Wasserstein contractions associated with the curvature-dimension condition

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1. Introduction

Framework

M: cpl. Riem. mfd., $\dim \geq 2$, $\partial M = \emptyset$
$P_t = e^{t\Delta}$: heat semigroup on M, $P_t 1 \equiv 1$

Goal

Characterize

$$\mathrm{Ric} \geq K \;\&\; \dim M \leq N$$

in terms of behavior of Wasserstein distances between heat distributions $P_t^* \mu$ ($\mu \in \mathcal{P}(M)$, $t > 0$)

lower Ricci bound on metric meas. sp.

Recent developments:

Generalization of "$\mathrm{Ric} \geq K$"

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Generalization of "$\mathrm{Ric} \geq K$"

- Equivalent conditions in terms of P_t
  or only "metric and measure"
- Same equivalence beyond Riem. mfds
  $\Rightarrow$ e.g. "Stable" sufficient cond.
  for Lipschitz regularity of $P_t f$

1. Introduction

2. Known results for lower Ricci bounds

3. Curvature-dimension conditions

4. Proofs & extensions
4.1 Duality
4.2 Questions

lower Ricci curv. bound

For $K \in \mathbb{R}$, TFAE ([von Renesse & Sturm '05] etc.):

(i) $\mathrm{Ric} \geq K$

(ii) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$,

(iii) $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2)$

(iv) $\frac{1}{2}(\Delta |\nabla f|^2 - 2\langle \nabla f, \nabla \Delta f \rangle) \geq K |\nabla f|^2$

(v) Ent is K-convex w.r.t. W_2

How important?

- **(iii)(iv)** has rich applications
  in functional ineq. & differential geometry,
  e.g. quantitative Lipschitz regularization of P_t
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  ⇒ extension of **(ii)(iii)(iv)** to singular spaces
  [Ambrosio, Gigli & Savaré] etc.

Implications

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(ii) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

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On non-smooth sp.:

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$\Downarrow$

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- Identification of $P_t^* \mu$ with the gradient flow of Ent in $(\mathcal{P}(M), W_2)$
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[Gigli, K. & Ohta]: cpt. Alexandrov sp.

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† TFAE [Ambrosio, Gigli & Savaré]

(v)* $\mathbf{Ent}$ is $\boldsymbol{K}$-convex w.r.t.$\boldsymbol{W_2}$ & linearity of heat flow

(vi) $^\forall \boldsymbol{\mu_0}$, $^\exists$sol $(\boldsymbol{\mu_t})_{t\geq 0}$ to **EVI**$_K$ of $\mathbf{Ent}$: for $^\forall \boldsymbol{\nu}$,

$$\frac{d}{dt}\left(\frac{W_2(\mu_t, \nu)^2}{2}\right) + \frac{K}{2} W_2(\mu_t, \nu)^2 + \mathrm{Ent}(\mu_t) \leq \mathrm{Ent}(\nu)$$

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★ **(iii)** $\Rightarrow$ **(v)** under a suitable ass.(incl. linearity) [Ambrosio, Gigli & Savaré]

Summary of implications (when the heat flow is linear)

$$\begin{array}{ccccc} \textbf{(vi)}\ \text{EVI} & \Longleftrightarrow & \textbf{(v)}\ \text{Ent} & & \\ \Downarrow & & \Uparrow & & \\ \textbf{(ii)}\ \text{Wass.} & \Longleftrightarrow & \textbf{(iii)}\ \boldsymbol{\nabla}\text{-est.} & \Longleftrightarrow & \textbf{(iv)}\ \boldsymbol{\Gamma_2} \end{array}$$

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$$

What we did for $\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$:

- Formulate a missing condition corresponding to **(ii)**
- Extension of the implication **(ii)** $\Leftrightarrow$ **(iii)** (even in an abstract setting)
- Another approach based on a coupling method

Known conditions

(i) $\mathrm{Ric} \geq K$

$\Updownarrow$

(iv) $\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$

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$\Updownarrow$ $\leftsquigarrow$ [F.-Y. Wang '11]

(iii)' $$|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) - \frac{1 - e^{-2Kt}}{NK}(\Delta P_t f)^2$$

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(i)' $\Leftrightarrow$ **(v)'**: $\mathsf{CD}(K, N)$ [Sturm '06 / Lott & Villani '09]

Theorem 1 ([K.])

