Time inhomogeneous couplings of diffusion processes on Riemannian manifolds

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partially joint work with R. Philipowski (Univ. Luxemburg)
and joint work with T. Amaba (Ritsumeikan Univ.)

Workshop on geometric aspects in probability and analysis
(Univ. Jena) Sep. 14, 2013

1. Introduction

Coupling by parallel transport on $\mathbb{R}^m$

$B_0(t)$: a BM, $B_1(t) := (B_1(0) - B_0(0)) + B_0(t)$

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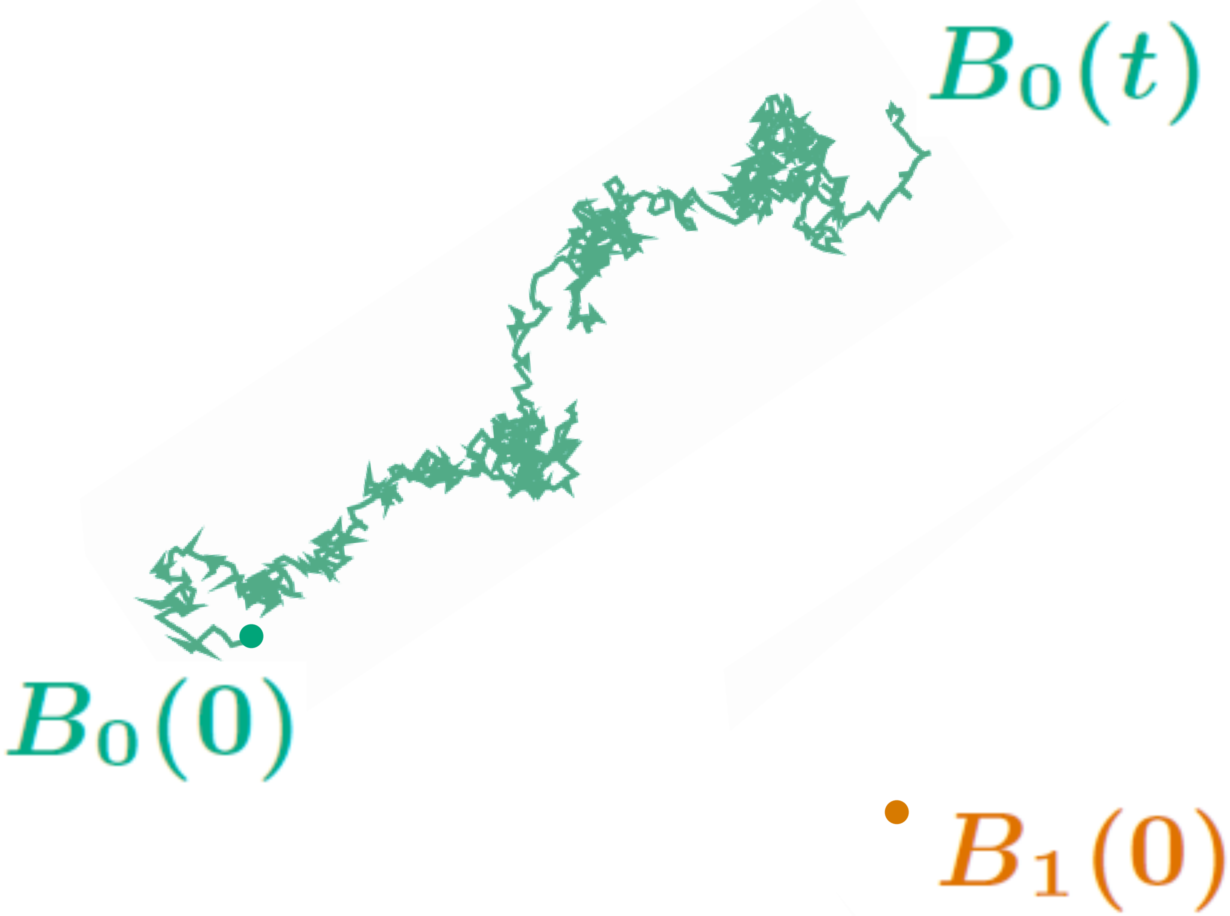

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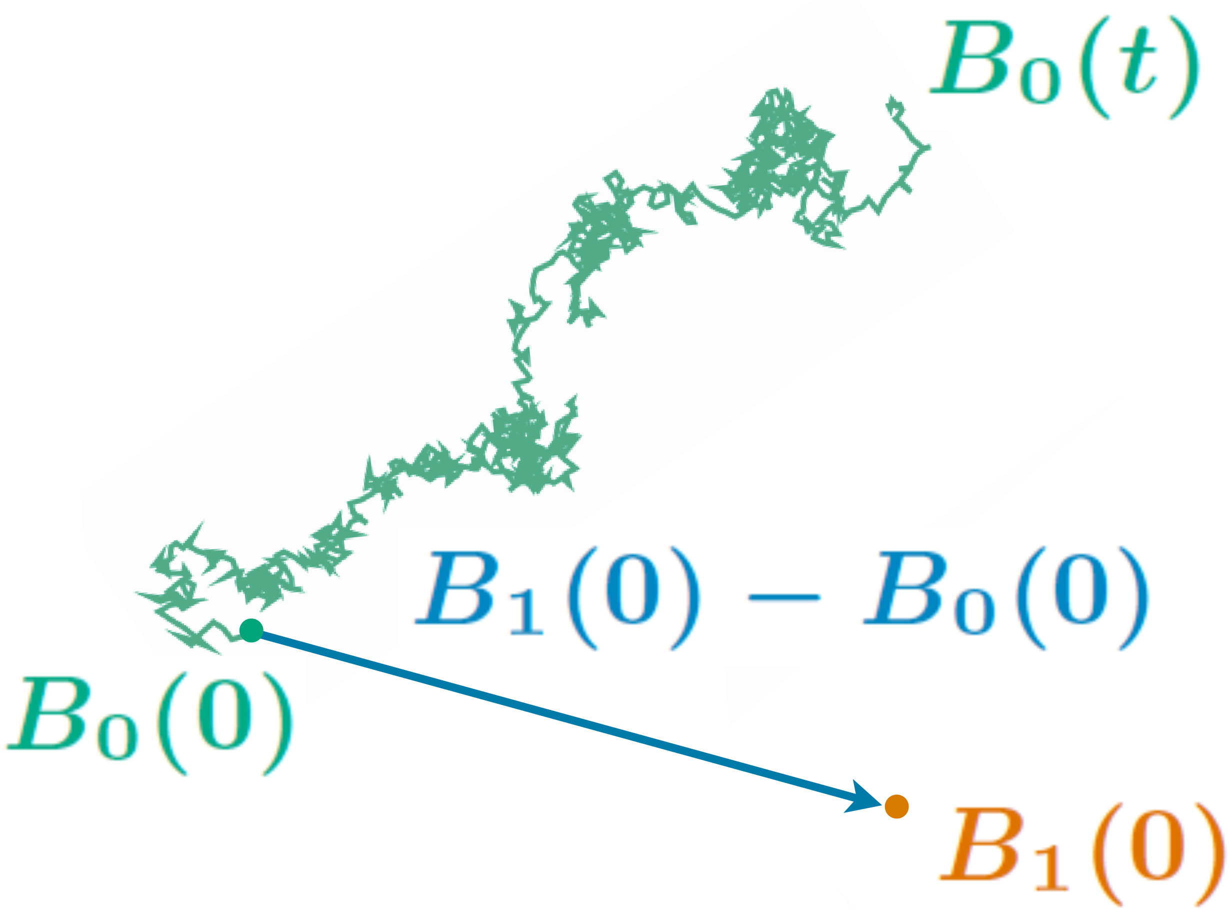

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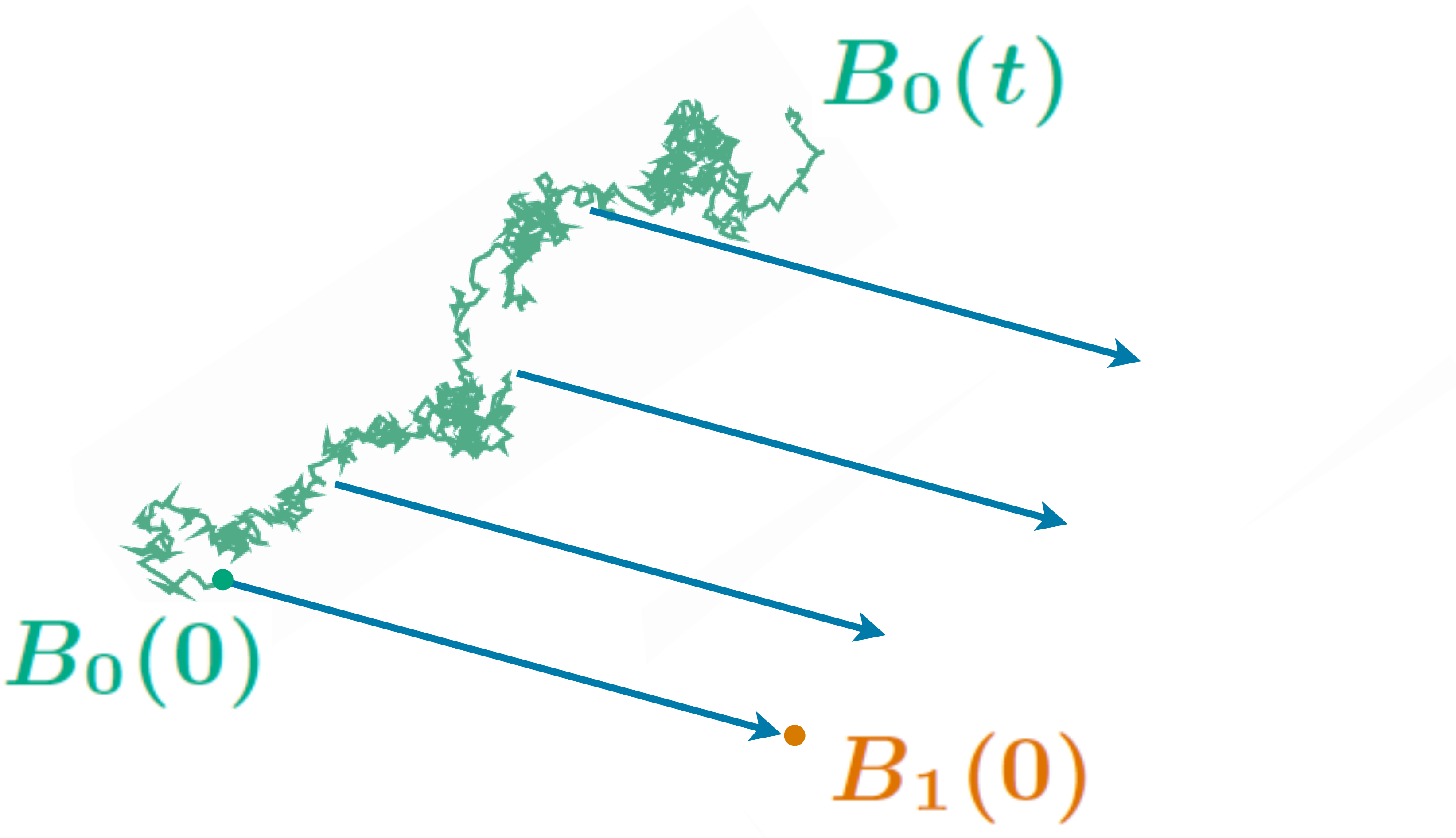

