The entropic curvature-dimension condition and Bochner's inequality

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1. Introduction

Framework

(M, g): complete Riem. mfd., $\partial M = \emptyset$, $\mathfrak{m} = \mathrm{vol}_g$,
$P_t = e^{t\Delta}$: heat semigroup

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or

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$P_t = e^{t\Delta}$: heat semigroup

or

$(M, d, \mathfrak{m})$: Polish geodesic metric measure sp.,
$\mathfrak{m}$: loc. finite, σ-finite, $\mathrm{supp}\,\mathfrak{m} = X$,
$P_t f = e^{t\Delta} f$: gradient curve in $L^2(\mathfrak{m})$
of Cheeger's Dirichlet energy functional

Framework

L^2-Wasserstein distance W_2: For $\mu_0, \mu_1 \in \mathcal{P}(M)$,

$$\boxed{W_2(\mu_0, \mu_1) := \inf_{\pi} \|d\|_{L^2(\pi)}}$$

$$\pi \in \mathcal{P}(M^2), \begin{cases} \pi(A \times M) = \mu_0(A), \\ \pi(M \times A) = \mu_1(A) \end{cases}$$

$$\mathcal{P}_2(M) := \{\mu \in \mathcal{P}(M) \mid W_2(\delta_{x_0}, \mu) < \infty\}$$

Framework

L^2-Wasserstein distance W_2: For $\mu_0, \mu_1 \in \mathcal{P}(M)$,

$$W_2(\mu_0, \mu_1) := \inf_\pi \|d\|_{L^2(\pi)}$$

- $^\forall \mu_0, \mu_1 \in \mathcal{P}_2(M)$, $^\exists (\mu_r)_{r\in[0,1]}$: W_2-geod.
- $^\exists \Gamma \in \mathcal{P}(\mathrm{Geo}(M))$ s.t.

$$\mu_t(A) = \int_{\mathrm{Geo}(M)} 1_A(\gamma(t))\Gamma(d\gamma),$$

$$W_2(\mu_s, \mu_t)^2 = \int_{\mathrm{Geo}(M)} d(\gamma(s), \gamma(t))^2\Gamma(d\gamma)$$

($\mathrm{Geo}(M)$: const. speed geod.'s)

Framework

Purpose

Unify the study of

$$\text{“}\mathbf{Ric} \geq \boldsymbol{K} \text{ and } \mathbf{dim} \leq \boldsymbol{N}\text{”}$$

in terms of optimal transportation / $\boldsymbol{P_t}$

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⋆ Analysis/Geometry on non-smooth sp.'s

⋆ Different viewpoints even on smooth sp.'s

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Known results

On a cpl. Riem. mfd (M, g), for $K \in \mathbb{R}$,
TFAE [von Renesse & Sturm '05]

(i) $\mathrm{Ric} \geq K$

(ii) "$\mathrm{Hess}\,\mathrm{Ent} \geq K$" on $(\mathcal{P}_2(M), W_2)$

(iii)

(iv) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

(v) $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2)$

(vi) $\frac{1}{2}\Delta |\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K |\nabla f|^2$

Known results

On a "nice" $(M, d, \mathfrak{m})$: mm sp., for $K \in \mathbb{R}$, TFAE [Ambrosio, Gigli & Savaré et al.]

(i) $\mathrm{Ric} \geq K$

(ii) "$\mathrm{Hess}\,\mathrm{Ent} \geq K$" on $(\mathcal{P}_2(M), W_2)$

(iii) $^{\exists}$sol. to the K-Evolution Variational Ineq. of Ent

(iv) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

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Entropy and heat distribution

$$\mathrm{Ent}(\mu) := \int_M \rho \log \rho \, d\mathfrak{m} \quad (\mu = \rho\mathfrak{m})$$

