The entropic curvature-dimension condition and Bochner's inequality

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Conference ProbaGeo 2013 (Univ. Luxembourg) Oct. 28–31, 2013

1. Introduction

Purpose/History

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Unify the study of

$$\text{“}\mathbf{Ric} \geq \boldsymbol{K} \text{ and } \mathbf{dim} \leq \boldsymbol{N}\text{”}$$

in terms of

optimal transportation / heat distribution

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- Established for "$\mathbf{Ric} \geq \boldsymbol{K}$"
  (Riem. mfd.: [von Renesse & Sturm '05]
  mm-sp.: [Ambrosio, Gigli & Savaré et al.]

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- Study via optimal transport
  &
  Study via heat distribution

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- Study via optimal transport & Study via heat distribution } separated

Purpose/History

- via $P_t = e^{t\Delta}$ (or Δ; Bochner inequality): [Bakry & Émery '84, Bakry & Ledoux '06]
- via Optimal transport: [Sturm '06, Lott & Villani '09, Sturm & Bacher '10]

in terms of $\mathbf{Ent}$

Purpose/History

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- via Optimal transport: [Sturm '06, Lott & Villani '09, Sturm & Bacher '10]

in terms of Ent

Key "fact" for $\mathrm{Ric} \geq K$

$P_t\mu$: gradient curve of Ent on $(\mathcal{P}(X), W_2)$

$$\left(\mathrm{Ent}(\mu) := \int_X \rho \log \rho \, d\mathfrak{m} \quad (\mu = \rho\mathfrak{m})\right)$$

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in terms of the Rényi entropy (NOT Ent)

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- via $P_t = e^{t\Delta}$ (or Δ; Bochner inequality):
  [Bakry & Émery '84, Bakry & Ledoux '06]
- via Optimal transport: [Sturm '06,
  Lott & Villani '09, Sturm & Bacher '10]
  in terms of the Rényi entropy (NOT $\mathbf{Ent}$)
- Another approach via porous medium eq.
  (gradient curve of the Rényi ent.)
  [Ambrosio, Savaré & Mondino]

Goal

- Characterize "$\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$" by $\mathbf{Ent}$

- Find missing conditions

$$\text{connecting} \begin{cases} \text{optimal transport approach} \\ \qquad\quad \& \\ \boldsymbol{P_t} \text{ approach} \end{cases}$$

- Establish the equivalence

Goal

- Characterize "$\mathrm{Ric} \geq K$ & $\dim \leq N$" by Ent
  $\Rightarrow$ (K, N)-convexity of Ent
- Find missing conditions

$$\text{connecting} \begin{cases} \text{optimal transport approach} \\ \& \\ P_t \text{ approach} \end{cases}$$

  $\Rightarrow$ (K, N)-evolution variational inequality
  Space-time W_2-control
- Establish the equivalence

Outline of the talk

1. **Introduction**
2. **Entropic curvature-dimension condition**
3. **Evolution variational inequalities**
4. **Connection with Bakry-Émery theory**
5. **Applications**

1. Introduction

2. Entropic curvature-dimension condition

3. Evolution variational inequalities

4. Connection with Bakry-Émery theory

5. Applications

Framework

$(X, d, \mathfrak{m})$: Polish geodesic metric measure sp.,

$\mathfrak{m}$: loc. finite, σ-finite, $\operatorname{supp}\mathfrak{m} = X$,

$P_t f = e^{t\Delta} f$: gradient curve in $L^2(\mathfrak{m})$ of

Cheeger's L^2-Dirichlet energy functional

$$\mathrm{Ch}(f) = \int |\nabla f|_w^2 \, d\mathfrak{m}$$

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$$\mathrm{Ch}(f) = \int |\nabla f|_w^2 \, d\mathfrak{m}$$
$$= \inf_{\substack{f_n:\ \text{Lip.} \\ f_n \to f \text{ in } L^2}} \liminf_n \int |\nabla f_n|^2 d\mathfrak{m}$$

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Cheeger's L^2-Dirichlet energy functional

$$\mathrm{Ch}(f) = \int |\nabla f|_w^2 \, d\mathfrak{m} = \inf_{\substack{f_n\text{: Lip.} \\ f_n \to f \text{ in } L^2}} \liminf_n \int |\nabla f_n|^2 d\mathfrak{m}$$