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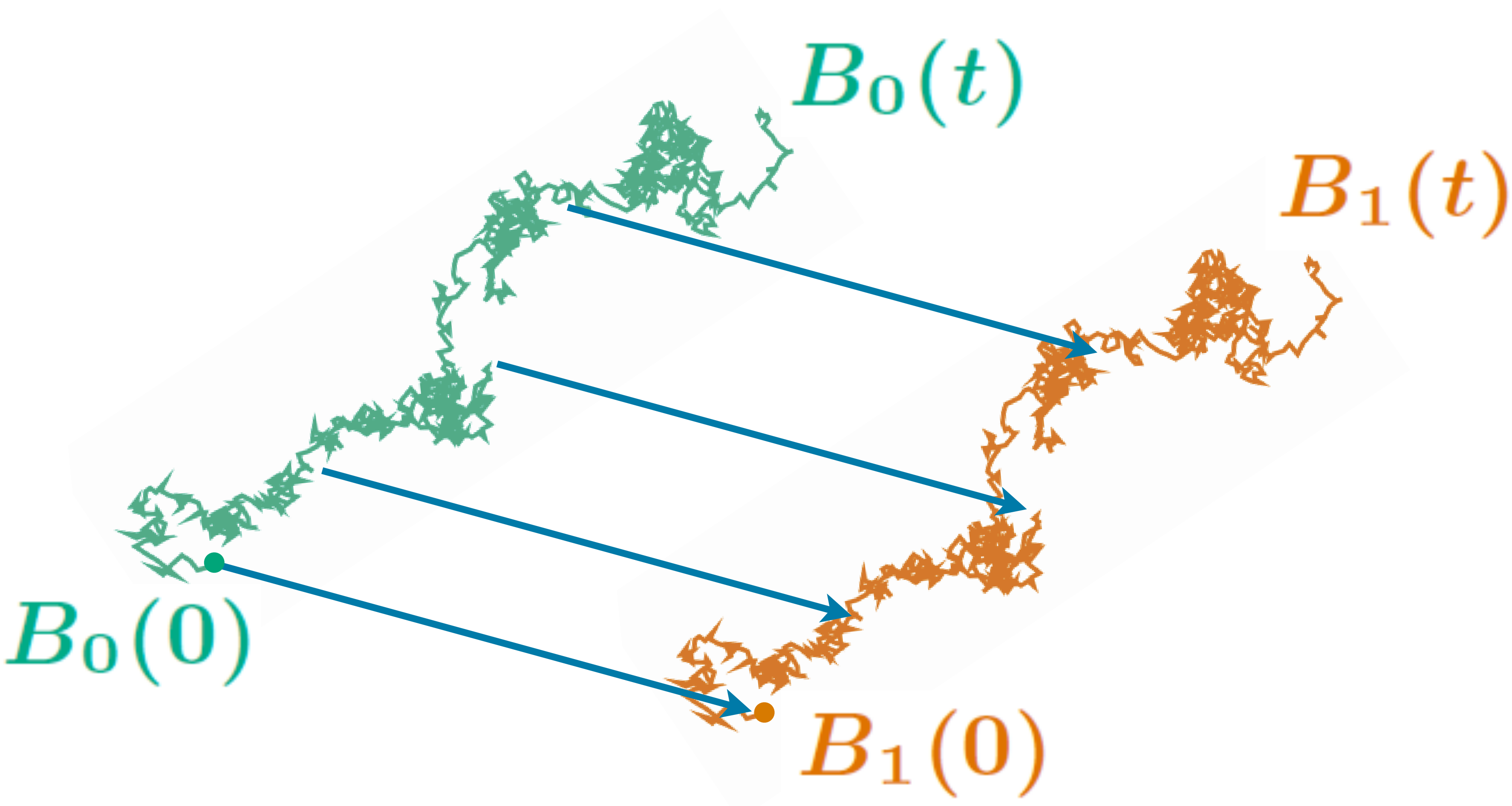

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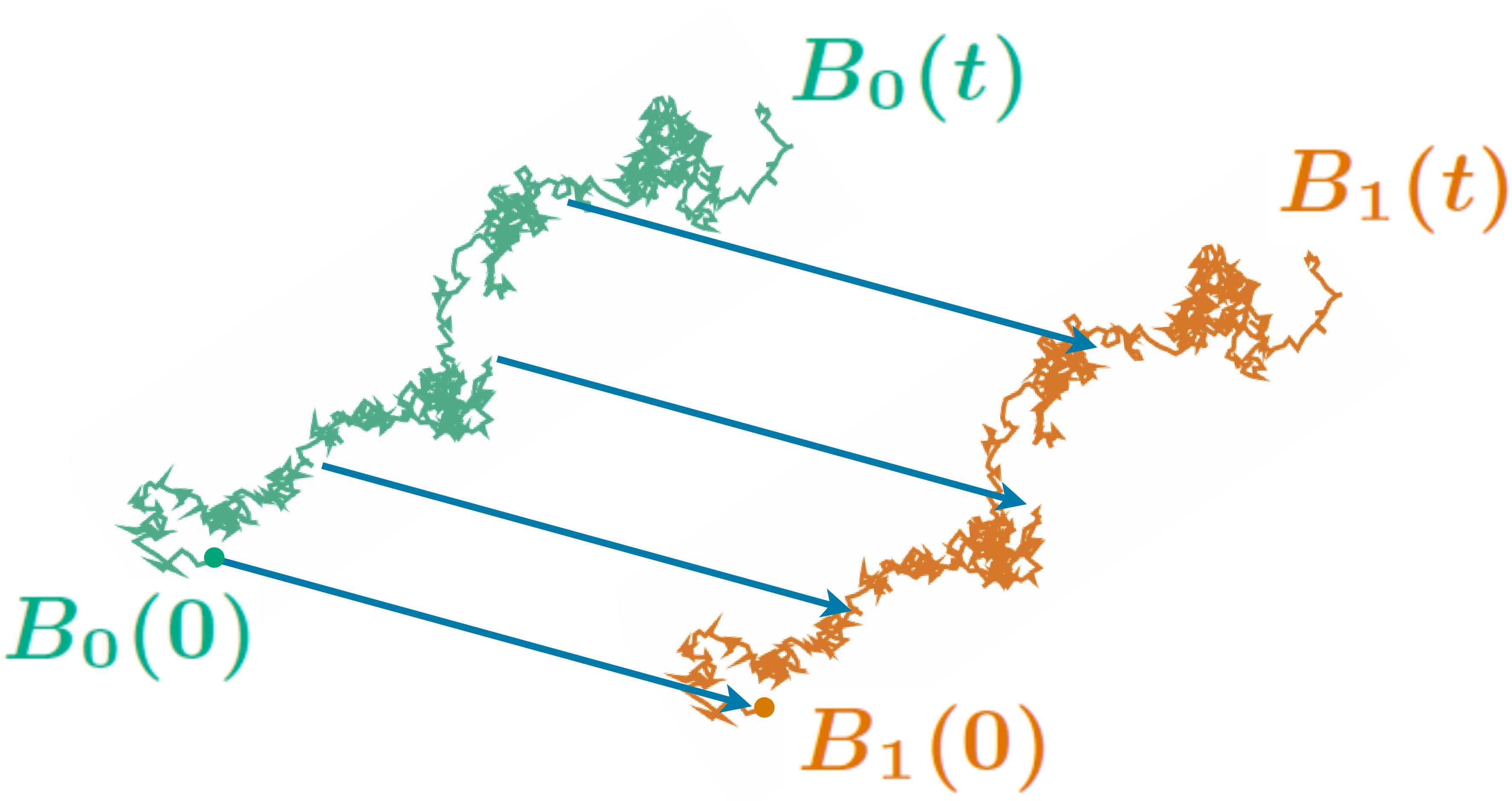

$\Rightarrow |B_0(t) - B_1(t)| = |B_0(0) - B_1(0)| \quad (^\forall t)$

Coupling by parallel transport on $\mathbb{R}^m$

$$|B_0(t) - B_1(t)| = |B_0(0) - B_1(0)|$$
$$\Downarrow$$
$$W_p(P_t^* \mu_0, P_t^* \mu_1) \le W_p(\mu_0, \mu_1),$$

† P_t: transition semigroup

† $W_p(\mu_0, \mu_1) := \inf_\pi \left\{ \|d\|_{L^p(\pi)} \,\middle|\, \begin{array}{l} (p_0)_\sharp \pi = \mu_0 \\ (p_1)_\sharp \pi = \mu_1 \end{array} \right\}$

$\mu_i \in \mathcal{P}(\mathbb{R}^m)$, $1 \le p \le \infty$

(Wasserstein distance)

Question

Estimate for $\boldsymbol{P}_{t_0}^* \boldsymbol{\mu}_0$ & $\boldsymbol{P}_{t_1}^* \boldsymbol{\mu}_1$ with $\boldsymbol{t}_0 \neq \boldsymbol{t}_1$?

Background

- On Riem. mfd., for $K \in \mathbb{R}$,

$$W_p(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_p(\mu_0, \mu_1) \quad ({}^\forall t)$$

$$\Updownarrow$$

$$\mathrm{Ric} \geq K$$

[von Renesse & Sturm '05]

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[von Renesse & Sturm '05]

$\Rightarrow$ Equivalence in metric measure spaces

[Ambrosio, Gigli & Savaré et. al.]

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[von Renesse & Sturm '05]

$\Rightarrow$ Equivalence in metric measure spaces

[Ambrosio, Gigli & Savaré et. al.]

- An estimate of $W_2(P_{t_0}^* \mu_0, P_{t_1}^* \mu_1)$

  $\Leftrightarrow$ "$\mathrm{Ric} \ge K$ & $\dim \le N$"

(even in met. meas. sp.) [K. / Erbar, K. & Sturm]

Background

Another situation: time-inhomogeneous processes

(Ex. BM on Riem. mfd. w.r.t. time-dependent metric)

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On (backward) Ricci flow,

Analoguous estimates for distance-like functions

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Monotonicity formulae along heat distribution
[Topping '09 / Lott '09] (via optimal transport)

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Monotonicity formulae along heat distribution

[Topping '09 / Lott '09] (via optimal transport)

How about couplings of $P^*_{t_0 \to t_0'} \mu_0$ & $P^*_{t_1 \to t_1'} \mu_1$?

(non-trivial even if $t_0' - t_0 = t_1' - t_1$)

Aim

A probabilistic approach to these problems
via coupling method
(generalizations of coupling by parallel transport)

Outline of the talk

1. Introduction

2. Space-time W_p-control

2.1 Framework and main result

2.2 Outline of the proof

2.3 Estimates involving comparison functions

3. Couplings on backward Ricci flow

3.1 $\mathcal{L}$-coupling

3.2 $\mathcal{L}_0$-coupling

Framework

† M: cpl. Riem. mfd., $\dim M = m$, $\partial M = \emptyset$

† Z: C^1-vector field on M

† $(X(t), \mathbb{P}_x)$: diffusion process $\leftrightsquigarrow$ $\Delta + Z$

† $P_t^* \mu := \int_M \mathbb{P}_x \circ X(t)^{-1} \mu(dx)$

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Bakry-Émery Ricci tensor

$$\mathrm{Ric}^{Z,N} := \mathrm{Ric} - (\nabla Z)^{\mathsf{sym}} - \frac{1}{N - m} Z \otimes Z$$

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Assumption 1

$\mathrm{Ric}^{Z,N} \geq K$ for some $K \in \mathbb{R}$ & $N \in [m, \infty]$

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⋆ Ass. 1 $\Rightarrow \mathbb{P}_x[X(t) \in M] = 1$

Theorem 1 ([K.])