(ii) "$\mathrm{Hess}\,\mathrm{Ent} \geq K$" on $(\mathcal{P}_2(M), W_2)$

$$\Downarrow$$

(iv) $W_2(P_t^*\mu_0, P_t^*\mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

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$\Downarrow$

- $P_t^* \mu$: gradient curve of Ent on $(\mathcal{P}_2(M), W_2)$
- $(\mathcal{P}_2(M), W_2)$: "infinitesimally Hilbertian"

(iv) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

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Known results $(\dim < \infty)$

On a cpl. Riem. mfd (M, g), for $K \in \mathbb{R}$, TFAE

(i) $\mathrm{Ric} \geq K$

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(iv) $W_2(P_t^*\mu_0, P_t^*\mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

(v) $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2)$

(vi) $\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$

Known results ($\dim < \infty$)

On a cpl. Riem. mfd (M, g), for $K \in \mathbb{R}$ & $N > 0$,
TFAE [Bakry & Émery '84, Bakry & Ledoux '06]

(i) $\mathrm{Ric} \geq K$ & $\dim M \leq N$

(ii) "$\mathrm{Hess}\,\mathrm{Ent} \geq K$" on $(\mathcal{P}_2(M), W_2)$

(iii) $^{\exists}$sol. to $\mathbf{EVI}_K$ of Ent

(iv) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

(v) $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) - \dfrac{1 - e^{-2Kt}}{NK} |\Delta P_t f|^2$

(vi) $\dfrac{1}{2} \Delta |\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K |\nabla f|^2 + \dfrac{1}{N} (\Delta f)^2$

Known results $(\dim < \infty)$

On a cpl. Riem. mfd (M, g), for $K \in \mathbb{R}$ & $N > 0$, TFAE [Sturm '06, Lott & Villani '09]

(i) $\mathrm{Ric} \geq K$ & $\dim M \leq N$

(ii) Curvature-dimension condition **CD**(K, N)

(iii) $^{\exists}$sol. to **EVI**$_K$ of Ent

(iv) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

(v) $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) - \dfrac{1 - e^{-2Kt}}{NK} |\Delta P_t f|^2$

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Known results ($\dim < \infty$)

On a cpl. Riem. mfd (M, g), for $K \in \mathbb{R}$ & $N > 0$, TFAE [Bacher & Sturm '10]

(i) $\mathrm{Ric} \geq K$ & $\dim M \leq N$

(ii) Reduced curv.-dim. condition **CD**$^*(K, N)$

(iii) $^\exists$sol. to **EVI**$_K$ of Ent

(iv) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

(v) $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) - \dfrac{1 - e^{-2Kt}}{NK} |\Delta P_t f|^2$

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Goal

- Characterize "$\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$" by $\mathbf{Ent}$
- Find missing **(iii) EVI**$_{\boldsymbol{K},\boldsymbol{N}}$
  & **(iv)** $\boldsymbol{W_2}$ estimate for heat distributions
- Make a connection between them
  & previously known conditions

Outline of the talk

1. **Introduction**

2. **Entropic curvature-dimension condition**

3. **(K, N)-evolution variational inequality**

4. **Connection with Bakry-Émery theory**

5. **Applications**

(K, N)-convexity

K-convexity of Ent :

$\forall \mu_0, \mu_1 \in \mathcal{P}_2(M)$, $\exists (\mu_t)_{t \in [0,1]}$: geod. s.t.

$$\mathrm{Ent}(\mu_t) \leq (1-t)\,\mathrm{Ent}(\mu_0) + t\,\mathrm{Ent}(\mu_1) - \frac{K}{2} t(1-t) W_2(\mu_0, \mu_1)^2$$

(K,N)-convexity

<u>K-convexity of Ent</u> ("$\mathrm{Hess}\,\mathrm{Ent} \geq K$"):

$\forall \mu_0, \mu_1 \in \mathcal{P}_2(M)$, $\exists (\mu_t)_{t\in[0,1]}$: geod. s.t.