⋆ (X, d, m): infinitesimally Hilbertian

$\overset{\text{def}}{\Leftrightarrow}$ Ch: quadratic form

($\Rightarrow$ Ch: Dirichlet form & P_t, Δ: linear)

Entropic curvature-dimension condition

$\boldsymbol{K}$-convexity of Ent

$$\text{``}\mathrm{Hess}\,\mathrm{Ent} \ \geq \boldsymbol{K}\text{''}$$

w.r.t. $\boldsymbol{W_2}$

Entropic curvature-dimension condition

K-convexity of Ent

$$\text{“}\mathrm{Hess}\,\mathrm{Ent} \;\geq K\text{”} \Leftrightarrow \mathrm{Ric} \geq K$$

Entropic curvature-dimension condition

(K, N)-convexity of Ent

$$\text{“}\mathrm{Hess}\,\mathrm{Ent} - \frac{1}{N} \nabla\,\mathrm{Ent}^{\otimes 2} \geq K\text{”}$$

Entropic curvature-dimension condition

$$\text{“Hess Ent} - \frac{1}{N} \nabla \text{Ent}^{\otimes 2} \geq K\text{”}$$

$$\Updownarrow$$

$$\text{“Hess}\, U_N \leq -\frac{K}{N} U_N\text{”},\; U_N := \exp\left(-\frac{1}{N}\,\text{Ent}\right)$$

Entropic curvature-dimension condition

$$\text{“}\mathrm{Hess}\, \boldsymbol{U_N} \leq -\frac{\boldsymbol{K}}{\boldsymbol{N}} \boldsymbol{U_N}\text{”}, \; \boldsymbol{U_N} := \exp\left(-\frac{1}{\boldsymbol{N}}\,\mathrm{Ent}\right)$$

Entropic curvature-dimension condition

$$\text{“Hess}\, U_N \le -\frac{K}{N} U_N\text{”}, \; U_N := \exp\left(-\frac{1}{N}\,\mathrm{Ent}\right)$$

$\mathsf{CD}^e(K,N)$: Entropic curvature-dimension condition

${}^\forall \mu_0, \mu_1 \in \mathcal{P}_2(X)$, ${}^\exists(\mu_t)_{t\in[0,1]}$: W_2-geod. s.t.

$$U_N(\mu_t) \ge \sigma^{(1-t)}_{K/N}(W_2(\mu_0,\mu_1))U_N(\mu_0) + \sigma^{(t)}_{K/N}(W_2(\mu_0,\mu_1))U_N(\mu_1),$$

$$\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}}, \quad \sigma^{(s)}_\kappa(r) := \frac{\mathfrak{s}_\kappa(sr)}{\mathfrak{s}_\kappa(r)}$$

Relation with known conditions

CD$^*(K, N)$: reduced curvature-dimension condition

[Bacher & Sturm '10]

$\Downarrow$ geod.'s on X are non-branching

$^\forall \mu_0, \mu_1 \in \mathcal{P}_2(X)$, $^\exists (\mu_r)_{r\in[0,1]}$: W_2-geod.,
$^\exists \Gamma \in \mathcal{P}(\mathrm{Geo}(X))$: "lift" of $(\mu_r)_{r\in[0,1]}$ s.t.

$$\rho_t(\gamma_t)^{-\frac{1}{N}} \geq \sigma_{K/N}^{(1-t)}(d(\gamma_0, \gamma_1))\rho_0(\gamma_0)^{-\frac{1}{N}} + \sigma_{K/N}^{(t)}(d(\gamma_0, \gamma_1))\rho_1(\gamma_1)^{-\frac{1}{N}}$$

for Γ-a.e. γ, where $\rho_t \mathfrak{m} = \mu_t$

Relation with known conditions

CD$^*(K,N)$: reduced curvature-dimension condition
[Bacher & Sturm '10]

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for Γ-a.e. γ, where $\rho_t\mathfrak{m}=\mu_t$

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for Γ-a.e. γ, where $\rho_t \mathfrak{m} = \mu_t$

$\Downarrow$

$$\rho_t(\gamma_t)^{-\frac{1}{N}} = \exp\left(-\frac{1}{N}\log \rho_t(\gamma_t)\right)$$

& Jensen's ineq.