Under Ass. 1, for $p \in [2, \infty)$,

$t_1 > t_0 > 0$ & $\mu_0, \mu_1 \in \mathcal{P}(M)$,

$$W_p(P_{t_0}^* \mu_0, P_{t_1}^* \mu_1)^2 \le \left(\fint_{t_0}^{t_1} e^{Kr} J(dr) \right)^{-2} W_p(\mu_0, \mu_1)^2 + \frac{N + p - 2}{2} J([t_0, t_1])^2, \qquad (*)$$

where $J(A) := \int_A \sqrt{\frac{2K}{e^{2Kr} - 1}} dr$

$\star$ $J([t_0, t_1])^2 = 4(\sqrt{t_1} - \sqrt{t_0})^2$ when $K = 0$

Connection with Bakry-Émery theory

$$(*) \quad W_p(P_t^* \mu_0, P_t^* \mu_1)^2 \leq e^{-2Kt} W_p(\mu_0, \mu_1)^2$$

$$(t_0 = t_1 = t)$$

$\Updownarrow$ [K. '10, ...]

$$|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^{p_*})^{\frac{2}{p_*}}$$

$\Updownarrow$ [Bakry & Émery '84] ($p_* = 2$)

$$\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K |\nabla f|^2$$

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$\Uparrow$ [Bakry & Émery '84 / Savaré] ($p_* = 1$)

$$\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f\rangle \ge K|\nabla f|^2$$

Connection with Bakry-Émery theory

$$(*)\ \ W_p(P^*_{t_0}\mu_0, P^*_{t_1}\mu_1)^2 \le A(t_0,t_1)^2 W_p(\mu_0,\mu_1)^2 + B(t_0,t_1)$$

$\Updownarrow$ [K.]

$$|\nabla P_t f|^2 \le e^{-2Kt} P_t(|\nabla f|^{p_*})^{\frac{2}{p_*}} - \frac{1-e^{-2Kt}}{(N+p-2)K}|\Delta P_t f|^2$$

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Future questions

- Applications of L^p-estimate?
- Sharpness of the estimate?
- Validity on non-smooth spaces?

Coupling by parallel transport

$(\boldsymbol{X}_0(t), \boldsymbol{X}_1(t))$: coupling of BMs moving parallely

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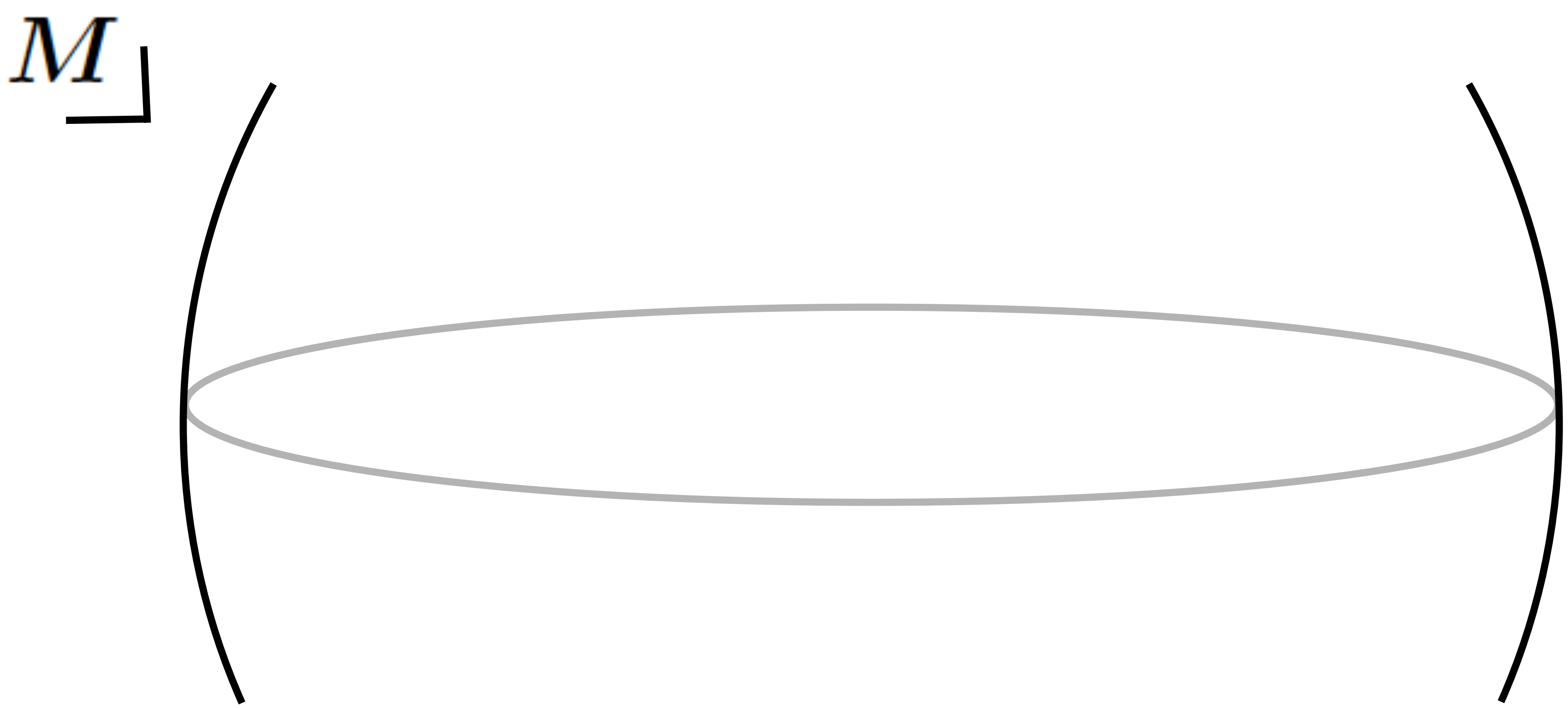

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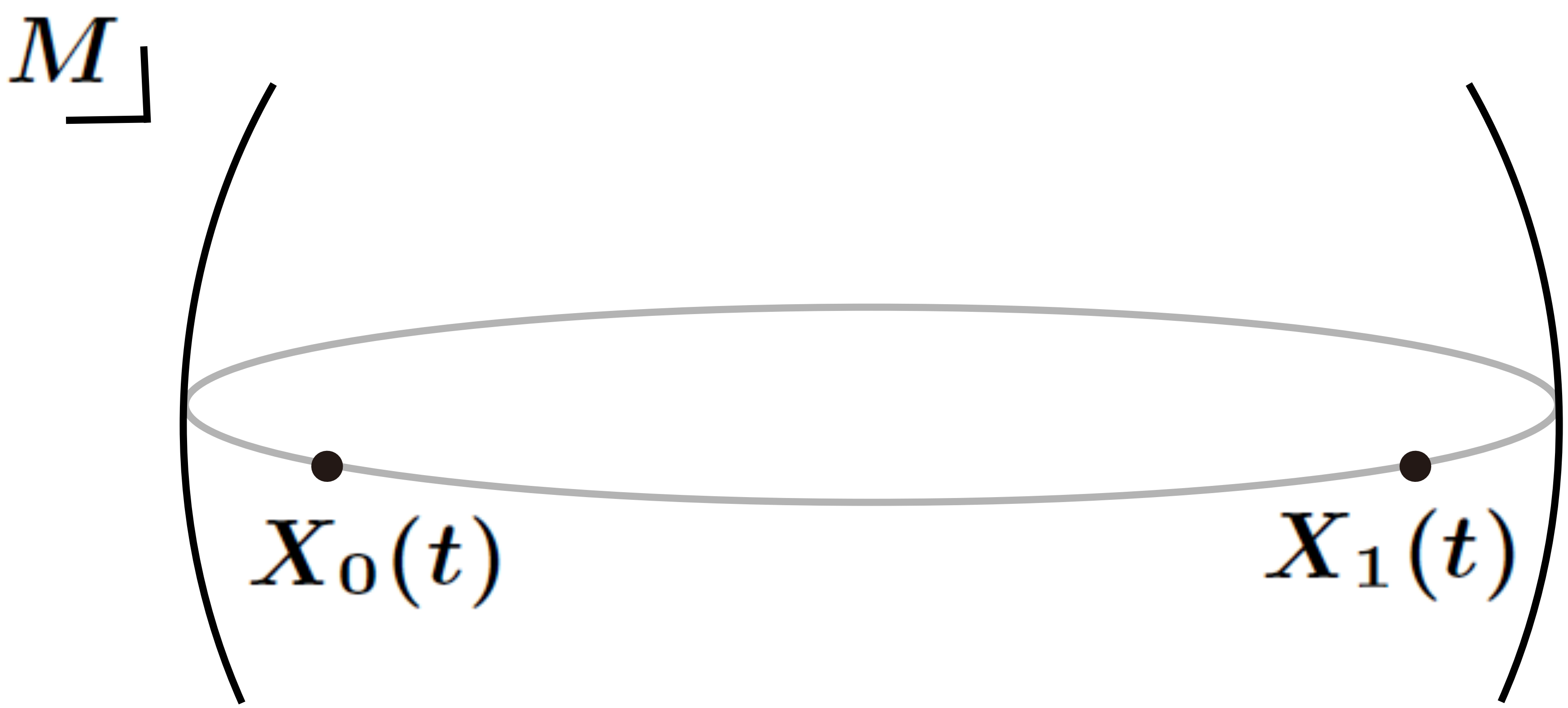

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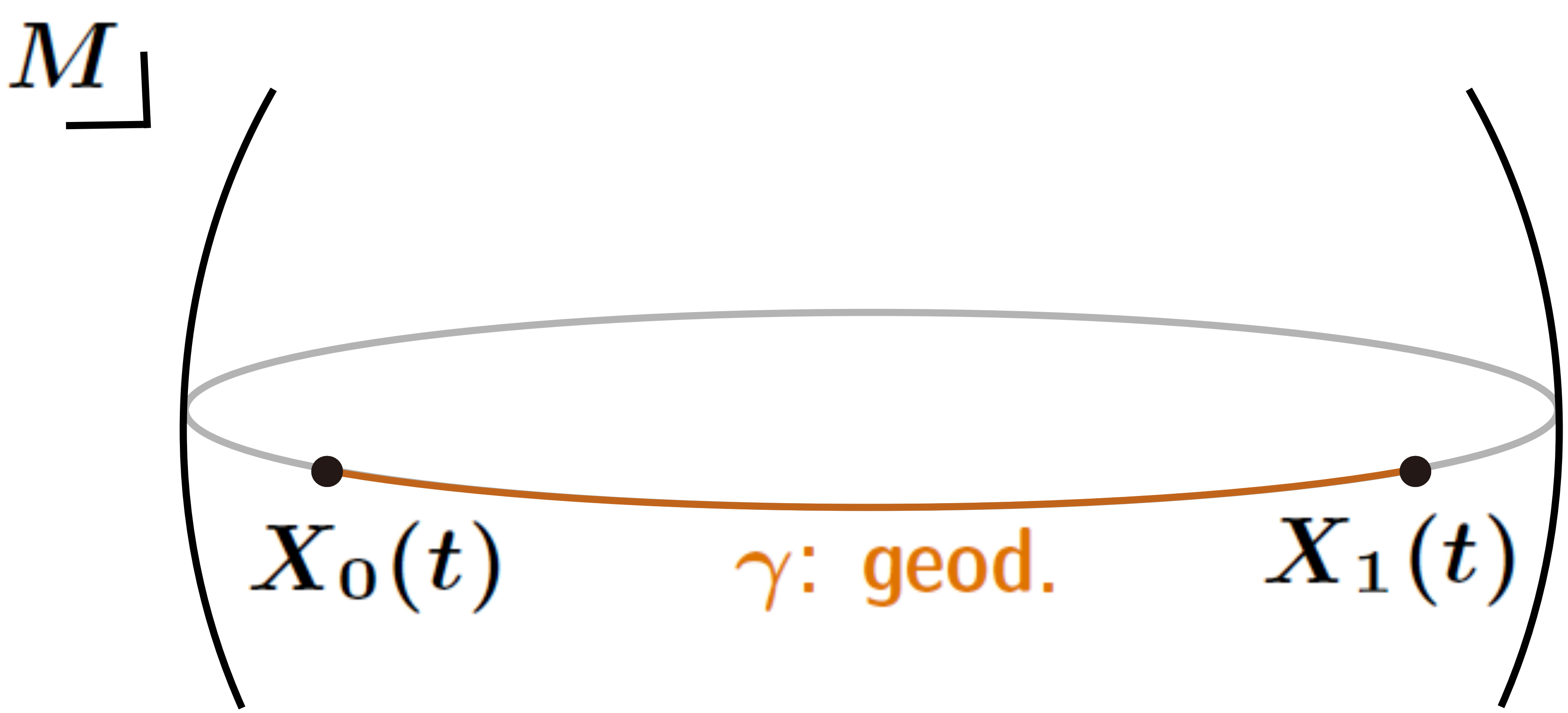

