Entropic curvature-dimension condition and Bochner's inequality

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1. Introduction

Framework

$(X, d, \mathfrak{m})$: Polish geodesic metric measure sp.,

$\mathfrak{m}$: loc. finite, σ-finite, $\operatorname{supp}\mathfrak{m} = X$,

$P_t f = e^{t\Delta} f$: gradient curve in $L^2(\mathfrak{m})$ of

Cheeger's L^2-Dirichlet energy functional

$$\mathrm{Ch}(f) = \int |\nabla f|_w^2 \, d\mathfrak{m}$$

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$\star$ (X, d, m): infinitesimally Hilbertian

$\overset{\text{def}}{\Leftrightarrow}$ Ch: quadratic form

($\Rightarrow$ Ch: Dirichlet form & P_t, Δ: linear)

Purpose/History

Purpose

Unify the study of

$$\text{“}\mathbf{Ric} \geq \boldsymbol{K} \text{ and } \mathbf{dim} \leq \boldsymbol{N}\text{”}$$

in terms of optimal transportation / $\boldsymbol{P_t}$

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  (Riem. mfd.: [von Renesse & Sturm '05]
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- Study via optimal transport & Study via $\boldsymbol{P_t}$ } separated

Purpose/History

- via P_t (or Δ; Bochner inequality): [Bakry & Émery '84, Bakry & Ledoux '06]
- via Optimal transport: [Sturm '06, Lott & Villani, '09, Sturm & Bacher '10]

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Key "fact" for $\mathrm{Ric} \geq K$

$P_t\mu$: gradient curve of Ent on $(\mathcal{P}(X), W_2)$

$$\left(\mathrm{Ent}(\mu) := \int_X \rho \log \rho \, d\mathfrak{m} \quad (\mu = \rho \mathfrak{m}) \right)$$

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- via P_t (or Δ; Bochner inequality):
  [Bakry & Émery '84, Bakry & Ledoux '06]
- via Optimal transport: [Sturm '06,
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  in terms of the Rényi entropy (NOT Ent)

- Another approach via porous medium eq.
  (gradient curve of the Rényi ent.)
  [Ambrosio, Savaré & Mondino]

Goal

- Characterize "$\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$" by $\mathbf{Ent}$

- Find missing conditions

$$\text{connecting} \begin{cases} \text{optimal transport approach} \\ \qquad\quad \& \\ \boldsymbol{P_t} \text{ approach} \end{cases}$$

- Establish the equivalence

Goal

- Characterize "$\mathbf{Ric} \geq K$ & $\mathbf{dim} \leq N$" by $\mathbf{Ent}$

  $\Rightarrow$ (K, N)-convexity of $\mathbf{Ent}$

- Find missing conditions

  $$\text{connecting} \begin{cases} \text{optimal transport approach} \\ \qquad\qquad \& \\ P_t \text{ approach} \end{cases}$$

  $\Rightarrow$ (K, N)-evolution variational inequality

  Space-time W_2-control

- Establish the equivalence

Outline of the talk

1. **Introduction**

2. **Entropic curvature-dimension condition**

3. **Connection with Bakry-Émery theory**

4. **Applications**

1. Introduction

2. Entropic curvature-dimension condition

3. Connection with Bakry-Émery theory

4. Applications

Entropic curvature-dimension condition

(K, N)-convexity of Ent

$$\text{“}\mathrm{Hess}\,\mathrm{Ent} - \frac{1}{N}\nabla\,\mathrm{Ent}^{\otimes 2} \geq K\text{”}$$

$$\Updownarrow$$

$$\text{“}\mathrm{Hess}\,U_N \leq -\frac{K}{N}U_N\text{”},\; U_N := \exp\left(-\frac{1}{N}\,\mathrm{Ent}\right)$$

Entropic curvature-dimension condition

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Entropic curvature-dimension condition

(K, N)-convexity of Ent

$$\text{“}\mathrm{Hess}\, U_N \le -\frac{K}{N} U_N\text{”},\ U_N := \exp\left(-\frac{1}{N}\,\mathrm{Ent}\right)$$

Entropic curvature-dimension cond. $\mathsf{CD}^e(K, N)$:

$^\forall \mu_0, \mu_1 \in \mathcal{P}_2(M)$, $^\exists (\mu_t)_{t\in[0,1]}$: W_2-geod. s.t.

$$U_N(\mu_t) \ge \sigma_{K/N}^{(1-t)}(W_2(\mu_0, \mu_1)) U_N(\mu_0) + \sigma_{K/N}^{(t)}(W_2(\mu_0, \mu_1)) U_N(\mu_1),$$

$$\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}}, \quad \sigma_\kappa^{(s)}(r) := \frac{\mathfrak{s}_\kappa(sr)}{\mathfrak{s}_\kappa(r)}$$

