# Notations and statements for the talk

Geometry Colloquium (Jun. 26, 2014)

- **Local Lipschitz constant**: For $f : X \to \mathbb{R}$, $|\nabla f|(x) := \limsup_{y \to x} \dfrac{|f(y) - f(x)|}{d(y,x)}$.
- **Cheeger's energy functional**:

$$\mathsf{Ch}(f) := \inf \left\{ \liminf_{n\to\infty} \int_X |\nabla f_n|^2 \, dm \;\middle|\; \begin{array}{l} f_n : X \to \mathbb{R}\text{: Lipschitz} \\ f_n \to f \text{ in } L^2(m) \end{array} \right\} = \int_X |\nabla f|_w^2 \, dm.$$

Note that the existence of such a function $|\nabla f|_w$ is non-trivial.

**Assumption 1:** $\mathsf{Ch}$ is quadratic, that is, it comes from a bilinear form.

**Assumption 2:** ${}^\exists c > 0$ such that $\displaystyle\int_X \exp\left(-cd(x_0,x)^2\right) m(dx) < \infty$.

**Assumption 3:** For any $f \in \mathcal{D}(\mathsf{Ch})$ with $|\nabla f|_w \le 1$, $f$ has 1-Lipschitz representative.

- **$L^2$-Wasserstein distance**: $L^2$-Wasserstein space $\mathcal{P}_2(X)$ consists of probability measures on $X$ with finite second moments:

$$\mathcal{P}_2(X) := \left\{ \mu \in \mathcal{P}(X) \;\middle|\; \int_X d(x_0,x)^2 \mu(dx) < \infty \text{ for some } x_0 \in X \right\}.$$

$L^2$-Wasserstein distance $W_2 : \mathcal{P}(X) \times \mathcal{P}(X) \to [0,\infty)$ is given as follows:

$$W_2(\mu,\nu) := \inf \left\{ \|d\|_{L^2(\pi)} \;\middle|\; \begin{array}{l} \pi \in \mathcal{P}(X \times X),\ {}^\forall A \in \mathcal{B}(X), \\ \pi(A \times X) = \mu(A),\ \pi(X \times A) = \nu(A) \end{array} \right\}.$$

Note that $(\mathcal{P}_2(X), W_2)$ is a Polish geodesic metric space since so is $(X,d)$.

- **Relative entropy**: $\mathrm{Ent} : \mathcal{P}(X) \to \bar{\mathbb{R}}$ is defined as follows:

$$\mathrm{Ent}(\mu) := \begin{cases} \displaystyle\int_X \rho \log \rho \, dm & (\mu \ll m,\ \mu = \rho m,\ (\rho \log \rho)_+ \in L^1(m)), \\ \infty & (\text{otherwise}). \end{cases}$$

Under Assumption 2, $\mathrm{Ent}(\mu) > -\infty$ holds for ${}^\forall \mu \in \mathcal{P}_2(X)$.

- **Comparison functions**: For $\kappa \in \mathbb{R}$ and $\kappa\theta^2 \le \pi^2$,

$$\mathfrak{s}_\kappa(\theta) := \frac{\sin(\sqrt{\kappa}\theta)}{\sqrt{\kappa}}, \quad \sigma_\kappa^{(t)}(\theta) := \frac{\mathfrak{s}_\kappa(t\theta)}{\mathfrak{s}_\kappa(\theta)}.$$

(i)' **CD**$^*(K,N)$ **(reduced curvature-dimension condition)**: For $\mu_0 = \rho_0 m, \mu_1 = \rho_1 m \in \mathcal{P}(X)$ with bounded supports, there exists an optimal coupling $q$ of them (i.e. a minimizer of $W_2(\mu_0,\mu_1)$) and a geodesic $\mu_t = \rho_t m \in \mathcal{P}_2(X)$ with bounded supports such that for all $t \in [0,1]$ and $N' \ge N$:

$$\int_X \rho_t^{-1/N'} d\mu_t \ge \int_{X\times X} \big[\sigma_{K/N'}^{(1-t)}(d(x_0,x_1))\rho_0(x_0)^{-1/N'} + \sigma_{K/N'}^{(t)}(d(x_0,x_1))\rho_1(x_1)^{-1/N'}\big] q(dx_0dx_1).$$

