The entropic curvature dimension condition and Bochner's inequality

桒田 和正

(お茶の水女子大学)

M. Erbar 氏，K.-Th. Sturm 氏 (Bonn 大学) との共同研究

確率論シンポジウム

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1. Introduction

Purpose/History

★ Many characterizations of "Ricci curvature $\geq K$" on a complete Riemannian manifold X:

- K-convexity of Ent (via optimal transportation)
- $^{\forall}$Given $B_1(0)$ & $B_2(0)$,
  $^{\exists}(B_1(t), B_2(t))$: coupling of BMs s.t.

$$d(B_1(t), B_2(t)) \leq e^{-Kt} d(B_1(0), B_2(0))$$

- $|\nabla e^{t\Delta} f|^2 \leq e^{-2Kt} e^{t\Delta}(|\nabla f|^2)$
- $\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$

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- $^\forall \mu, \nu \in \mathcal{P}(X)$,

$$W_2((e^{t\Delta})^*\mu, (e^{t\Delta})^*\nu)^2 \leq e^{-2Kt} W_2(\mu, \nu)^2$$

- $|\nabla e^{t\Delta} f|^2 \leq e^{-2Kt} e^{t\Delta}(|\nabla f|^2)$
- $\dfrac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$

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Unify characterizations of

$$\text{“}\mathbf{Ric} \geq \boldsymbol{K} \text{ and } \mathbf{dim} \leq \boldsymbol{N}\text{”}$$

in terms of optimal transportation / $\boldsymbol{T_t = e^{t\Delta}}$

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- Established for “$\mathbf{Ric} \geq \boldsymbol{K}$”
  ([von Renesse & Sturm '05] $\boldsymbol{X}$: Riem. mfd.
  [Ambrosio, Gigli & Savaré et al.] $\boldsymbol{X}$: mm sp.

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- Study via optimal transport
  &
  Study via $\boldsymbol{T_t}$

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- Study via optimal transport & Study via T_t } separated when $N < \infty$

Purpose/History

The case $N = \infty$

- via T_t, $\boldsymbol{\Delta}$ & $\boldsymbol{\nabla}$ (or Δ & ∇): [Bakry & Émery '84]
- via Optimal transport: [Sturm '06,
  Lott & Villani, '09, Sturm & Bacher '10]

  in terms of the relative entropy $\mathbf{Ent}$

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The case $N = \infty$

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in terms of the relative entropy Ent

Key "fact" when $N = \infty$

$T_t^* \mu$: gradient curve of Ent on $(\mathcal{P}(X), W_2)$

$$\left(\mathrm{Ent}(\mu) := \int_X \rho \log \rho \, d\mathfrak{m} \quad (\mu = \rho \mathfrak{m}) \right)$$

Purpose/History

The case $N < \infty$

- Bochner's inequality (via T_t [Bakry & Ledoux '06])
$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2$$
- via Optimal transport: [Sturm '06,
  Lott & Villani, '09, Sturm & Bacher '10]
  in terms of the Rényi entropy (NOT Ent)

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Goal

- Characterize "$\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$" by $\mathbf{Ent}$

- Find missing conditions

$$\text{connecting} \begin{cases} \text{optimal transport approach} \\ \qquad\qquad \& \\ T_t \text{ approach} \end{cases}$$

- Establish the equivalence

Goal

- Characterize "$\mathbf{Ric} \geq K$ & $\mathbf{dim} \leq N$" by $\mathbf{Ent}$

  $\Rightarrow$ (K, N)-convexity of $\mathbf{Ent}$
- Find missing conditions

  $$\text{connecting} \begin{cases} \text{optimal transport approach} \\ \qquad\qquad \& \\ T_t \text{ approach} \end{cases}$$

  $\Rightarrow$ (K, N)-evolution variational inequality,

  Space-time W_2-control
- Establish the equivalence

Applications

- Stability under mGH (or Sturm's $\mathbb{D}$)-conv.
- Tensorization
- From local to global
- Measure contraction property $\mathbf{MCP}(K, N)$
  – (sharp) Bishop-Gromov volume comparison
  – volume doubling property
  – (local unif.) Poincaré ineq. [Rajala '11]
  $\Rightarrow$ $^{\exists}$heat kernel, two-sided Gaussian bound
  $\Rightarrow$ Ultracontractivity of T_t

Applications

- N-precision of f'nal ineq.'s
  - N-HWI ineq.
    - $\Rightarrow$ N-log Sobolev ineq.
    - $\Rightarrow$ N-Talagrand ineq.
- Lipschitz regularity of the heat kernel/eigenfn.'s
- Lichnerowicz bound of λ_1 [Ketterer]
- Li-Yau's ineq. [Garofalo & Mondino]
- Maximal diameter theorem [Ketterer]