$\mathsf{CD}^e(K, N)$

Relation with known conditions

$$\rho_t(\gamma_t)^{-\frac{1}{N}} \geq \sigma_{K/N}^{(1-t)}(d(\gamma_0,\gamma_1))\rho_0(\gamma_0)^{-\frac{1}{N}} + \sigma_{K/N}^{(t)}(d(\gamma_0,\gamma_1))\rho_1(\gamma_1)^{-\frac{1}{N}}$$

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$\Updownarrow$ geod.'s on X are non-branching

$\mathsf{CD}^e(K,N)$

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3. Evolution variational inequalities

4. Connection with Bakry-Émery theory

5. Applications

K-evolution variational ineq.

EVI$_K$ of Ent

$^\forall \mu_0$, $^\exists(\mu_t)_{t\geq 0}$: abs. conti. s.t. for $^\forall \nu$,

$$\frac{d}{dt}\left(\frac{W_2(\mu_t,\nu)^2}{2}\right) + \frac{K}{2}W_2(\mu_t,\nu)^2 + \mathrm{Ent}(\mu_t) \leq \mathrm{Ent}(\nu)$$

⋆ Heuristically, $^\forall$geod.$(\sigma_s)_{s\in[0,1]}$ with $\sigma_0 = \mu_t$, **EVI$_K$** for $\nu = \sigma_s$:

$$-\langle \partial_t \mu_t, \dot{\sigma}_0 \rangle \leq \frac{\mathrm{Ent}(\sigma_s) - \mathrm{Ent}(\sigma_0)}{s} + o(1)$$

$\Rightarrow \partial_t \mu_t = -\nabla\,\mathrm{Ent}(\mu_t)$

K-evolution variational ineq.

Properties of $\mathbf{EVI}_K$ [Ambrosio, Gigli & Savaré]
[Ambrosio, Gigli, Mondino & Rajala]

K-evolution variational ineq.

The volume growth cond. **(V)**

$$\int_X \exp\left(-^{\exists}c\, d(x_0, x)^2\right) \mathfrak{m}(dx) < \infty$$

Properties of **EVI**$_K$ [Ambrosio, Gigli & Savaré]
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Properties of $\mathbf{EVI}_K$ [Ambrosio, Gigli & Savaré]
[Ambrosio, Gigli, Mondino & Rajala]

- ${}^{\forall}\mu_0$, ${}^{\exists}(\mu_t)_{t\ge 0}$ sol. to $\mathbf{EVI}_K$ & **(V)**

  $\Leftrightarrow$ $\mathrm{Hess}\,\mathrm{Ent} \ge K$ & $P_t\mu_0$: linear w.r.t. μ_0

  $\Leftrightarrow$ $\mathrm{Hess}\,\mathrm{Ent} \ge K$ & infin. Hilb.

- $(\mu_t)_{t\ge 0}$ sol. to $\mathbf{EVI}_K$ $\Rightarrow$ $\mu_t = P_t\mu_0$

$\mathsf{EVI}_{K,N}$ and Riemannian $\mathsf{CD}^e(K,N)$

$\mathsf{EVI}_{K,N}$ of Ent

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$$\frac{d}{dt}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 + K\,\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 \le \frac{N}{2}\left(1-\frac{U_N(\nu)}{U_N(\mu_t)}\right)$$

$$\left(\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}},\ U_N := \exp\left(-\frac{1}{N}\mathrm{Ent}\right)\right)$$

$\mathsf{EVI}_{K,N}$ and Riemannian $\mathsf{CD}^e(K,N)$

Theorem 1 ($\mathsf{RCD}^*(K,N)$ cond.)

For $K \in \mathbb{R}$ and $N > 0$, TFAE:

(i) $\mathsf{CD}^*(K,N)$ *& infinitesimally Hilbertian*

(ii) $\mathsf{CD}^e(K,N)$ *& infinitesimally Hilbertian*

(iii) ${}^\forall \mu_0$, ${}^\exists (\mu_t)_{t\ge 0}$: $\mathsf{EVI}_{K,N}$*-curve &* **(V)**

$$\textbf{(V)} \quad \int_X \exp\left(-{}^\exists c\, d(x_0,x)^2\right) \mathfrak{m}(dx) < \infty$$

EVI$_{K,N}$ and Riemannian CD$^e(K,N)$

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★ RCD$^*(K,N) \Rightarrow$ RCD(K,∞)

$\mathsf{EVI}_{K,N}$ and Riemannian $\mathsf{CD}^e(K,N)$

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★ $\mathsf{RCD}^*(K,N) \Rightarrow \mathsf{RCD}(K,\infty)$

$\Rightarrow$ geod.'s on X are essentially non-branching

[Rajala & Sturm]

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Regularity results on $\mathbf{RCD}(K, \infty)$ sp.