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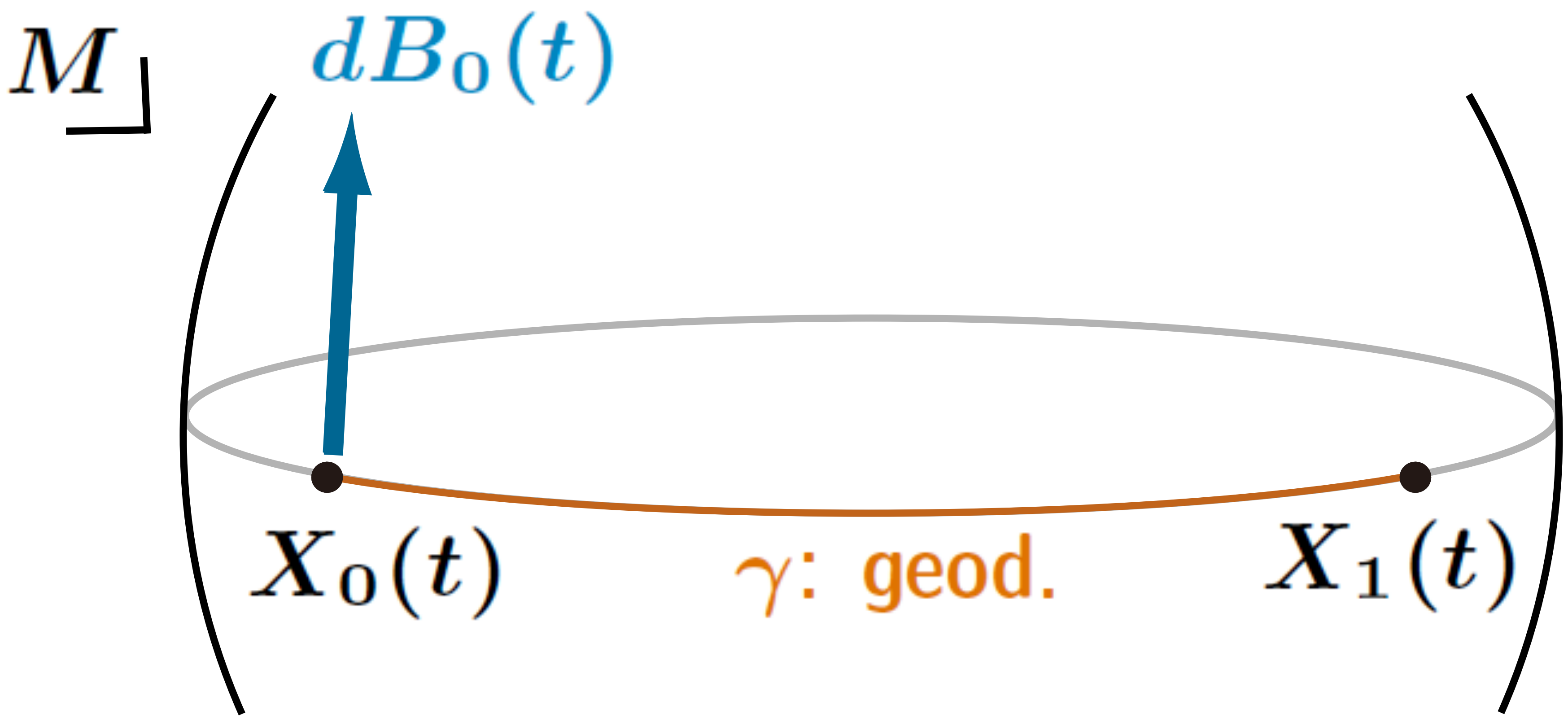

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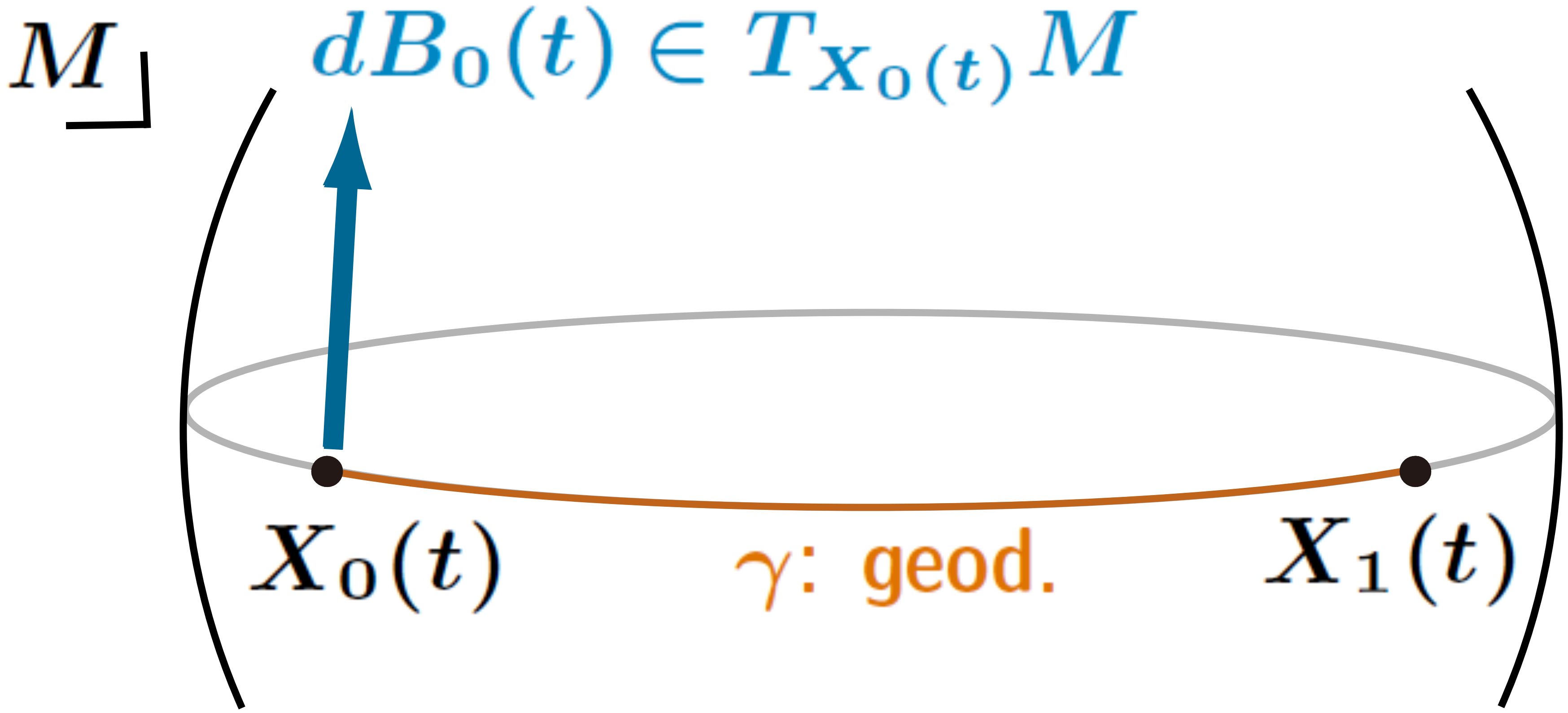

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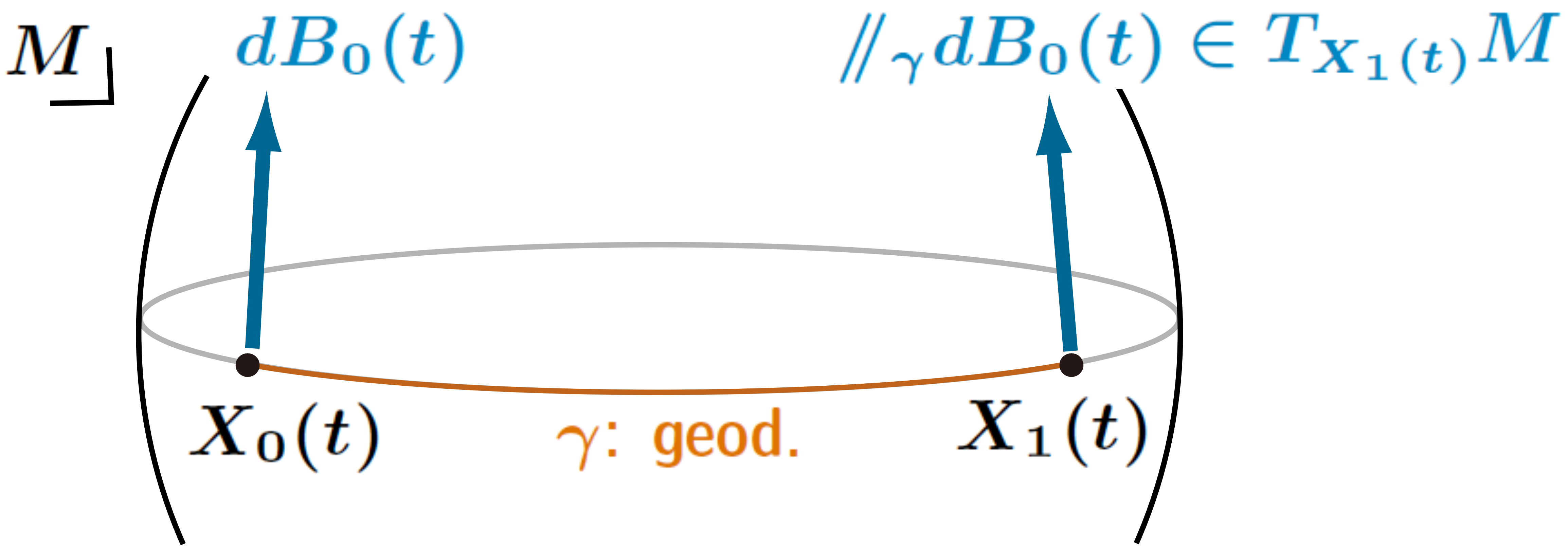