(ii)' $\mathsf{CD}^e(K,N)$ **(entropic curvature-dimension condition)**: For each $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, there is a minimal $W_2$-geodesic $(\mu_t)_{t\in[0,1]}$ joining $\mu_0$ and $\mu_1$ s.t.

$$U_N(\mu_t) \geq \sigma_{K/N}^{(1-t)}(W_2(\mu_0,\mu_1))U_N(\mu_0) + \sigma_{K/N}^{(t)}(W_2(\mu_0,\mu_1))U_N(\mu_1),$$

where $U_N := \exp\left(-\frac{1}{N}\operatorname{Ent}\right)$. We also call this property "(weak) $(K,N)$-convexity of Ent on $(\mathcal{P}_2(X), W_2)$". This is an integral formulation of the following inequality in the distributional sense:

$$\partial_t^2 U_N(\mu_t) \leq -\frac{K}{N} W_2(\mu_0,\mu_1)^2 U_N(\mu_t).$$

If we can regard $(\mathcal{P}_2(X), W_2)$ as a Riemannian manifold (i.e. $W_2$ is the Riemannian distance) and Ent is $C^2$-function on it, then Ent is $(K,N)$-convex if and only if

$$\operatorname{Hess}\operatorname{Ent} - \frac{1}{N}\nabla\operatorname{Ent}\otimes\nabla\operatorname{Ent} \geq K.$$

(iii)' $\mathsf{EVI}_{K,N}$ **(evolution variational inequality)**: $^\forall\mu \in \mathcal{D}(\mathrm{Ent})$, $^\exists$locally absolutely continuous curve $\mu_t \in \mathcal{P}_2(X)$ with $\mu_0 = \mu$ s.t., for $^\forall\nu \in \mathcal{P}_2(X)$,

$$\frac{d}{dt}\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_t,\nu)}{2}\right) + K\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_t,\nu)}{2}\right) \leq \frac{N}{2}\left(1 - \exp\left(-\frac{1}{N}(\mathrm{Ent}(\nu) - \mathrm{Ent}(\mu_t))\right)\right).$$

(iv) **Space-time $W_2$-control**: $^\forall\mu_0, \mu_1 \in \mathcal{P}_2(X)$, $^\forall t, s \geq 0$,

$$\mathfrak{s}_{K/N}^2\left(\frac{W_2(P_t^*\mu_0, P_s^*\mu_1)}{2}\right) \leq \mathrm{e}^{-K(s+t)}\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_0,\mu_1)}{2}\right) + \frac{N}{2}\frac{1-\mathrm{e}^{-K(s+t)}}{K(s+t)}\left(\sqrt{t}-\sqrt{s}\right)^2.$$

(v) **Bakry-Ledoux gradient estimate**: $^\forall f \in \mathcal{D}(\mathsf{Ch})$, $^\forall t > 0$,

$$|\nabla P_t f|_w^2 + \frac{2tC(t)}{N}|\Delta P_t f|^2 \leq \mathrm{e}^{-2Kt}P_t(|\nabla f|_w^2) \quad m\text{-a.e.},$$

where $C(t) > 0$ is a function satisfying $C(t) = 1 + O(t)$ as $t \to 0$.

(vi) **Bochner's inequality (weak form)**: $^\forall f \in \mathcal{D}(\Delta)$ with $\Delta f \in \mathcal{D}(\mathsf{Ch})$, $^\forall g \in \mathcal{D}(\Delta) \cap L^\infty(X,m)$ with $g \geq 0$ and $\Delta g \in L^\infty(X,m)$,

$$\frac{1}{2}\int_X \Delta g|\nabla f|_w^2\,dm - \int_X g\langle\nabla f, \nabla\Delta f\rangle\,dm \geq K\int_X g|\nabla f|_w^2\,dm + \frac{1}{N}\int_X g(\Delta f)^2\,dm,$$

where $\langle\cdot,\cdot\rangle$ means

$$\langle\nabla f, \nabla g\rangle = \frac{1}{4}\left(|\nabla(f+g)|_w^2 - |\nabla(f-g)|_w^2\right).$$

This is bilinear in $f$ and $g$ under Assumption 1. This is a weak form of the following inequality:

$$\frac{1}{2}\Delta|\nabla f|_w^2 - \langle\nabla f, \nabla\Delta f\rangle \geq K|\nabla f|_w^2 + \frac{1}{N}(\Delta f)^2.$$