[Ambrosio, Gigli, Savaré et al.]: $\mathbf{RCD}(K, \infty)$ yields

- The volume growth bound **(V)**
- P_t: strong Feller, i.e. $P_t f \in C_b^{\mathrm{Lip}}$ for $f \in L^\infty(\mathfrak{m})$
- $|\nabla f|_w \leq 1$ $\mathfrak{m}$-a.e. $\Rightarrow$ f: 1-Lip.

Regularity results on RCD(K, ∞) sp.

Assumption 1 (cf. [Ambrosio, Gigli & Savaré])

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- $|\nabla f|_w \leq 1$ $\mathfrak{m}$-a.e. $\Rightarrow$ f: 1-Lip.

W_2-control

EVI$_K$-curve

$$\frac{d}{dt}\left(\frac{W_2(\mu_t,\nu)^2}{2}\right)+\frac{K}{2}W_2(\mu_t,\nu)^2 + \mathrm{Ent}(\mu_t)\le \mathrm{Ent}(\nu)$$

$\Downarrow$ $\nu=\nu_t$: another **EVI$_K$**-curve

W_2-control

$$W_2(\mu_t,\nu_t)\le e^{-Kt}W_2(\mu_0,\nu_0)$$

Space-time W_2-control

EVI$_{K,N}$-curve

$$\frac{d}{dt}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 + K\,\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 \le \frac{N}{2}\left(1-\frac{U_N(\nu)}{U_N(\mu_t)}\right)$$

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$$\Downarrow$$

Space-time W_2-control:

$$\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu_s)}{2}\right)^2 \le e^{-K(s+t)}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_0,\nu_0)}{2}\right)^2 + \frac{N}{2}\frac{1-e^{-K(s+t)}}{K(s+t)}(\sqrt{t}-\sqrt{s})^2$$

Space-time W_2-control

Heuristics ($K = 0$)

$$\mathrm{Hess\,Ent} \geq \frac{1}{N} \nabla\,\mathrm{Ent}^{\otimes 2} \Rightarrow \text{Sp.-t. } W_2\text{-control}$$

$(\sigma_r)_{r\in[0,1]}$: W_2-geod. from $\mu_{t_0 u}$ to $\nu_{t_1 u}$,
$\varphi_r := \langle \nabla\,\mathrm{Ent}(\sigma_r), \dot{\sigma}_r \rangle$, $(t_r)_{r\in[0,1]}$: interpolation

$$\frac{\partial}{\partial u} \frac{W_2(\mu_{t_0 u}, \nu_{t_1 u})^2}{2} = t_0 \varphi_0 - t_1 \varphi_1$$

$$= -\int_0^1 \frac{\partial}{\partial r}(t_r \varphi_r) dr \leq -\int_0^1 \dot{t}_r \varphi_r + \frac{1}{N} t_r \varphi_r^2 \, dr$$

$$\leq \frac{N}{4} \int_0^1 \frac{\dot{t}_r^2}{t_r} \, dr$$

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$$= -\int_0^1 \frac{\partial}{\partial r}(t_r\varphi_r)dr \leq -\int_0^1 \dot{t}_r\varphi_r + \frac{1}{N}t_r\varphi_r^2\, dr$$

$$\leq \frac{N}{4}\int_0^1 \frac{\dot{t}_r^2}{t_r}\, dr = N(\sqrt{t_1} - \sqrt{t_0})^2$$

Bakry-Ledoux gradient estimate

Space-time W_2-control:

$$\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu_s)}{2}\right)^2 \le e^{-K(s+t)}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_0,\nu_0)}{2}\right)^2 + \frac{N}{2}\frac{1-e^{-K(s+t)}}{K(s+t)}(\sqrt{t}-\sqrt{s})^2$$

Bakry-Ledoux gradient estimate

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$\Downarrow$ Derivative in space & time

Bakry-Ledoux gradient est.: For $f \in W^{1,2}$, $\mathfrak{m}$-a.e.,

$$|\nabla P_t f|_w^2 \le e^{-2Kt}P_t(|\nabla f|_w^2) - \frac{2tC(t)}{N}|\Delta P_t f|^2,$$

$$C(t) = 1 + O(t)\ (t \to 0)$$

RCD$^*(K, N)$ ⇒ Bakry-Ledoux

Theorem 2

For $K \in \mathbb{R}$ *&* $N > 0$, **(iii)** ⇒ **(iv)** ⇒ **(v)**

(iii) $^{\forall}\mu_0$, $^{\exists}$**EVI**$_{K,N}$*-curve* $(\mu_t)_t$ *&* **(V)**