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⋆ (RHS) solves

$$\varphi''(t) = KW_2(\mu_0,\mu_1)^2,$$
$$\varphi(0) = \mathrm{Ent}(\mu_0),$$
$$\varphi(1) = \mathrm{Ent}(\mu_1)$$

(K, N)-convexity

(K, N)-convexity of Ent

$$\text{“}\mathrm{Hess}\,\mathrm{Ent} - \frac{1}{N}\nabla\,\mathrm{Ent}^{\otimes 2} \geq K\text{”}$$

$$\Updownarrow$$

$$\text{“}\mathrm{Hess}\,U_N \leq -\frac{K}{N}U_N\text{”},\ U_N := \exp\left(-\frac{1}{N}\,\mathrm{Ent}\right)$$

(K,N)-convexity

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$$\text{“}\mathrm{Hess}\, U_N \le -\frac{K}{N} U_N\text{”},\; U_N := \exp\left(-\frac{1}{N}\,\mathrm{Ent}\right)$$

Entropic curvature-dimension cond. $\mathbf{CD}^e(K,N)$:

$\forall \mu_0, \mu_1 \in \mathcal{P}_2(M)$, $\exists (\mu_t)_{t\in[0,1]}$: geod. s.t.

$$U_N(\mu_t) \ge \sigma_{K/N}^{(1-t)}(W_2(\mu_0,\mu_1)) U_N(\mu_0) + \sigma_{K/N}^{(t)}(W_2(\mu_0,\mu_1)) U_N(\mu_1),$$

$$\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}}, \quad \sigma_\kappa^{(s)}(r) := \frac{\mathfrak{s}_\kappa(sr)}{\mathfrak{s}_\kappa(r)}$$

Relation with known conditions

On cpl. Riem. mfd.,

Volume distortion est. for optimal transport

$$\Downarrow$$

reduced curvature-dimension condition $\mathbf{CD}^*(K, N)$

Relation with known conditions

$$\mathsf{CD}^*(K,N)$$

$\Downarrow$ geod.'s on M are non-branching

$^\forall\mu_0,\mu_1 \in \mathcal{P}_2(M)$, $^\exists(\mu_r)_{r\in[0,1]}$: W_2-geod.,
$^\exists\Gamma \in \mathcal{P}(\mathrm{Geo}(M))$: "lift" of $(\mu_r)_{r\in[0,1]}$ s.t.

$$\rho_t(\gamma_t)^{-\frac{1}{N}} \geq \sigma_{K/N}^{(1-t)}(d(\gamma_0,\gamma_1))\rho_0(\gamma_0)^{-\frac{1}{N}} + \sigma_{K/N}^{(t)}(d(\gamma_0,\gamma_1))\rho_1(\gamma_1)^{-\frac{1}{N}}$$

for Γ-a.e. γ, where $\rho_t \mathfrak{m} = \mu_t$

Relation with known conditions

$$\mathsf{CD}^*(K,N)$$

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Relation with known conditions

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for Γ-a.e. γ, where $\rho_t \mathfrak{m} = \mu_t$

$\Downarrow$

$$\rho_t(\gamma_t)^{-\frac{1}{N}} = \exp\left(-\frac{1}{N}\log \rho_t(\gamma_t)\right)$$

& Jensen's ineq.

CD$^e(K,N)$

Relation with known conditions

$$\rho_t(\gamma_t)^{-\frac{1}{N}} \geq \sigma_{K/N}^{(1-t)}(d(\gamma_0,\gamma_1))\rho_0(\gamma_0)^{-\frac{1}{N}} + \sigma_{K/N}^{(t)}(d(\gamma_0,\gamma_1))\rho_1(\gamma_1)^{-\frac{1}{N}}$$

for Γ-a.e. γ, where $\rho_t \mathfrak{m} = \mu_t$

$\Updownarrow$ geod.'s on M are non-branching

$\mathsf{CD}^e(K,N)$

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5. Applications

K-evolution variational ineq.