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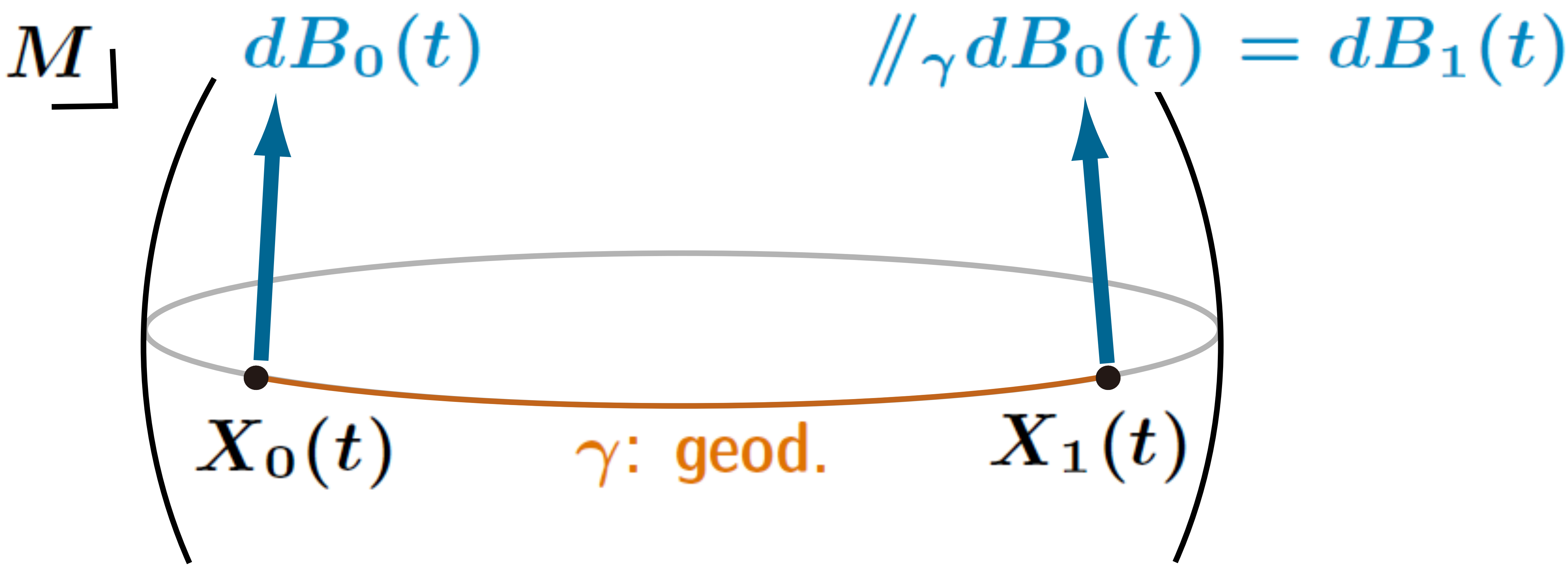

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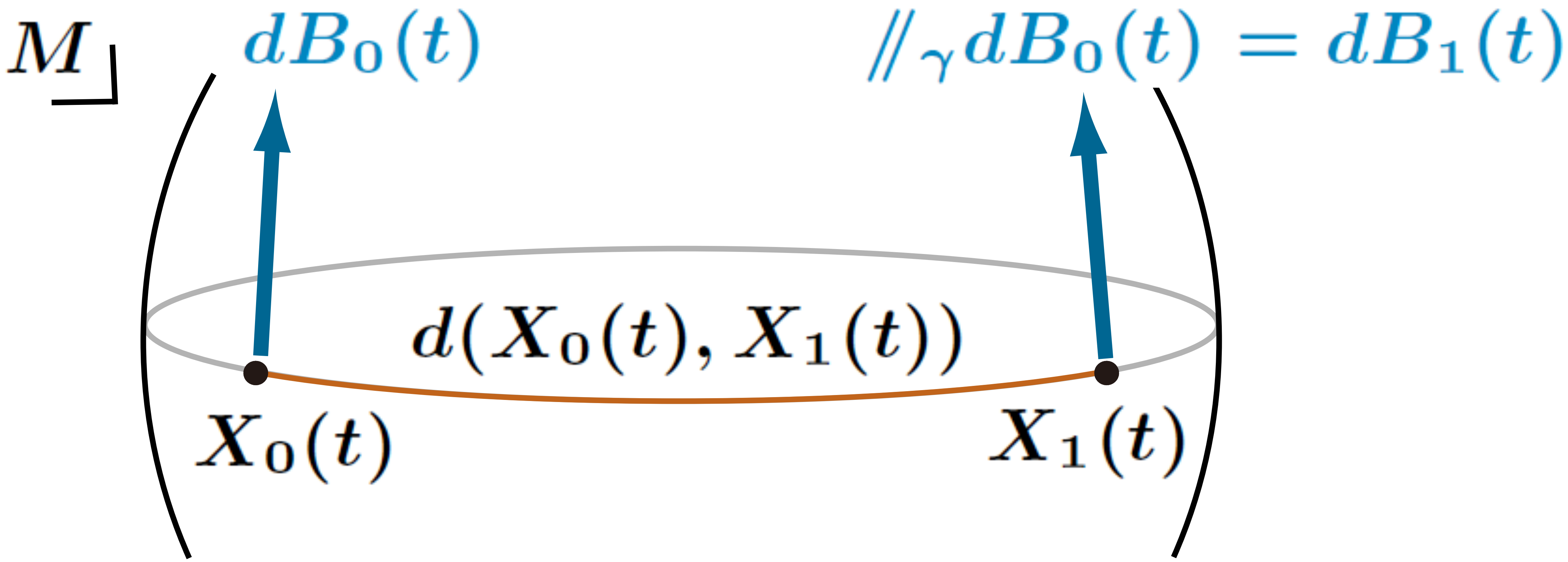

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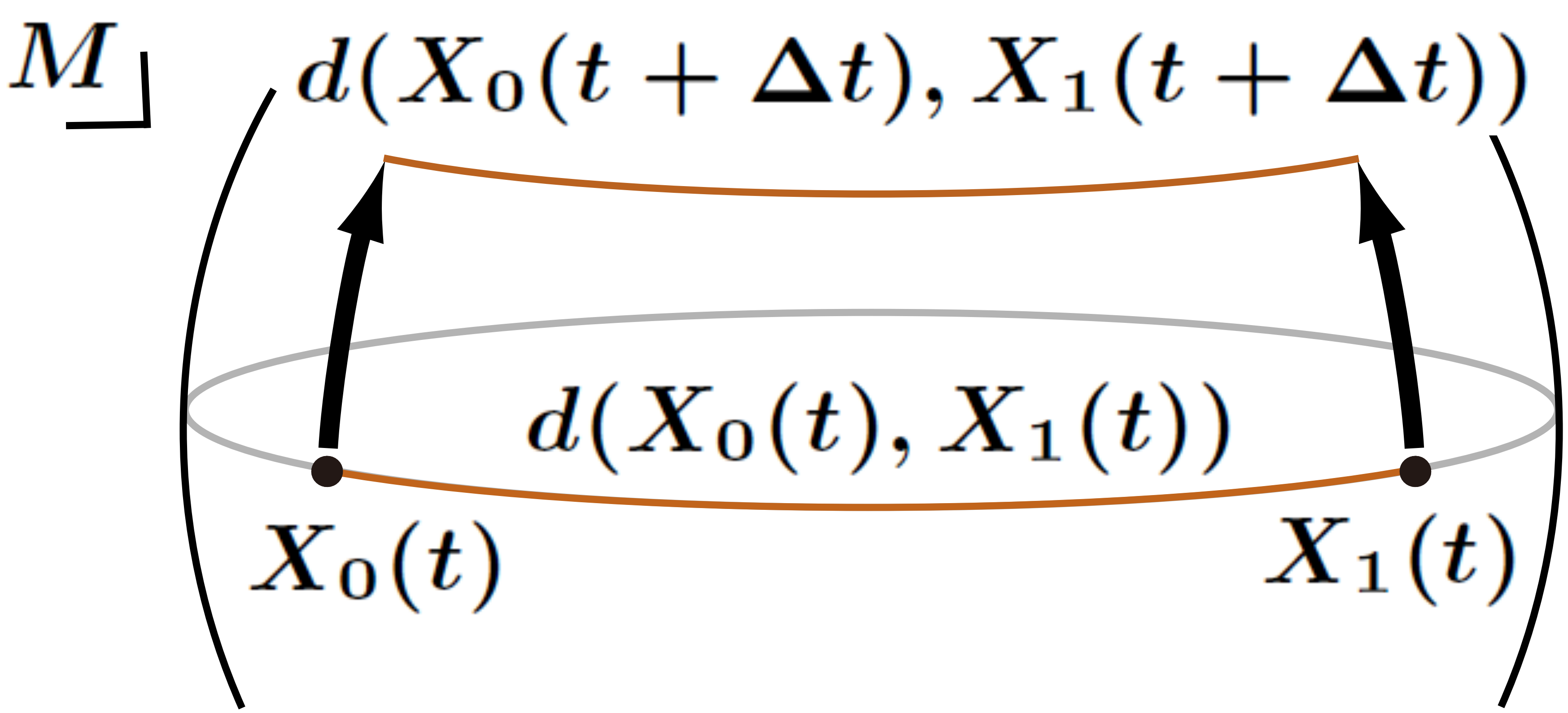

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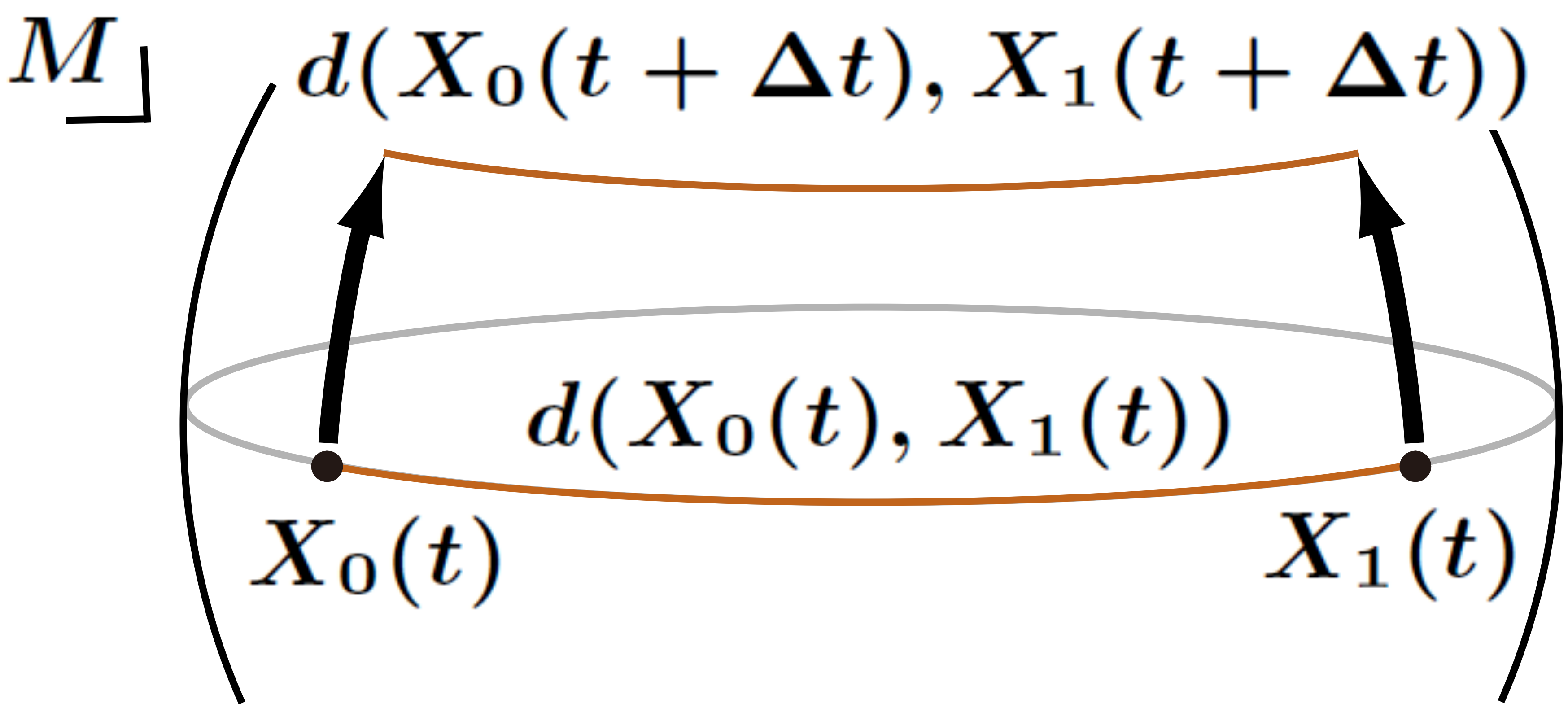