EVI$_{K,N}$ and Riemannian CD$^e(K,N)$

A (strong) formulation of "$\partial_t \mu_t = -\nabla \operatorname{Ent}(\mu_t)$"

$\rightsquigarrow$ (K,N)-Evolution Variational Inequality of Ent:

$\mathsf{EVI}_{K,N}$ and Riemannian $\mathsf{CD}^e(K,N)$

A (strong) formulation of "$\partial_t \mu_t = -\nabla \operatorname{Ent}(\mu_t)$"

$\rightsquigarrow$ (K,N)-Evolution Variational Inequality of Ent:

$^\forall \mu_0$, $^\exists (\mu_t)_{t\ge 0}$: abs. conti. s.t. for $^\forall \nu$,

$$\frac{d}{dt}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 + K\,\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 \le \frac{N}{2}\left(1-\frac{U_N(\nu)}{U_N(\mu_t)}\right)$$

$$\left(\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}}, \quad U_N := \exp\left(-\frac{1}{N}\operatorname{Ent}\right)\right)$$

$\mathsf{EVI}_{K,N}$ and Riemannian $\mathsf{CD}^e(K,N)$

Theorem 1 ($\mathsf{RCD}^*(K,N)$ cond.)

For $K \in \mathbb{R}$ and $N > 0$, TFAE:

(i) $\mathsf{CD}^*(K,N)$ *& infinitesimally Hilbertian*

(ii) $\mathsf{CD}^e(K,N)$ *& infinitesimally Hilbertian*

(iii) ${}^\forall \mu_0$, ${}^\exists (\mu_t)_{t \ge 0}$: $\mathsf{EVI}_{K,N}$*-curve &* **(V)**

- ⋆ $\mathsf{CD}^*(K,N)$: reduced curvature-dimension cond.

  [Bacher & Sturm '10]

- ⋆ For Riem. mfds,

  (i)–(iii) $\Leftrightarrow \mathrm{Ric} \ge K$ and $\dim \le N$

$\mathsf{EVI}_{K,N}$ and Riemannian $\mathsf{CD}^e(K,N)$

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$$\textbf{(V)} \quad \int_X \exp\left(-{}^\exists c\, d(x_0,x)^2\right) \mathfrak{m}(dx) < \infty$$

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Space-time W_2-control

EVI$_{K,N}$-curve

$$\frac{d}{dt}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 + K\,\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 \leq \frac{N}{2}\left(1-\frac{U_N(\nu)}{U_N(\mu_t)}\right)$$

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$$\Downarrow$$

Space-time W_2-control:

$$\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu_s)}{2}\right)^2 \le e^{-K(s+t)}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_0,\nu_0)}{2}\right)^2 + \frac{N}{2}\frac{1-e^{-K(s+t)}}{K(s+t)}(\sqrt{t}-\sqrt{s})^2$$

Space-time W_2-control

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Space-time W_2-control (NOTE: $\mu_t = P_t\mu_0$):

$$\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu_s)}{2}\right)^2 \le e^{-K(s+t)}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_0,\nu_0)}{2}\right)^2 + \frac{N}{2}\frac{1-e^{-K(s+t)}}{K(s+t)}(\sqrt{t}-\sqrt{s})^2$$

Bakry-Ledoux gradient estimate

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$\Downarrow$ Derivative in space & time

Bakry-Ledoux gradient estimate

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$\Downarrow$ Derivative in space & time

Bakry-Ledoux gradient est.: For $f \in W^{1,2}$, $\mathfrak{m}$-a.e.,

$$|\nabla P_t f|_w^2 \le e^{-2Kt} P_t(|\nabla f|_w^2) - \frac{2tC(t)}{N}|\Delta P_t f|^2,$$

$$(C(t) = 1 + O(t)\ (t \to 0))$$

$\mathsf{RCD}^*(K, N) \Rightarrow$ Bakry-Ledoux

Theorem 2

For $K \in \mathbb{R}$ *&* $N > 0$, **(iii)** $\Rightarrow$ **(iv)** *&* **(v)**

(iii) ${}^\forall\mu_0$, ${}^\exists\mathsf{EVI}_{K,N}$*-curve* $(\mu_t)_t$ *&* **(V)**