Outline of the talk

1. Introduction

2. Entropic curvature dimension condition

3. Connection with Bakry-Émery theory

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Framework

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Cheeger's L^2-Dirichlet energy functional

$$\mathrm{Ch}(f) := \inf_{\substack{f_n\text{: Lip.} \\ f_n \to f \text{ in } L^2}} \liminf_{n} \int_X |\nabla f_n|^2 d\mathfrak{m}$$

$$= \int_X |\nabla f|_w^2 \, d\mathfrak{m}$$

($|\nabla f|_w$: minimal weak upper gradient)

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★ Suppose Ch: quadratic ($\Rightarrow$ Ch: Dirichlet form)

Framework

$(X, d, \mathfrak{m})$: Polish geodesic metric measure sp.,
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L^2-Wasserstein disntace

For $\mu, \nu \in \mathcal{P}(X)$,

$$W_2(\mu, \nu) := \inf\{\|d\|_{L^2(\pi)} \mid \pi\text{: coupling of } \mu \text{ \& } \nu\}$$

$$\mathcal{P}_2(X) := \{\mu \mid {}^{\forall/\exists}x \in X, W_2(\mu, \delta_x) < \infty\}$$

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★ $(\mathcal{P}_2(X), W_2)$: Polish geodesic met. sp.
(conv. in $W_2 \Leftrightarrow$ weak conv. with unif. L^2-moment)

Framework

Examples

- $X = \mathbb{R}^n$, d: Eucl. dist., $\mathfrak{m}(dx) = e^{-V(x)}dx$
- X: cpl. Riem. mfd, d: Riem. dist., $\mathfrak{m}$: vol. meas.

In both cases, $\mathrm{Ch}(f) = \int_X \langle \nabla f, \nabla f \rangle d\mathfrak{m}$

Entropic curvature dimension condition

$K \in \mathbb{R}$, $N > 0$

(K, N)-convexity of Ent

$$\text{“Hess Ent} - \frac{1}{N} \nabla \operatorname{Ent}^{\otimes 2} \geq K\text{”}$$

$$\Updownarrow$$

$$\text{“Hess}\, U_N \leq -\frac{K}{N} U_N\text{”},\ U_N := \exp\left(-\frac{1}{N} \operatorname{Ent}\right)$$

Entropic curvature dimension condition

(K, N)-convexity of Ent

$$\text{“}\mathrm{Hess}\, U_N \leq -\frac{K}{N} U_N\text{”}, \; U_N := \exp\left(-\frac{1}{N}\,\mathrm{Ent}\right)$$

Entropic curvature dimension cond.

$(X, d, \mathfrak{m})$ satsifies strong $\mathsf{CD}^e(K, N)$:

$\overset{\text{def}}{\Leftrightarrow}$ Ent is (K, N)-convex along $^\forall$min. geod.'s in $(\mathcal{P}_2(X), W_2)$

Equivalence of CD cond'ns

Theorem 1

For $K \in \mathbb{R}$ *and* $N > 0$*, TFAE:*

(i) *strong* $\mathsf{CD}^*(K, N)$

(ii) *strong* $\mathsf{CD}^e(K, N)$

(iii) $^\forall\mu_0$, $^\exists(\mu_t)_{t\geq 0}$*:* $\mathsf{EVI}_{K,N}$*-curve &* **(V)**

- $\star$ $\mathsf{CD}^*(K, N)$: reduced curvature-dimension cond. [Bacher & Sturm '10]
- $\star$ For Riem. mfds,

$$\textbf{(i)} \Leftrightarrow \mathrm{Ric} \geq K \text{ and } \dim \leq N$$

Equivalence of CD cond'ns

Relation between **CD**$^e(K,N)$ & **CD**$^*(K,N)$:

CD$^e(K,N)$ $\rightsquigarrow$ relative entropy $\displaystyle\int_X \log\rho \, d\mu$

CD$^*(K,N)$ $\rightsquigarrow$ Rényi entropy $\displaystyle\int_X \rho^{-1/N} \, d\mu$

$(\mu = \rho\mathfrak{m})$

$$\star \int_X \rho^{-1/N} \, d\mu = \int_X \exp\left(-\frac{1}{N}\log\rho\right) \, d\mu$$

$$\geq \exp\left(-\frac{1}{N}\int_X \log\rho \, d\mu\right) = U_N(\mu)$$

Equivalence of CD cond'ns

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(iii) ${}^\forall \mu_0$, ${}^\exists(\mu_t)_{t \ge 0}$: $\mathbf{EVI}_{K,N}$-*curve & (V)*

$$\text{(V)} \quad \int_X \exp\left(-{}^\exists c\, d(x_0, x)^2\right) \mathfrak{m}(dx) < \infty$$

(K, N)-Evolution Variational Inequality of Ent:

A variational formulation of "$\partial_t \mu_t = -\nabla \, \mathrm{Ent}(\mu_t)$"

Space-time W_2-control

EVI$_{K,N}$: $\forall \nu$, a bound of $\dfrac{d}{dt} W_2(\mu_t, \nu)$

Space-time W_2-control

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$$\Downarrow$$

Space-time W_2-control

$$\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu_s)}{2}\right)^2 \le e^{-K(s+t)}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_0,\nu_0)}{2}\right)^2 + \frac{N}{2}\frac{1-e^{-K(s+t)}}{K(s+t)}(\sqrt{t}-\sqrt{s})^2$$

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(NOTE: $\mu_t = T_t^* \mu_0$ ($T_t = e^{t\Delta}$, $\Delta \leftrightsquigarrow$ Ch) [Ambrosio, Gigli & Savaré])

Bakry-Ledoux's gradient estimate

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$\Downarrow$ Derivative in space & time (cf. [K.])