- [mart. part of $d(X_0(t), X_1(t))$] $= 0$

Coupling by parallel transport

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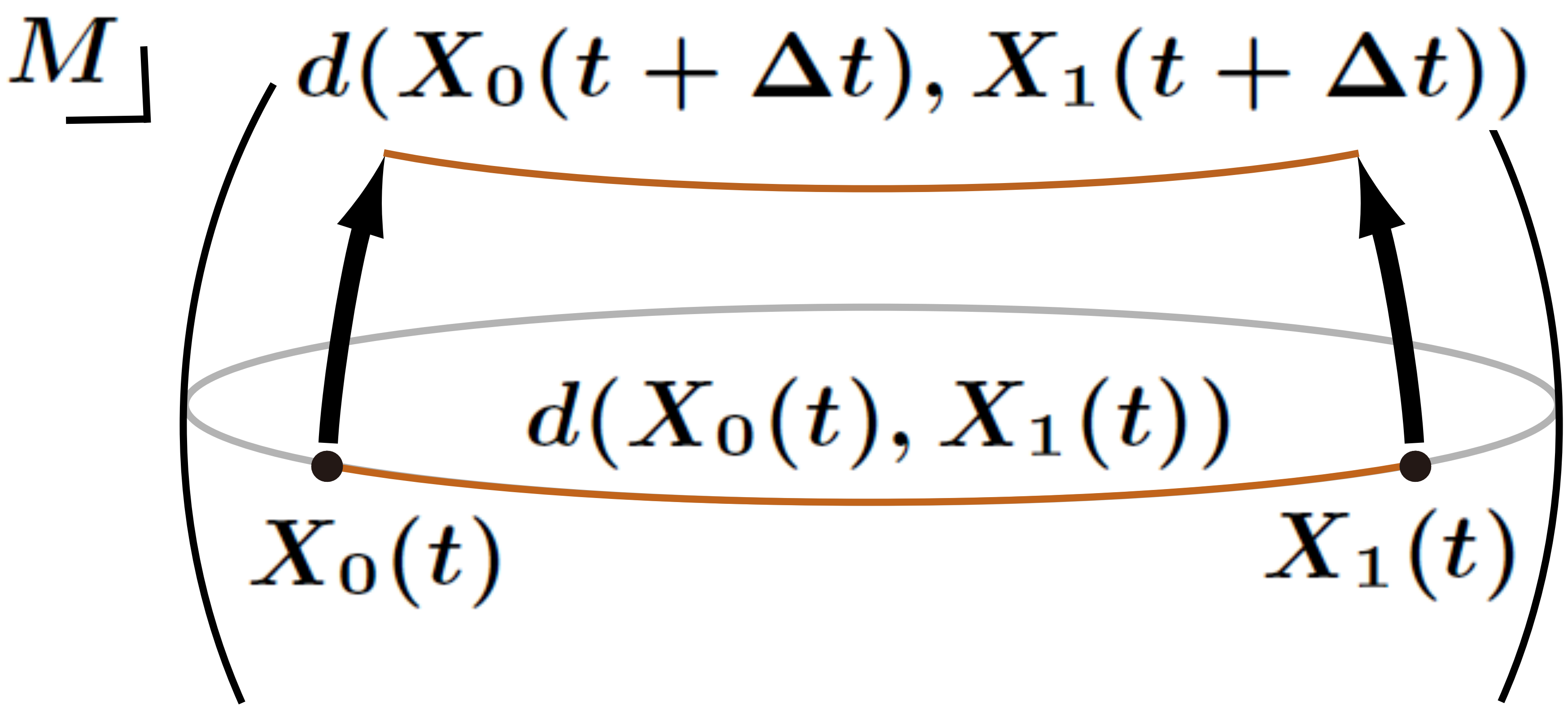

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- ["bdd. var." part]: Controlled by $\mathrm{Ric} \geq K$

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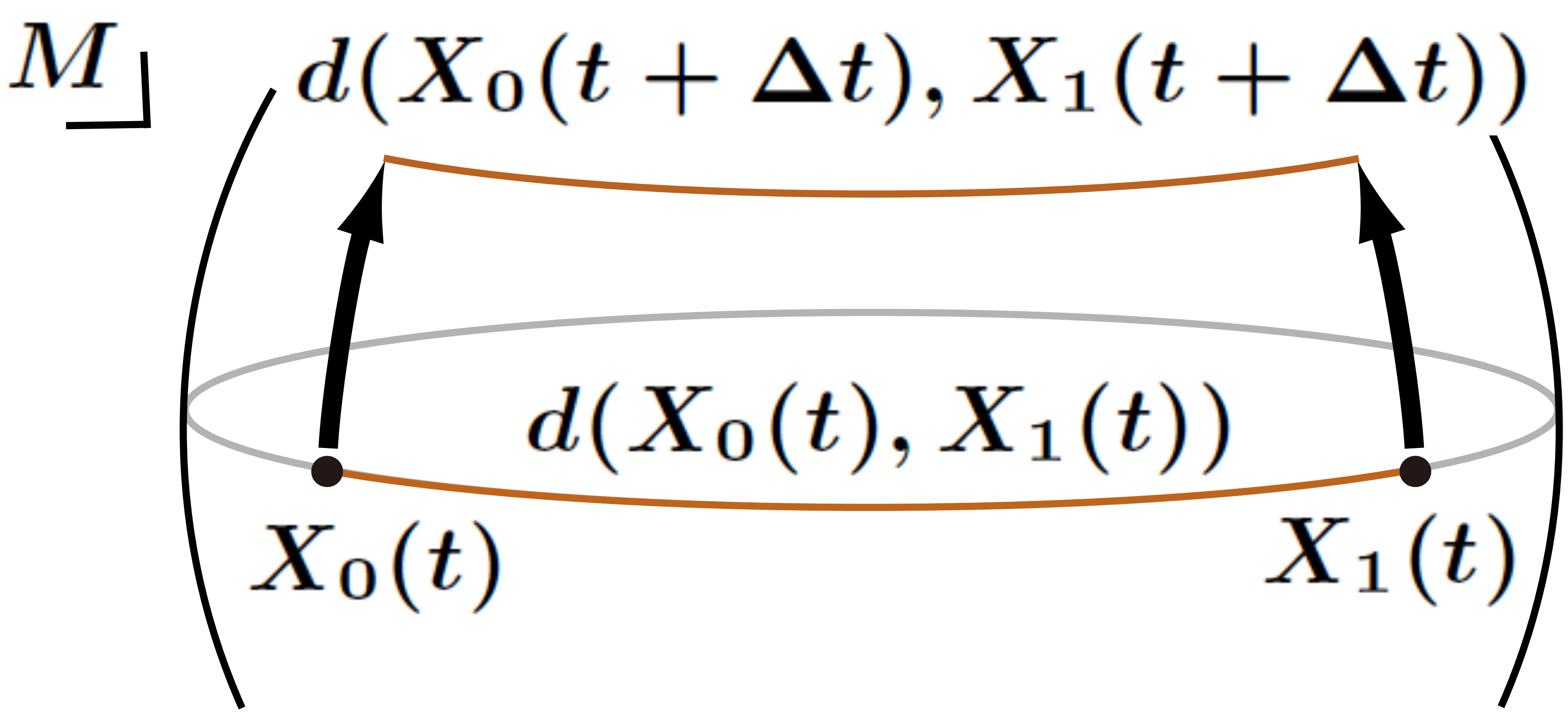

- [mart. part of $d(X_0(t), X_1(t))$] $= 0$
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$$\Downarrow$$

$$\text{“}\frac{\partial}{\partial t} d(X_0(t), X_1(t)) \leq -K d(X_0(t), X_1(t))\text{”}$$

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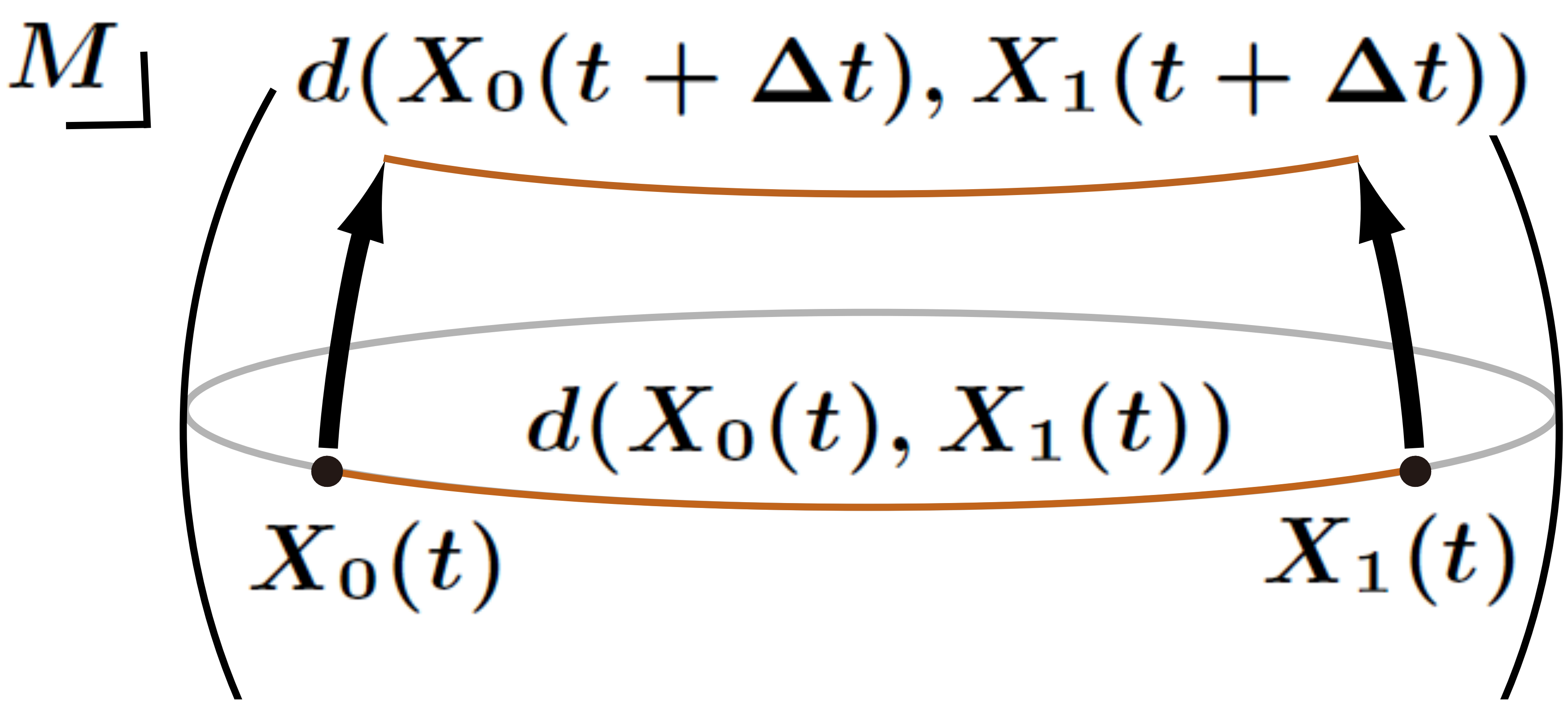