(iv) *Space-time* W_2*-control*

(v) *Bakry-Ledoux gradient estimate*

Bakry-Ledoux $\Leftrightarrow$ Bochner

Theorem 3

Suppose $(M, d, \mathfrak{m})$*: infinitesimally Hilbertian*

For $K \in \mathbb{R}$ *&* $N > 0$, **(v) $\Leftrightarrow$ (vi)**

(v) $|\nabla P_t f|_w^2 \leq e^{-2Kt} P_t(|\nabla f|_w^2) - \frac{2C(t)}{N} |\Delta P_t f|^2$

(vi) $^\forall f \in W^{1,2}$ *with* $\Delta f \in W^{1,2}$ *&*

$g \in D(\Delta) \cap L^\infty$ *with* $g \geq 0$ *&* $\Delta g \in L^\infty$

$$\int_M \left(\frac{1}{2} \Delta g |\nabla f|_w^2 - g \langle \nabla f, \nabla \Delta f \rangle \right) d\mathfrak{m} \geq \int_M g \left(K |\nabla f|_w^2 + \frac{1}{N} |\Delta f|^2 \right) d\mathfrak{m}$$

Bakry-Ledoux $\Leftrightarrow$ Bochner

Theorem 3

Suppose $(M, d, \mathfrak{m})$: *infinitesimally Hilbertian*

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$$\geq \int_M g \left(K |\nabla f|_w^2 + \frac{1}{N} |\Delta f|^2 \right) d\mathfrak{m}$$

Summary

Theorem 4

$K \in \mathbb{R}$, $N > 0$

(1) *TFAE*

- **(i)** $\mathbf{CD}^*(K, N)$ *& infin. Hilb.*
- **(ii)** $\mathbf{CD}^e(K, N)$ *& infin. Hilb.*
- **(iii)** $^\exists\mathbf{EVI}_{K,N}$*-curves &* **(V)**

(2) *Under* $(M, d, \mathfrak{m})$*: infin. Hilb. & Ass. 1 below, either* **(iv)–(vi)** *is also equiv. to* **(i)–(iii)**

- **(iv)** *Space-time* W_2*-control*
- **(v)** *Bakry-Ledoux gradient estimate*
- **(vi)** *Bochner inequality*

Assumption 1

- The volume growth bound **(V)**
- $|\nabla f|_w \le 1$ $\mathfrak{m}$-a.e. $\Rightarrow f$: 1-Lip.

Remark

$(M, d, \mathfrak{m})$: infin. Hilb., Ass. 1
& $|\nabla P_t f|_w^2 \le e^{-2Kt} P_t(|\nabla f|_w^2)$
(Bakry-Émery's gradient estimate)

$\Rightarrow$ **CD**(K, ∞)

[Ambrosio, Gigli & Savaré]

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4. Applications

Properties of $\mathbf{RCD}^*(K, N)$

- Stability under mGH (or Sturm's $\mathbb{D}$)-conv.
- Tensorization
- From local to global
- Measure contraction property $\mathbf{MCP}(K, N)$
  (via $\mathbf{CD}^*(K, N)$; [Cavalletti & Sturm '12])
  – (sharp) Bishop-Gromov volume comparison
  – volume doubling property
  – (local unif.) Poincaré ineq. [Rajala '11]
  $\Rightarrow$ $^{\exists}$heat kernel, two-sided Gaussian bound
  $\Rightarrow$ Ultracontractivity of P_t

Properties of $\mathsf{RCD}^*(K, N)$

- N-precision of f'nal ineq.'s
  - N-HWI ineq.
    $\Rightarrow$ N-log Sobolev ineq.
    $\Rightarrow$ N-Talagrand ineq.
- Lipschitz regularity of the heat kernel/eigenfn.'s
- Lichnerowicz bound of λ_1 [Ketterer]
- Li-Yau's ineq. [Garofalo & Mondino]

Examples

- $N \log \mathfrak{s}_{K/N}(x)$ or $N \log \mathfrak{s}'_{K/N}(x)$
  are (K, N)-convex fn's on $\mathbb{R}$ (or an interval)
- When $(M, d, e^{-V}\mathbf{vol})$: cpl. Riem. mfd, $\partial M = \emptyset$,
  $\mathsf{RCD}^*(K, N) \Leftrightarrow$
  $$\mathrm{Ric} + \mathrm{Hess}\, V - \frac{1}{N - m} \nabla V^{\otimes 2} \geq K$$
  $$(m = \dim M \leq N)$$
- More generally,
  $(M, d, \mathfrak{m})$: $\mathsf{RCD}^*(K, N)$
  & V: (K', N')-convex on M
  $\Rightarrow (M, d, e^{-V}\mathfrak{m})$: $\mathsf{RCD}^*(K + K', N + N')$