(iv) *Ass. 1, infin. Hilb. & Space-time* W_2*-control*

(v) *Ass. 1, infin. Hilb. & Bakry-Ledoux grad. est.*

Bakry-Ledoux ⇔ Bochner

Theorem 3

$(X, d, \mathfrak{m})$: *infinitesimally Hilbertian*

For $K \in \mathbb{R}$ & $N > 0$, **(v) ⇔ (vi)**

(v) $|\nabla P_t f|_w^2 \leq e^{-2Kt} P_t(|\nabla f|_w^2) - \frac{2tC(t)}{N} |\Delta P_t f|^2$

(vi) $^\forall f \in W^{1,2}$ *with* $\Delta f \in W^{1,2}$ &

$g \in D(\Delta) \cap L^\infty$ *with* $g \geq 0$ & $\Delta g \in L^\infty$

$$\int_X \left(\frac{1}{2} \Delta g |\nabla f|_w^2 - g \langle \nabla f, \nabla \Delta f \rangle \right) d\mathfrak{m} \geq \int_X g \left(K |\nabla f|_w^2 + \frac{1}{N} |\Delta f|^2 \right) d\mathfrak{m}$$

Bakry-Ledoux $\Leftrightarrow$ Bochner

Theorem 3

$(X, d, \mathfrak{m})$: *infinitesimally Hilbertian*

For $K \in \mathbb{R}$ & $N > 0$, **(v) $\Leftrightarrow$ (vi)**

(v) $|\nabla P_t f|_w^2 \le e^{-2Kt} P_t(|\nabla f|_w^2) - \dfrac{2tC(t)}{N} |\Delta P_t f|^2$

(vi) $^\forall f \in W^{1,2}$ *with* $\Delta f \in W^{1,2}$ &

$g \in D(\Delta) \cap L^\infty$ *with* $g \ge 0$ & $\Delta g \in L^\infty$

$$\int_X \left(\frac{1}{2} \Delta g |\nabla f|_w^2 - g \langle \nabla f, \nabla \Delta f \rangle \right) d\mathfrak{m}$$

$$\ge \int_X g \left(K |\nabla f|_w^2 + \frac{1}{N} |\Delta f|^2 \right) d\mathfrak{m}$$

Bakry-Ledoux $\Rightarrow$ RCD$^*(K,N)$

Idea: Action estimate (as in [Ambrosio, Gigli, Savaré])

$$\frac{W_2(\mu_0, P_\tau \mu_1)^2}{2} - \frac{1}{2}\int_0^1 |\dot{\mu}_s|^2 e^{-2K\tau} ds \leq Nt(U_N(P_\tau\mu_1) - U_N(\mu_0)) \qquad (\spadesuit)$$

for $t \ll 1$, where $\tau = \tau_{s,t}$: $\partial_t \tau = s\, U_N(P_\tau \mu_s)$,

$$\tau_{s,0} = 0$$

$$(\Rightarrow \partial_t P_\tau \mu_s = sN\nabla U_N(P_\tau \mu_s))$$

Ingredients of the proof

Kantorovich duality, approximations

Bakry-Ledoux $\Rightarrow$ RCD$^*(K, N)$

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for $t \ll 1$, where $\tau = \tau_{s,t}$: $\partial_t \tau = s\, U_N(P_\tau \mu_s)$,
$\tau_{s,0} = 0$

$$(\Rightarrow \partial_t P_\tau \mu_s = sN\nabla U_N(P_\tau \mu_s))$$

Ingredients of the proof

Kantorovich duality, approximations & detailed calc.