$\therefore \mathrm{Ric} \geq K$

$$\Rightarrow W_p(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_p(\mu_0, \mu_1)$$

$$(1 \leq {}^\forall p \leq \infty)$$

Different speed case

$(\boldsymbol{X}_0(\alpha_0 t), \boldsymbol{X}_1(\alpha_1 t))$: coupling of BMs

Driving noise $\sqrt{\alpha_1}d\boldsymbol{B}_1(t)$ of $\boldsymbol{X}_1(t)$
$=$ parallel transport of $\sqrt{\alpha_0}d\boldsymbol{B}_0(t)$ & scaling

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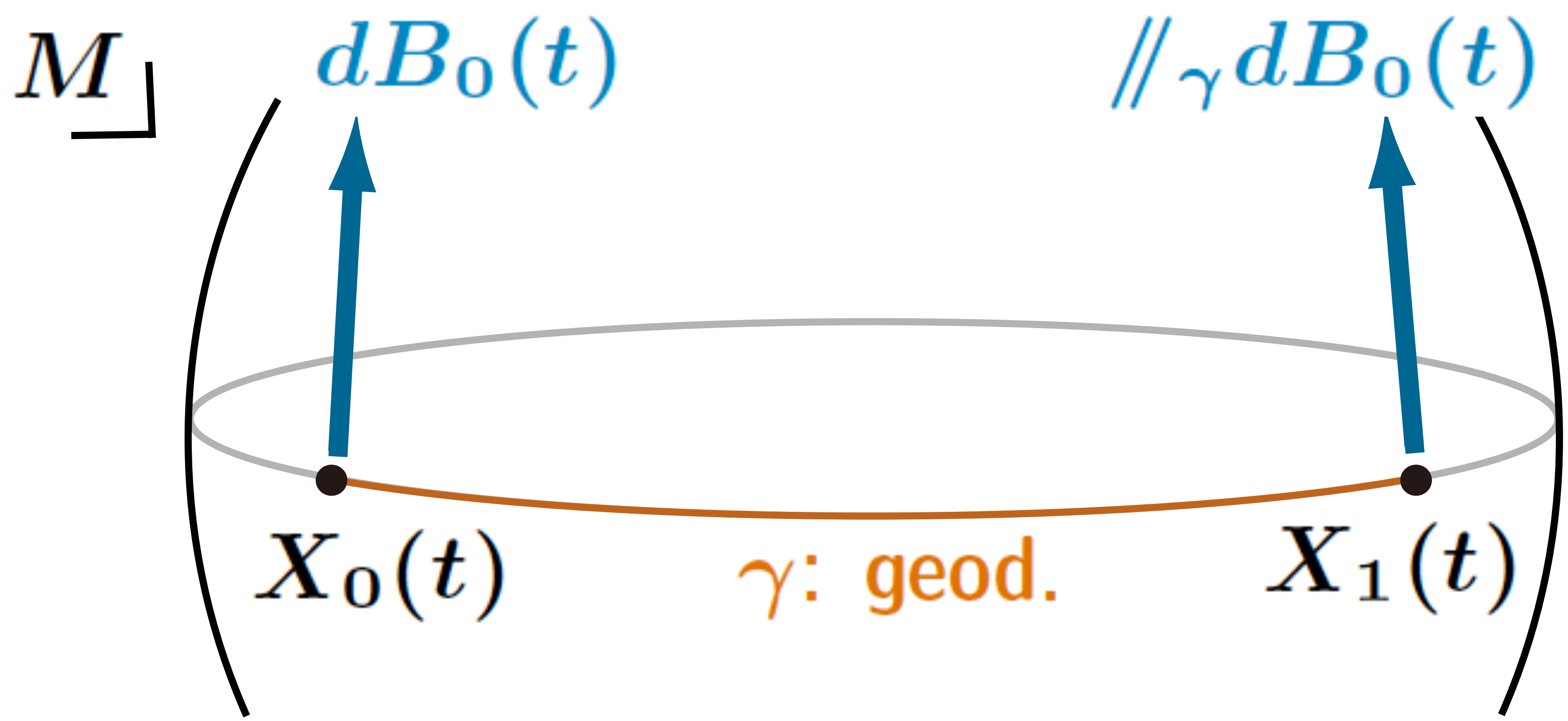

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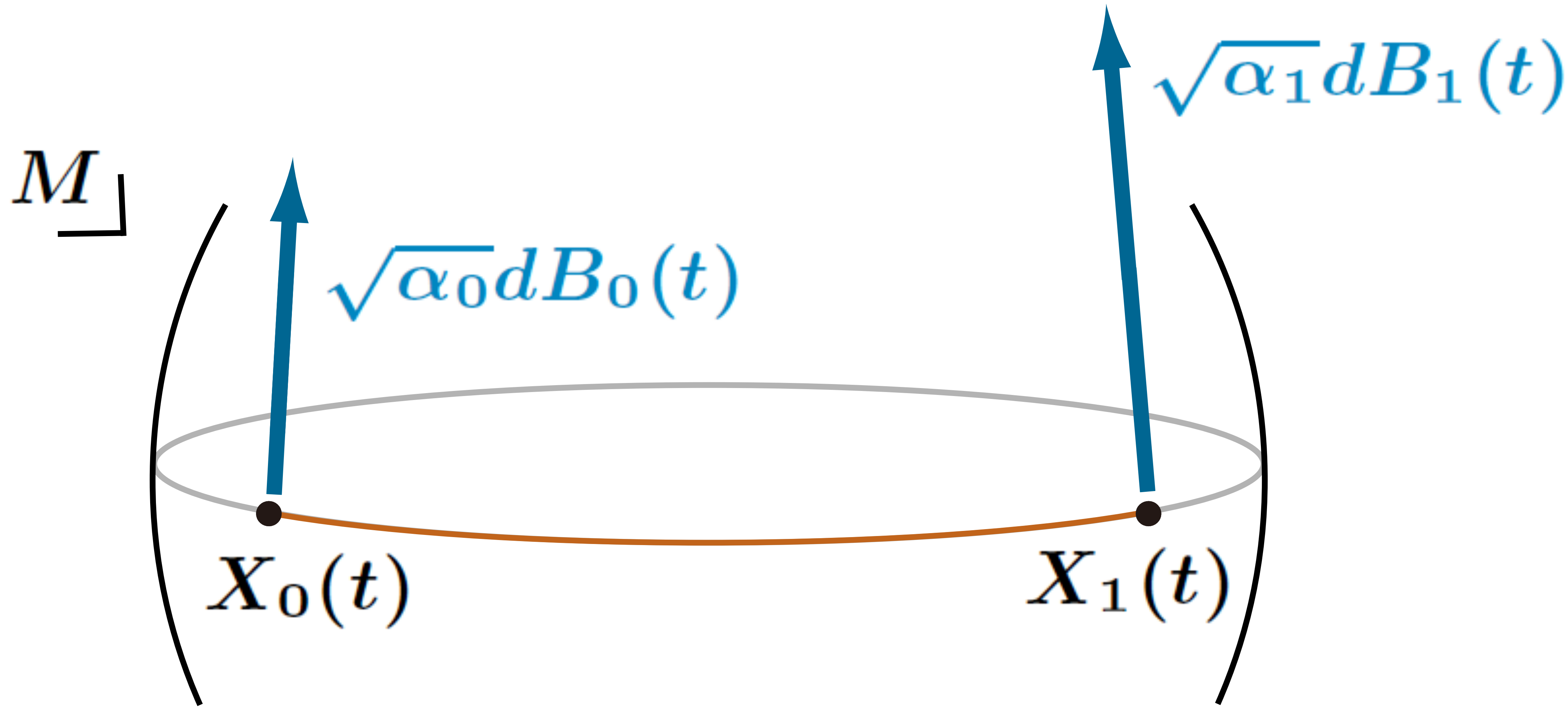

Different speed case

Proposition 2

$^\exists X_i(r) \overset{d}{=} X(t_i r)$ $(i = 0, 1)$: *coupling s.t.*

$$\mathbb{E}[d(X_0(1), X_1(1))^p]^{2/p}$$
$$\leq e^{-2Kt_*}\mathbb{E}[d(X_0(0), X_1(0))^p]^{2/p}$$
$$+ \frac{(N + p - 2)(1 - e^{-2Kt_*})}{Kt_*}(\sqrt{t_1} - \sqrt{t_0})^2,$$

where $t_* := \begin{cases} \sqrt{t_0 t_1} & (K \geq 0), \\ \dfrac{t_0 + t_1}{2} & (K < 0). \end{cases}$

Different speed case

$t_0 < t_1$ fixed
$(t_r)_{r \in [0,\ell]}$: interpolation

$(\mu_r)_{r \in [0,1]}$: W_p-geod. in $\mathcal{P}(M)$
$\eta : [0, \ell] \longrightarrow [0, 1]$: $\nearrow$, surj.