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Bakry-Ledoux's gradient est.: For $f \in W^{1,2}$, $\mathfrak{m}$-a.e.,

$$|\nabla T_t f|_w^2 \le e^{-2Kt}T_t(|\nabla f|_w^2) - \frac{2tC(t)}{N}|\Delta T_t f|^2,$$

$$(C(t) = 1 + O(t)\ (t \to 0))$$

$\mathsf{CD}^e(K,N) \Rightarrow$ Bakry-Ledoux

Theorem 2

For $K \in \mathbb{R}$ *&* $N > 0$, **(iii)** $\Rightarrow$ **(iv)** *&* **(v)**

(iii) $^\forall\mu_0$, $^\exists$**EVI**$_{K,N}$*-curve* $(\mu_t)_t$ *&* **(V)**

(iv) *Space-time* W_2*-control*

(v) *Bakry-Ledoux's gradient estimate*

Bakry-Ledoux $\Leftrightarrow$ Bochner

Theorem 3

For $K \in \mathbb{R}$ *&* $N > 0$, **(v) $\Leftrightarrow$ (vi)**

(v) $$|\nabla T_t f|_w^2 \le e^{-2Kt} T_t(|\nabla f|_w^2) - \frac{2tC(t)}{N} |\Delta T_t f|^2$$

(vi) $\forall f \in W^{1,2}$ *with* $\Delta f \in W^{1,2}$ *&*

$g \in D(\Delta) \cap L^\infty$ *with* $g \ge 0$ *&* $\Delta g \in L^\infty$

$$\int_X \left(\frac{1}{2} \Delta g |\nabla f|_w^2 - g \langle \nabla f, \nabla \Delta f \rangle \right) d\mathfrak{m}$$

$$\ge \int_X g \left(K |\nabla f|_w^2 + \frac{1}{N} |\Delta f|^2 \right) d\mathfrak{m}$$

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Theorem 3

For $K \in \mathbb{R}$ & $N > 0$, **(v) $\Leftrightarrow$ (vi)**

(v) $|\nabla T_t f|_w^2 \le e^{-2Kt} T_t(|\nabla f|_w^2) - \dfrac{2tC(t)}{N} |\Delta T_t f|^2$

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Summary

Theorem 4

$K \in \mathbb{R}$, $N > 0$ *(Recall: Ch is quadratic)*

(1) *TFAE*

- **(i)** *strong* **CD**$^*(K, N)$
- **(ii)** *strong* **CD**$^e(K, N)$
- **(iii)** $^\exists$**EVI**$_{K,N}$*-curves &* **(V)**

(2) *Under Ass. below,*
either **(iv)–(vi)** *is also equiv. to* **(i)–(iii)**

- **(iv)** *Space-time* W_2*-control*
- **(v)** *Bakry-Ledoux's gradient estimate*
- **(vi)** *Bochner's inequality*

Assumption

- The volume growth bound **(V)**
- $|\nabla f|_w \le 1$ $\mathfrak{m}$-a.e. $\Rightarrow$ f: 1-Lip.

Remark

Ass. 1 & $|\nabla T_t f|_w^2 \le e^{-2Kt} T_t(|\nabla f|_w^2)$

$\Rightarrow$ $\mathbf{CD}(K, \infty)$

[Ambrosio, Gigli & Savaré]

Examples

- $-N \log \mathfrak{s}_{K/N}(x)$ or $-N \log \mathfrak{s}'_{K/N}(x)$ are (K, N)-convex fn's on $\mathbb{R}$ (or an interval)
- When $(X, d, e^{-V}\mathbf{vol})$: cpl. Riem. mfd, $\partial X = \emptyset$,

  $\mathsf{CD}^e(K, N) \Leftrightarrow$

  $$\mathrm{Ric} + \mathrm{Hess}\, V - \frac{1}{N-n} \nabla V^{\otimes 2} \geq K$$

  $$(n = \dim X \leq N)$$

- More generally,

  $(X, d, \mathfrak{m})$: strong $\mathsf{CD}^e(K, N)$

  & V: strong (K', N')-convex on X

  $\Rightarrow (X, d, e^{-V}\mathfrak{m})$: strong $\mathsf{CD}^e(K + K', N + N')$