EVI$_K$ of Ent

$^\forall \mu_0$, $^\exists (\mu_t)_{t \ge 0}$: abs. conti. s.t. for $^\forall \nu$,

$$\frac{d}{dt}\left(\frac{W_2(\mu_t,\nu)^2}{2}\right) + \frac{K}{2} W_2(\mu_t,\nu)^2 + \mathrm{Ent}(\mu_t) \le \mathrm{Ent}(\nu)$$

⋆ Heuristically, $^\forall$geod.$(\sigma_s)_{s\in[0,1]}$ with $\sigma_0 = \mu_t$,
EVI$_K$ for $\nu = \sigma_s$:

$$-\langle \partial_t \mu_t, \dot{\sigma}_0 \rangle \le \frac{\mathrm{Ent}(\sigma_s) - \mathrm{Ent}(\sigma_0)}{s} + o(1)$$

$\Rightarrow \partial_t \mu_t = -\nabla\, \mathrm{Ent}(\mu_t)$

K-evolution variational ineq.

Properties of **EVI**$_K$ [Ambrosio, Gigli & Savaré]
[Ambrosio, Gigli, Mondino & Rajala]

K-evolution variational ineq.

The volume growth cond. **(V)**

$$\int_M \exp\left(-{}^{\exists}c\, d(x_0, x)^2\right) \mathfrak{m}(dx) < \infty$$

Properties of **EVI**$_K$ [Ambrosio, Gigli & Savaré]
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K-evolution variational ineq.

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Properties of $\mathbf{EVI}_K$ [Ambrosio, Gigli & Savaré]
[Ambrosio, Gigli, Mondino & Rajala]

- $(\mu_t)_{t \geq 0}$ sol. to $\mathbf{EVI}_K$ of $\mathrm{Ent} \Rightarrow \mu_t = P_t^* \mu_0$
- ${}^{\forall}\mu_0$, ${}^{\exists}(\mu_t)_{t \geq 0}$ sol. to $\mathbf{EVI}_K$ of Ent & **(V)**

  $\Leftrightarrow \mathrm{Hess}\,\mathrm{Ent} \geq K$ & $P_t^* \mu_0$: linear w.r.t. μ_0

  $\Leftrightarrow \mathrm{Hess}\,\mathrm{Ent} \geq K$

  & Cheeger's L^2-energy f'nal is quadratic

  (infinitesimally Hilbertian)

EVI$_{K,N}$ and Riemannian CD$^e(K,N)$

EVI$_{K,N}$ of Ent

$^\forall\mu_0$, $^\exists(\mu_t)_{t\geq 0}$: abs. conti. s.t. for $^\forall\nu$,

$$\frac{d}{dt}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 + K\,\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 \leq \frac{N}{2}\left(1-\frac{U_N(\nu)}{U_N(\mu_t)}\right)$$

$$\left(\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}},\ U_N := \exp\left(-\frac{1}{N}\mathrm{Ent}\right)\right)$$

$\mathsf{EVI}_{K,N}$ and Riemannian $\mathsf{CD}^e(K,N)$

Theorem 1 ($\mathsf{RCD}^*(K,N)$ cond.)

For $K \in \mathbb{R}$ and $N > 0$, TFAE:

(i) $\mathsf{CD}^*(K,N)$ *& infinitesimally Hilbertian*

(ii) $\mathsf{CD}^e(K,N)$ *& infinitesimally Hilbertian*

(iii) ${}^\forall\mu_0$, ${}^\exists(\mu_t)_{t\geq 0}$: $\mathsf{EVI}_{K,N}$*-curve &* **(V)**

$$\textbf{(V)} \quad \int_M \exp\left(-{}^\exists c\, d(x_0,x)^2\right) \mathfrak{m}(dx) < \infty$$

$\mathsf{EVI}_{K,N}$ and Riemannian $\mathsf{CD}^e(K,N)$

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★ $\mathsf{RCD}^*(K,N) \Rightarrow \mathsf{RCD}(K,\infty)$

EVI$_{K,N}$ and Riemannian CD$^e(K,N)$

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★ RCD$^*(K,N) \Rightarrow$ RCD(K,∞)

$\Rightarrow$ geod.'s on M are essentially non-branching

[Rajala & Sturm]

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Regularity results on $\mathsf{RCD}(K,\infty)$ sp.

