\forall x, y \in Y$ ${}^\exists\gamma : [0,1] \to Y$: constant speed geodesic from x to y s.t.
$$U_N(\gamma_t) \ge \sigma_{K/N}^{(1-t)}(d_Y(x,y))U_N(\gamma_0) + \sigma_{K/N}^{(t)}(d_Y(x,y))U_N(\gamma_1), \quad \text{where } U_N := \exp\left(-\frac{1}{N}V\right).$$
This is an integral formulation of the following inequality in the distributional sense:
$$\partial_t^2 V_N(\gamma_t) \le -\frac{K}{N} d(x,y)^2 V_N(\gamma_t).$$
If V is C^2-function on a Riemannian manifold, then V is (K,N)-convex if and only if
$$\mathrm{Hess}\, V - \frac{1}{N} \nabla V \otimes \nabla V \ge K.$$
- Relative entropy: The (imprecise) definition of $\mathrm{Ent} : \mathscr{P}(X) \to \bar{\mathbb{R}}$ is as follows:
$$\mathrm{Ent}(\mu) := \begin{cases} \displaystyle\int_X \rho \log \rho \, dm & (\mu \ll m,\ \mu = \rho m), \\ \infty & (\text{otherwise}). \end{cases}$$
Note that Ent is well-defined for ${}^\forall \mu \in \mathscr{P}_2(X)$ and $\mathrm{Ent}(\mu) > -\infty$ under Assumption 1.

(i)' $\underline{\mathsf{CD}^*(K,N)\text{ (reduced curvature-dimension condition)}}$: For $\mu_0 = \rho_0 m, \mu_1 = \rho_1 m \in \mathscr{P}(X)$ with bounded supports, there exists an optimal coupling q of them (i.e. minimizer of $W_2(\mu_0,\mu_1)$) and a geodesic $\mu_t = \rho_t m \in \mathscr{P}_2(X)$ with bounded supports such that for all $t \in [0,1]$ and $N' \geq N$:

$$\int_X \rho_t^{-1/N'} d\mu_t \geq \int_{X\times X} \big[\sigma_{K/N'}^{(1-t)}(d(x_0,x_1))\rho_0(x_0)^{-1/N'} \\ + \sigma_{K/N'}^{(t)}(d(x_0,x_1))\rho_1(x_1)^{-1/N'}\big] q(dx_0dx_1).$$

(ii)' $\underline{\mathsf{CD}^e(K,N)\text{ (entropic curvature-dimension condition)}}$: The relative entropy Ent is (K,N)-convex on $(\mathscr{P}_2(X), W_2)$.

(iii)' $\underline{\mathsf{EVI}_{K,N}\text{ (evolution variational inequality)}}$: $^\forall\mu \in D(\mathrm{Ent})$, $^\exists$locally absolutely continuous curve $\mu_t \in \mathscr{P}_2(X)$ with $\mu_0 = \mu$ s.t.

$$\frac{d}{dt}\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_t,\nu)}{2}\right) + K\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_t,\nu)}{2}\right) \leq \frac{N}{2}\left(1-\exp\left(-\frac{1}{N}(\mathrm{Ent}(\nu)-\mathrm{Ent}(\mu_t))\right)\right)$$

for $^\forall\nu \in \mathscr{P}_2(X)$.

(iv) Space-time W_2-control: $^\forall\mu_0,\mu_1 \in \mathscr{P}_2(X)$, $^\forall t,s \geq 0$,

$$\mathfrak{s}_{K/N}^2\left(\frac{W_2(T_t\mu_0, T_s\mu_1)}{2}\right) \leq \mathrm{e}^{-K(s+t)}\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_0,\mu_1)}{2}\right) + \frac{N}{2}\frac{1-\mathrm{e}^{-K(s+t)}}{K(s+t)}\left(\sqrt{t}-\sqrt{s}\right)^2.$$

(v) Bakry-Ledoux gradient estimate: $^\forall f \in D(\mathsf{Ch})$, $^\forall t > 0$,

$$|\nabla T_t f|_w^2 + \frac{2tC(t)}{N}|\Delta T_t f|^2 \leq \mathrm{e}^{-2Kt}T_t(|\nabla f|_w^2) \quad m\text{-a.e.},$$

where $C(t) > 0$ is a function satisfying $C(t) = 1 + O(t)$ as $t \to 0$.

(vi) Bochner's inequality (weak form): $^\forall f \in D(\Delta)$ with $\Delta f \in D(\mathsf{Ch})$, $^\forall g \in D(\Delta)\cap L^\infty(X,m)$ with $g \geq 0$ and $\Delta g \in L^\infty(X,m)$,

$$\frac{1}{2}\int_X \Delta g|\nabla f|_w^2\,dm - \int_X g\langle\nabla f, \nabla\Delta f\rangle\,dm \geq K\int_X g|\nabla f|_w^2\,dm + \frac{1}{N}\int_X g(\Delta f)^2\,dm.$$

This is a weak form of

$$\frac{1}{2}\Delta|\nabla f|_w^2 - \langle\nabla f,\nabla\Delta f\rangle \geq K|\nabla f|_w^2 + \frac{1}{N}(\Delta f)^2.$$

Here $\langle\cdot,\cdot\rangle$ means

$$\langle\nabla f,\nabla g\rangle = \frac{1}{4}\left(|\nabla(f+g)|_w^2 - |\nabla(f-g)|_w^2\right).$$

This is bilinear in f and g under Assumption 1.