Bakry-Ledoux $\Rightarrow$ RCD$^*(K,N)$

Idea: Action estimate (as in [Ambrosio, Gigli, Savaré])

$$\frac{W_2(\mu_0, P_\tau \mu_1)^2}{2} - \frac{1}{2}\int_0^1 |\dot{\mu}_s|^2 e^{-2K\tau} ds \leq Nt(U_N(P_\tau \mu_1) - U_N(\mu_0)) \qquad (\spadesuit)$$

for $t \ll 1$, where $\tau = \tau_{s,t} : \partial_t \tau = s\, U_N(P_\tau \mu_s)$,

$\Downarrow$

For $(\sigma_s)_{s\in[0,1]}$: W_2-geod.,
$(\spadesuit)$ for $(\mu_0, \mu_1) = (\sigma_0, \sigma_r)$ or (σ_1, σ_r)
& $t \to 0$

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$$U_N(\sigma_r) - (1-r)U_N(\sigma_0) - rU_N(\sigma_1) \ge \frac{K}{N}|\dot\sigma|^2 \int_0^1 (s(1-r)) \wedge ((1-s)r)\, U_N(\sigma_r) dr$$

Bakry-Ledoux $\Rightarrow$ RCD$^*(K, N)$

Theorem 4

$(X, d, \mathfrak{m})$: *infinitesimally Hilbertian & Ass. 1*

For $K \in \mathbb{R}$ *&* $N > 0$, **(v)** $\Rightarrow$ **(ii)**:

(v) *Bakry-Ledoux gradient estimate*

$$|\nabla P_t f|^2 \le e^{-2Kt} P_t(|\nabla f|^2) - \frac{2tC(t)}{N} |\Delta P_t f|^2$$

(ii) *Entropic curv.-dim.* **RCD**$^*(K, N)$:

$$\text{“} \operatorname{Hess} \operatorname{Ent} - \frac{1}{N} \nabla \operatorname{Ent}^{\otimes 2} \ge K \text{”}$$

Summary

Theorem 5

$K \in \mathbb{R}$, $N > 0$

(1) *TFAE*

- **(i)** $\mathsf{CD}^*(K, N)$ *& infin. Hilb.*
- **(ii)** $\mathsf{CD}^e(K, N)$ *& infin. Hilb.*
- **(iii)** $^\exists\mathsf{EVI}_{K,N}$*-curves &* **(V)**

(2) *Under* $(X, d, \mathfrak{m})$*: infin. Hilb. & Ass. 1,*
either **(iv)–(vi)** *is also equiv. to* **(i)–(iii)**

- **(iv)** *Space-time* W_2*-control*
- **(v)** *Bakry-Ledoux gradient estimate*
- **(vi)** *Bochner inequality*

5. Applications

Properties of $\mathbf{RCD}^*(K, N)$

- Stability under mGH (or Sturm's $\mathbb{D}$)-conv.
- Tensorization
- From local to global
- Measure contraction property $\mathbf{MCP}(K, N)$
  (via $\mathbf{CD}^*(K, N)$; [Cavalletti & Sturm])
  – (sharp) Bishop-Gromov volume comparison
  – volume doubling property
  – (local unif.) Poincaré ineq. [Rajala]
  $\Rightarrow$ $^\exists$heat kernel, two-sided Gaussian bound
  $\Rightarrow$ Ultracontractivity of P_t

Properties of $\mathsf{RCD}^*(K, N)$

- N-precision of f'nal ineq.'s
  - N-HWI ineq.
    $\Rightarrow$ N-log Sobolev ineq.
    $\Rightarrow$ N-Talagrand ineq.
- Lipschitz regularity of the heat kernel/eigenfn.'s
- Li-Yau's ineq. [Garofalo & Mondino]

Examples

- $-N \log \mathfrak{s}_{K/N}(x)$ or $-N \log \mathfrak{s}'_{K/N}(x)$ are (K, N)-convex fn's on $\mathbb{R}$ (or an interval)
- When $(X, d, e^{-V}\mathbf{vol})$: cpl. Riem. mfd, $\partial X = \emptyset$,
  RCD$^*(K, N) \Leftrightarrow$
  $$\mathrm{Ric} + \mathrm{Hess}\, V - \frac{1}{N-m}\nabla V^{\otimes 2} \geq K$$
  $$(m = \dim X \leq N)$$
- More generally,
  $(X, d, \mathfrak{m})$: **RCD**$^*(K', m)$
  & V: $(K'', N-m)$-convex on X
  $\Rightarrow (X, d, e^{-V}\mathfrak{m})$: **RCD**$^*(K' + K'', N)$

Examples

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- More generally,

  $(X,d,\mathfrak{m})$: RCD$^*(K',m)$

  & V: $(K'',N-m)$-convex on X

  $\Rightarrow (X,d,e^{-V}\mathfrak{m})$: RCD$^*(K'+K'',N)$