Cheeger's L^2-energy f'nal

$$\mathsf{Ch}(f) := \frac{1}{2}\int_M |\nabla f|_w^2 \, d\mathfrak{m}$$

$|\nabla f|_w$: minimal weak upper gradient of f, i.e.

$$\left| \int_{\partial\gamma} f \right| \le \int_\gamma |\nabla f|_w \quad \text{for } \Gamma\text{-a.e. } \gamma, \ ({}^\forall \text{test plan } \Gamma)$$

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[Ambrosio, Gigli, Savaré et al.]: $\mathbf{RCD}(K,\infty)$ yields

- The volume growth bound **(V)**
- P_t: strong Feller, i.e. $P_t f \in C_b^{\mathrm{Lip}}$ for $f \in L^\infty(\mathfrak{m})$
- $|\nabla f|_w \le 1$ $\mathfrak{m}$-a.e. $\Rightarrow$ f: 1-Lip.

Regularity results on RCD(K, ∞) sp.

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Assumption 1 (cf. [Ambrosio, Gigli & Savaré])

- The volume growth bound **(V)**
- P_t: strong Feller, i.e. $P_t f \in C_b^{\mathrm{Lip}}$ for $f \in L^\infty(\mathfrak{m})$
- $|\nabla f|_w \leq 1$ $\mathfrak{m}$-a.e. $\Rightarrow f$: 1-Lip.

W_2-control

EVI$_K$-curve

$$\frac{d}{dt}\left(\frac{W_2(\mu_t,\nu)^2}{2}\right) + \frac{K}{2} W_2(\mu_t,\nu)^2 + \mathrm{Ent}(\mu_t) \le \mathrm{Ent}(\nu)$$

$\Downarrow$ $\nu = \nu_t$: another **EVI$_K$**-curve

W_2-control

$$W_2(\mu_t,\nu_t) \le e^{-Kt} W_2(\mu_0,\nu_0)$$

Space-time W_2-control

EVI$_{K,N}$-curve

$$\frac{d}{dt}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 + K\,\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 \leq \frac{N}{2}\left(1-\frac{U_N(\nu)}{U_N(\mu_t)}\right)$$

Space-time W_2-control

EVI$_{K,N}$-curve

$$\frac{d}{dt}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 + K\,\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu)}{2}\right)^2 \le \frac{N}{2}\left(1-\frac{U_N(\nu)}{U_N(\mu_t)}\right)$$

$$\Downarrow$$

Space-time W_2-control:

$$\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu_s)}{2}\right)^2 \le e^{-K(s+t)}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_0,\nu_0)}{2}\right)^2 + \frac{N}{2}\frac{1-e^{-K(s+t)}}{K(s+t)}(\sqrt{t}-\sqrt{s})^2$$

Space-time W_2-control

Heuristics ($K = 0$)

$$\mathrm{Hess\,Ent} \geq \frac{1}{N} \nabla\,\mathrm{Ent}^{\otimes 2} \Rightarrow \text{Sp.-t. } W_2\text{-control}$$