For $K \in \mathbb{R}$ *and* $N \in [2, \infty)$,
(iii)' *is equivalent to the following* **(ii)'**:

(ii)' $W_2(P_s^* \mu_0, P_t^* \mu_1)^2$

$$\leq \left(\int_s^t e^{Kr} \xi(dr) \right)^{-2} W_2(\mu_0, \mu_1)^2 + \frac{N}{2} \xi([s, t])^2$$

where $\xi(dr) = \left(\frac{2K}{1 - e^{-2Kr}} \right)^{-1/2} dr$

The case $K = 0$

Corollary 2 ([K.])

For $N \in [2, \infty)$, TFAE:

(i)' $\mathrm{Ric} \geq 0$ & $\dim M \leq N$

(ii)' $$W_2(P_s^* \mu_0, P_t^* \mu_1)^2 \leq W_2(\mu_0, \mu_1)^2 + 2N(\sqrt{t} - \sqrt{s})^2$$

(iii)' $$|\nabla P_t f|^2 \leq P_t(|\nabla f|^2) - \frac{2}{N}(\Delta P_t f)^2$$

Remark

[Bakry, Gentil & Ledoux]: **(iv)'** $\Rightarrow$ **(iii)"** $\Rightarrow$ **(ii)'**
(**(iii)"** an intertwining ineq. for P_t and Hopf-Lax semigr.)

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$$P_t(d(x_0, \cdot)^2)(x_1) \leq d(x_0, x_1)^2 + 2Nt$$

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$$P_t(d(x_0,\cdot)^2)(x_1) \leq d(x_0,x_1)^2 + 2Nt$$
$$\Rightarrow \Delta(d(x_0,\cdot)^2)(x_1) \leq 2N$$

(sharp Laplacian comparison)

Idea of the proof

(ii)' $\Rightarrow$ **(iii)'**: Differentiation

(iii)' $\Rightarrow$ **(ii)'**: Kantorovich duality
& analysis of the Hopf-Lax semigroup
(cf. [K. '10 / K.] when $N = \infty$)

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Sketch of proof: (ii)' $\Rightarrow$ (iii)'

(ii)' $W_2(P_s^*\mu_0, P_t^*\mu_1)^2$

$$\leq \left(\int_s^t e^{Kr}\xi(dr) \right)^{-2} W_2(\mu_0,\mu_1)^2 + \frac{N}{2}\xi([s,t])^2$$

(iii)' $\dfrac{2\Psi(t)}{N}(\Delta P_t f)^2 + |\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2)$

where $\xi(dr) = \left(\dfrac{2K}{1-e^{-2Kr}} \right)^{-1/2} dr =: \dfrac{dr}{\sqrt{\Psi(r)}}$

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———— ∘ ———— ∘ ————

For π: coupling of $P_t^* \delta_x$ and $P_s^* \delta_y$,

$$P_t f(x) - P_s f(y) = \int (f(z) - f(w)) \pi(dz dw)$$

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Take $t - s = a\, d(x, y)$ for "suitable" $a \in \mathbb{R}$:

$\Rightarrow \dfrac{\text{(LHS)}}{d(x,y)} \to a \Delta P_t f(x) + |\nabla P_t f|(x)$ as $s \to t$

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$$\Rightarrow \text{(RHS)} = \int \frac{f(z) - f(w)}{d(z, w)} d(z, w) \pi(dz dw)$$

$$\text{"}\le\text{"} \ P_t(|\nabla f|^2)(x)^{1/2} W_2(P_t^* \delta_x, P_s^* \delta_y) \cdots \square$$

Sketch of proof: (iii)' $\Rightarrow$ (ii)'

Ingredients

- Kantorovich duality:

$$\frac{W_2(\nu,\mu)^2}{2} = \sup_f \left[\int Q_1 f \, d\mu - \int f \, d\nu \right]$$

- Hopf-Lax semigroup:

$$Q_r f(x) := \inf_{y \in M} \left[f(y) + \frac{d(x,y)^2}{2r} \right]$$

$\star$ $\partial_r Q_r f = -\frac{1}{2} |\nabla Q_r f|^2$ (Hamilton-Jacobi eq.)