$(X_r(s), X_{r'}(s))_{t \in [0,1]}$: coupling by parallel transport of $(X(t_r s), \mathbb{P}_{\mu_{\eta(r)}})$ and $(X(t_{r'} s), \mathbb{P}_{\mu_{\eta(r')}})$

Different speed case

$\star\ W_2(P^*_{t_r}\mu_r, P^*_{t_{r'}}\mu_{r'})^2 \le \mathbb{E}\left[d(X_r(1), X_{r'}(1))^2\right]$

$\Rightarrow |\dot{P^*_{t_r}\mu_r}|_{W_2} \le \cdots,$

where $|\dot{P^*_{t_r}\mu_r}|_{W_2} = \overline{\lim_{r'\downarrow r}} \frac{W_2(P^*_{t_r}\mu_r, P^*_{t_{r'}}\mu_{r'})}{r'-r}$

$\star\ W_2(P^*_{t_0}\mu_0, P^*_{t_1}\mu_1)^2 \le \int_0^\ell |\dot{P^*_{t_r}\mu_r}|^2_{W_2} dr$

$\star$ Good choice of ℓ, η and $(t_r)_{r\in[0,\ell]}$

$\Rightarrow$ Conclusion

Different speed case

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- $\star$ Good choice of ℓ, η and $(t_r)_{r\in[0,\ell]}$

  $\Rightarrow$ Conclusion

Technical difficulties

- Singularity of d at cut locus

  $\leadsto$ Coupling via approximation by geodesic RWs [von Renesse '04 / K. '10 / K. '12]

  (cf. Other approaches:
  [F.-Y. Wang '05]
  [Arnaudon, Coulibaly & Thalmaier '09])

Speciality of this coupling

- [mart. part of $d(X_r(s), X_{r'}(s))$] $\neq 0$

  $\Rightarrow p < \infty$

- Singularity of d at diagonal

  $\star$ $X_r(s)$ & $X_{r'}(s)$ cannot coalesce

  $\Rightarrow p \geq 2$

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  (cf. Other approaches:
  [F.-Y. Wang '05]
  [Arnaudon, Coulibaly & Thalmaier '09])

Speciality of this coupling

- [mart. part of $d(X_r(s), X_{r'}(s))$] $\neq 0$

  $\Rightarrow p < \infty$

- Singularity of d at diagonal

  $\star$ $X_r(s)$ & $X_{r'}(s)$ cannot coalesce

  $\Rightarrow p \geq 2$

Theorem 3 ([Erbar, K. & Sturm])

Suppose $Z \equiv 0$. *For* $K \in \mathbb{R}$ & $N > 0$, *TFAE:*

- $\mathrm{Ric} \geq K$ & $\dim M \leq N$
- $$\mathfrak{s}_{K/N}\left(\frac{W_2(P_{t_0}^* \mu_0, P_{t_1}^* \mu_1)}{2}\right)^2 \leq e^{-K(t_0+t_1)} \mathfrak{s}_{K/N}\left(\frac{W_2(\mu_0, \mu_1)}{2}\right)^2 + \frac{N}{2} \cdot \frac{1 - e^{-K(t_0+t_1)}}{K(t_0+t_1)} \left(\sqrt{t_1} - \sqrt{t_0}\right)^2$$

$$\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}} \quad (\kappa \in \mathbb{R})$$

Theorem 4 ([K.])

Under Ass. 1,

$$\mathcal{T}_{\mathfrak{s}^p_{K^*}(d/2)}(P^*_{t_0}\mu_0, P^*_{t_1}\mu_1)^{2/p} \leq e^{-\theta}\mathcal{T}_{\mathfrak{s}^p_{K^*}(d/2)}(\mu_0,\mu_1)^{2/p} + \frac{(N+p-2)(1-e^{-\theta})}{2\theta}(\sqrt{t_1}-\sqrt{t_0})^2,$$

where $\theta := K(t_0+t_1) + \frac{p}{2}K^*(\sqrt{t_1}-\sqrt{t_0})^2$,

$$K^* := \frac{K}{N-1}$$

$$\mathcal{T}_c(\mu,\nu) := \inf_\pi \|c\|_{L^1(\pi)} \text{ for } c : M^2 \to \mathbb{R}$$

Framework and basic results

$(M, g(t))_{t \in [0,T]}$: cpl. Riem. mfds

$(X(t), \mathbb{P}_x)$: $g(t)$-BM, i.e. diffusion process $\leftrightsquigarrow$ $\Delta_{g(t)}$

<u>A time-dep. analog of "$\mathrm{Ric} \geq K$"</u>

$$\partial_t g(t) \leq 2\,\mathrm{Ric}_{g(t)} - 2Kg(t) \quad (\natural)$$

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[K. & Philipowski '11 / K. '12]

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- $(\natural) \Rightarrow \mathbb{P}_x[X(t) \in M] = 1$
  [K. & Philipowski '11 / K. '12]
- $(\natural) \Rightarrow \mathcal{T}_{d^p_{g(t)}}(P^*_{s \to t}\mu_0, P^*_{s \to t}\mu_1)^{1/p}$
  $\leq e^{-K(t-s)} \mathcal{T}_{d^p_{g(s)}}(\mu_0, \mu_1)^{1/p}$
  [McCann & Topping '10 /
  Arnaudon, Coulibaly & Thalmaier '09 / K. '12]

Perel'man's $\mathcal{L}$-distance

$\gamma : [\tau_1, \tau_2] \to M$, $[\tau_1, \tau_2] \subset [0, T]$

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left(|\dot{\gamma}(\tau)|^2_{g(\tau)} + R_{g(\tau)}(\gamma(\tau)) \right) d\tau$$

(R: scalar curvature)

$$L(\tau_1, x; \tau_2, y) := \inf \left\{ \mathcal{L}(\gamma) \;\middle|\; \begin{array}{l} \gamma(\tau_1) = x, \\ \gamma(\tau_2) = y \end{array} \right\}$$

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<u>Normalization</u>

Given $0 \le \bar{\tau}_0 < \bar{\tau}_1 \le T$,

$$\Theta^t(x, y) := 2(\sqrt{\bar{\tau}_1 t} - \sqrt{\bar{\tau}_0 t}) L(\bar{\tau}_0 t, x; \bar{\tau}_1 t, y) - 2m(\sqrt{\bar{\tau}_1 t} - \sqrt{\bar{\tau}_0 t})^2$$

($m = \dim M$)

Theorem 5 ([K. & Philipowski '11])

Suppose
$$\begin{cases} \partial_t g(t) = 2\,\mathrm{Ric}_{g(t)}, \\ \inf_{\substack{X \in TM \\ t \in [0,T]}} \dfrac{\mathrm{Ric}_{g(t)}(X,X)}{g(t)(X,X)} > -\infty \end{cases}$$

$$\Downarrow$$

$^\exists(X_0(\tau), X_1(\tau))$: *coupling of* $g(\tau)$*-BMs s.t.*

$(\Theta^t(X_0(\bar{\tau}_0 t), X_1(\bar{\tau}_1 t)))_{t \in [1, T/\bar{\tau}_1]}$: *supermartingale*

Corollary 6 ([K. & Philipowski '11])

$\forall \varphi$: $\nearrow$, *concave* & $\forall \mu_t, \nu_t$: *heat distributions,*
$\mathcal{T}_{\varphi(\Theta^t)}(\mu_{\bar{\tau}_0 t}, \nu_{\bar{\tau}_1 t}) \searrow$

- [Topping '09]: $\mathcal{T}_{\Theta^t}(\mu_{\bar{\tau}_0 t}, \nu_{\bar{\tau}_1 t}) \searrow$
  when M:cpt, via optimal transport techniques
  ($\Rightarrow$ Monotonicity of Perelman's $\mathcal{W}$-entropy)

Strategy of the Proof

- Properties of $\mathcal{L}$-distance

  being analoguous to the Riem. dist.

  $\left(\begin{array}{l} \mathcal{L}\text{-geodesic, 1st \& 2nd variation of } \mathcal{L}\text{-length,} \\ \mathcal{L}\text{-index lemma, } \mathcal{L}\text{-cut locus} \end{array} \right)$

- Coupling of $dX_0(\bar{\tau}_0 t)$ and $dX_1(\bar{\tau}_1 t)$

  by space-time parallel transport along $\mathcal{L}$-geodesic

  & scaling

- Approximation by geodesic RWs

Strategy of the Proof

Space-time parallel transport

For $\gamma : [s, t] \to M$ & V: vector field along γ,

V: space-time parallel

$$\nabla^{g(u)}_{\dot{\gamma}(u)} V(u) = -\frac{1}{2}\partial_u g(u)^{\#} V(u)$$

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[mart. part of $\Theta^t(X_0(\bar{\tau}_0 t), X_1(\bar{\tau}_1 t)] \neq 0$

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Remark

[mart. part of $\Theta^t(X_0(\bar\tau_0 t), X_1(\bar\tau_1 t)] \neq 0$

- Driving noises have different speeds
- $\sqrt{u}\dot\gamma_u$ is NOT space-time parallel to γ

Lott's $\mathcal{L}_0$-distance

$$\gamma \ : \ [\tau_1, \tau_2] \to M,\ [\tau_1, \tau_2] \subset [0, T]$$

$$\mathcal{L}_0(\gamma) := \int_{\tau_1}^{\tau_2} \left(|\dot{\gamma}(\tau)|^2_{g(\tau)} + R_{g(\tau)}(\gamma(\tau)) \right) d\tau$$

$$L_0(\tau_1, x; \tau_2, y) := \inf \left\{ \mathcal{L}_0(\gamma) \ \middle| \ \begin{array}{l} \gamma(\tau_1) = x, \\ \gamma(\tau_2) = y \end{array} \right\}$$

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Notation

$0 < \bar{\tau}_0 < \bar{\tau}_1 < T$ fixed,

$$\bar{L}_0^t(x, y) := L_0(\bar{\tau}_0 + t, x; \bar{\tau}_1 + t, y)$$

Theorem 7 ([K. & Amaba])

Suppose $\begin{cases} \partial_t g(t) = 2\,\mathrm{Ric}_{g(t)}, \\ \displaystyle\inf_{\substack{X \in TM \\ t \in [0,T]}} \frac{\mathrm{Ric}_{g(t)}(X,X)}{g(t)(X,X)} > -\infty \end{cases}$

$\Downarrow$

$^{\exists}(X_0(\tau), X_1(\tau))$: *coupling of* $g(\tau)$*-BMs s.t.*
$(\bar{L}_0^t(X_0(\bar{\tau}_0 + t), X_1(\bar{\tau}_1 + t)))_{t \in [0, T - \bar{\tau}_1]}$: *supermart.*

Corollary 8 ([K. & Amaba])

$\forall\varphi$: $\nearrow$, *concave* & $\forall \boldsymbol{\mu}_t, \boldsymbol{\nu}_t$: *heat distributions,*
$\mathcal{T}_{\varphi(\bar{L}_0^t)}(\boldsymbol{\mu}_{\bar{\tau}_0+t}, \boldsymbol{\nu}_{\bar{\tau}_1+t}) \searrow$

- [Lott '09]: $\mathcal{T}_{\bar{L}_0^t}(\boldsymbol{\mu}_{\bar{\tau}_0+t}, \boldsymbol{\nu}_{\bar{\tau}_1+t}) \searrow$
  when M:cpt, via optimal transport techniques
  ($\Rightarrow$ Monotonicity of Perelman's $\mathcal{F}$-functional)

Strategy of the Proof

- Properties of $\mathcal{L}_0$-distance being analoguous to the Riem. dist.
  $\left(\begin{array}{l}\mathcal{L}_0\text{-geodesic, 1st \& 2nd variation of } \mathcal{L}_0\text{-length,} \\ \mathcal{L}_0\text{-index lemma, } \mathcal{L}_0\text{-cut locus}\end{array}\right)$
- Coupling of $dX_0(\bar{\tau}_0 + t)$ and $dX_1(\bar{\tau}_1 + t)$ by spacetime-parallel transport along $\mathcal{L}_0$-geodesic (without scaling)
- Approximation by geodesic RWs

Strategy of the Proof

Remark

$$[\text{mart. part of } \bar{L}_0^t(X_0(\bar{\tau}_0 + t), X_1(\bar{\tau}_1 + t)] \neq 0$$

Strategy of the Proof

Remark

[mart. part of $\bar{L}_0^t(X_0(\bar{\tau}_0 + t), X_1(\bar{\tau}_1 + t)] \neq 0$

- $\dot{\gamma}_u$ is NOT space-time parallel to γ

$\mathcal{L}_0$-geodesic:

$$\nabla^{g(u)}_{\dot{\gamma}_u} \dot{\gamma}_u = \frac{1}{2} \nabla^{g(u)} R_{g(u)} - 2 \, \mathrm{Ric}^{\#}_{g(u)}(\dot{\gamma}_u)$$