$(\sigma_r)_{r\in[0,1]}$: W_2-geod. from $\mu_{t_0 u}$ to $\nu_{t_1 u}$,
$\varphi_r := \langle \nabla\,\mathrm{Ent}(\sigma_r), \dot{\sigma}_r \rangle$, $(t_r)_{r\in[0,1]}$: interpolation

$$\frac{\partial}{\partial s} \frac{W_2(\mu_{t_0 u}, \nu_{t_1 u})^2}{2} = t_0 \varphi_0 - t_1 \varphi_1$$

$$= -\int_0^1 \frac{\partial}{\partial r}(t_r \varphi_r) dr \leq -\int_0^1 \dot{t}_r \varphi_r + \frac{1}{N} t_r \varphi_r^2 \, dr$$

$$\leq \frac{N}{4} \int_0^1 \frac{\dot{t}_r^2}{t_r} \, dr$$

Space-time W_2-control

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$$\frac{\partial}{\partial s} \frac{W_2(\mu_{t_0 u}, \nu_{t_1 u})^2}{2} = t_0 \varphi_0 - t_1 \varphi_1$$

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$$\leq \frac{N}{4} \int_0^1 \frac{\dot{t}_r^2}{t_r} dr = N(\sqrt{t_1} - \sqrt{t_0})^2$$

Bakry-Ledoux gradient estimate

Space-time W_2-control:

$$\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_t,\nu_s)}{2}\right)^2 \le e^{-K(s+t)}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu_0,\nu_0)}{2}\right)^2 + \frac{N}{2}\frac{1-e^{-K(s+t)}}{K(s+t)}(\sqrt{t}-\sqrt{s})^2$$

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$\Downarrow$ Derivative in space & time

Bakry-Ledoux gradient est.: For $f \in W^{1,2}$, $\mathfrak{m}$-a.e.,

$$|\nabla P_t f|_w^2 \le e^{-2Kt} P_t(|\nabla f|_w^2) - \frac{2tC(t)}{N}|\Delta P_t f|^2,$$

$$C(t) = 1 + O(t) \ (t \to 0)$$

RCD$^*(K,N) \Rightarrow$ Bakry-Ledoux

Theorem 2

$(M, d, \mathfrak{m})$: *infinitesimally Hilbertian*

For $K \in \mathbb{R}$ *&* $N > 0$, **(iii)** $\Rightarrow$ **(iv)** $\Rightarrow$ **(v)**

(iii) $^\forall \mu_0$, $^\exists$**EVI**$_{K,N}$-*curve* $(\mu_t)_t$ *&* **(V)**

(iv) *Ass. 1 & Space-time* W_2*-control*

(v) *Ass. 1 & Bakry-Ledoux gradient estimate*

Bakry-Ledoux $\Leftrightarrow$ Bochner

Theorem 3

$(M, d, \mathfrak{m})$: *infinitesimally Hilbertian*

For $K \in \mathbb{R}$ & $N > 0$, **(v)** $\Leftrightarrow$ **(vi)**

(v) $|\nabla P_t f|_w^2 \le e^{-2Kt} P_t(|\nabla f|_w^2) - \dfrac{2C(t)}{N} |\Delta P_t f|^2$

(vi) $^\forall f \in W^{1,2}$ *with* $\Delta f \in W^{1,2}$ &

$g \in D(\Delta) \cap L^\infty$ *with* $g \ge 0$ & $\Delta g \in L^\infty$

$$\int_M \left(\frac{1}{2} \Delta g |\nabla f|_w^2 - g \langle \nabla f, \nabla \Delta f \rangle \right) d\mathfrak{m} \ge \int_M g \left(K |\nabla f|_w^2 + \frac{1}{N} |\Delta f|^2 \right) d\mathfrak{m}$$

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Bakry-Ledoux ⇒ RCD$^*(K, N)$

Idea: Action estimate (as in [Ambrosio, Gigli, Savaré])

$$\frac{W_2(\mu_0, P_\tau^* \mu_1)^2}{2} - \frac{1}{2}\int_0^1 |\dot{\mu}_s|^2 e^{-2K\tau} ds \le Nt(U_N(P_\tau \mu_1) - U_N(\mu_0)) \qquad (\spadesuit)$$

for $t \ll 1$, where $\tau = \tau_{s,t} : \partial_t \tau = s\, U_N(P_\tau^* \mu_s)$,

$$\tau_{s,0} = 0$$

$$\left(\Rightarrow \partial_t P_\tau^* \mu_s = sN\nabla U_N(P_\tau^* \mu_s)\right)$$

Ingredients of the proof

Kantorovich duality, approximations

Bakry-Ledoux $\Rightarrow$ RCD$^*(K, N)$

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Ingredients of the proof

Kantorovich duality, approximations & detailed calc.