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—— ○ —— ○ ——

For simplicity, $\mu_0 = \delta_{x_0}$, $\mu_1 = \delta_{x_1}$

$$\frac{W_2(P_s^* \delta_{x_0}, P_t^* \delta_{x_1})^2}{2} = \sup_f \left[P_t Q_1 f(x_1) - P_s f(x_0) \right]$$

Idea: give an upper bound of $[\cdots]$ being uniform in f

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$\gamma : [0,1] \to M$: geod. joining x_0 & x_1

$\alpha : [0,1] \to [s,t]$, $\eta : [0,1] \to [0,1]$: $\nearrow$, surj.

(suitably chosen)

$$\Rightarrow P_t Q_1 f(x_1) - P_s f(x_0)$$

$$= P_{\alpha(1)} Q_1 f(\gamma(\eta(1))) - P_{\alpha(0)} Q_0 f(\gamma(\eta(0)))$$

$$= \int_0^1 \partial_r P_{\alpha(r)} Q_r f(\gamma(\eta(r))) dr$$

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$$\begin{aligned}\partial_r P_{\alpha(r)} Q_r f(\gamma(\eta(r))) &\leq \alpha'(r)\Delta P_{\alpha(r)} Q_r f(\gamma(\eta(r))) \\ &\quad - \frac{1}{2} P_{\alpha(r)}(|\nabla Q_r f|^2)(\gamma(\eta(r))) \\ &\quad + \eta'(r)|\nabla P_{\alpha(r)} Q_r f|(\gamma(\eta(r))) \\ &\leq \cdots\end{aligned}$$

□

Remarks

(ii) / (ii)' Wass. contr.	differentiation $\Rightarrow$ $\Leftarrow$ integration	**(iii) / (iii)'** gradient est.

- If an est. like **(ii)'** is "infinitesimally sharp", then it implies **(iii)'**

  $\Rightarrow$ An weaker est. than **(ii)'** can be equiv. to **(iii)'**

  $\Rightarrow$ Self-improvements in Wass. contr.'s

Extended duality

Theorem 3 ([K.])

M: *Polish geod. sp.*, $P_t = e^{t\mathcal{L}}$: *Feller or str. Feller*
Then for $a, b : [0,\infty) \to (0,\infty)$, *TFAE:*

(A) $$W_p(P_s^*\mu_0, P_t^*\mu_1)^2 \le \left(\fint_s^t \frac{\xi(dr)}{\sqrt{a(r)}} \right)^{-2} W_p(\mu_0,\mu_1)^2 + \xi([s,t])^2$$

(B) $$|\nabla P_t f|^2 \le a(t) \Big[P_t(|\nabla f|^q)^{2/q} + b(t)(\mathcal{L}P_t f)^2 \Big]$$

where $p^{-1} + q^{-1} = 1$, $\xi(dr) := b(r)^{-1/2} dr$,
$|\nabla f|(x)$: *loc. Lip. const. of* f *at* x

1. **Introduction**

2. **Known results for lower Ricci bounds**

3. **Curvature-dimension conditions**

4. **Proofs & extensions**

4.1 Duality

4.2 Questions

Questions

- CD$(K,N) \Rightarrow$ **(ii)'** ?
  Are there an alternative of EVI?
  (Work in progress with M. Erbar & K.-T. Sturm)
- How sharp **(ii)'** is?
  ★ probably sharp when $K = 0$
  (Laplacian comparison)
- Connection with the monotonicity of normalized $\mathcal{L}$-transp. cost under backward Ricci flow?
  [cf. Topping '09, K.-Philipowski '11]