Bakry-Ledoux $\Rightarrow$ RCD$^*(K,N)$

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$\Downarrow$

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($\spadesuit$) for $(\mu_0, \mu_1) = (\sigma_0, \sigma_r)$ or (σ_1, σ_r)
& $t \to 0$

Bakry-Ledoux $\Rightarrow$ RCD$^*(K,N)$

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$(\spadesuit)$ for $(\mu_0, \mu_1) = (\sigma_0, \sigma_r)$ or (σ_1, σ_r)
& $t \to 0$

$$U_N(\sigma_r) - (1-r)U_N(\sigma_0) - rU_N(\sigma_1) \ge \frac{K}{N}\int_0^1 (s(1-r)) \wedge ((1-s)r)\,U_N(\sigma_r)\,dr$$

Bakry-Ledoux $\Rightarrow$ RCD$^*(K, N)$

Theorem 4

$(M, d, \mathfrak{m})$: *infinitesimally Hilbertian & Ass. 1*

For $K \in \mathbb{R}$ *&* $N > 0$, **(v) $\Rightarrow$ (ii)**:

(v) *Bakry-Ledoux gradient estimate*

$$|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) - \frac{2C(t)}{N} |\Delta P_t f|^2$$

(ii) *Entropic curv.-dim.* **RCD**$^*(K, N)$:

$$\text{“}\,\mathrm{Hess}\,\mathrm{Ent} - \frac{1}{N}\mathrm{Ent}^{\otimes 2} \geq K\,\text{”}$$

Summary

Theorem 5

$K \in \mathbb{R}$, $N > 0$

(1) *TFAE*

- **(i)** $\mathbf{CD}^*(K, N)$ *& infin. Hilb.*
- **(ii)** $\mathbf{CD}^e(K, N)$ *& infin. Hilb.*
- **(iii)** $^\exists\mathbf{EVI}_{K,N}$*-curves &* **(V)**

(2) *Under* $(M, d, \mathfrak{m})$*: infin. Hilb. & Ass. 1, either* **(iv)–(vi)** *is also equiv. to* **(i)–(iii)**

- **(iv)** *Space-time* W_2*-control*
- **(v)** *Bakry-Ledoux gradient estimate*
- **(vi)** *Bochner inequality*

Properties of $\mathbf{RCD}^*(K,N)$

- Stability under mGH (or Sturm's $\mathbb{D}$)-conv.
- Tensorization
- From local to global
- Measure contraction property $\mathbf{MCP}(K,N)$
  (via $\mathbf{CD}^*(K,N)$; [Cavalletti & Sturm])
  – (sharp) Bishop-Gromov volume comparison
  – volume doubling property
  – (local unif.) Poincaré ineq. [Rajala]
  $\Rightarrow$ $^{\exists}$heat kernel, two-sided Gaussian bound
  $\Rightarrow$ Ultracontractivity of P_t

Properties of $\mathsf{RCD}^*(K, N)$

- N-precision of f'nal ineq.'s
  - N-HWI ineq.
    $\Rightarrow$ N-log Sobolev ineq.
    $\Rightarrow$ N-Talagrand ineq.
- Lipschitz regularity of the heat kernel/eigenfn.'s
- Lichnerowicz bound of λ_1 [Ketterer]
- Li-Yau's ineq. [Garofalo & Mondino]