The entropic curvature dimension condition and Bochner's inequality

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This talk is based on a joint work with M. Erbar and K.-Th. Sturm (Universität Bonn) [6].

There are several different ways to characterize "(Ricci curvature) $\geq K$ & $\dim X \leq N$" on a Riemannian manifold X, where $K \in \mathbb{R}$ and $N \in (0, \infty)$. Among them, the curvature-dimension condition introduced by Sturm [8], Lott and Villani [7] works well even in the framework of abstract metric measure spaces. It is described in terms of optimal transportation and it possesses many nice geometric stability properties. On the other hand, Bochner's inequality introduced by Bakry and Émery is formulated for an abstract diffusion generator. As Bochner's formula has played significant roles in Riemannian geometry, Bochner's inequality provides enormous important functional inequalities in geometric analysis. The purpose of this talk is to unify these two concepts by introducing new conditions equivalent to either (and hence both) of them on metric measure spaces. When $N = \infty$, this program was essentially finished by Ambrosio, Gigli, Savaré and their collaborators [1, 2, 3, 4] and our main focus is in the case $N < \infty$.

Let (X, d, m) be a Polish geodesic metric measure space, where the measure m is locally finite, σ-finite and $\operatorname{supp} m = X$. Let us introduce comparison functions: for $\kappa \in \mathbb{R}$ and $\kappa\theta^2 \leq \pi^2$,

$$\mathfrak{s}_\kappa(\theta) := \frac{\sin(\sqrt{\kappa}\theta)}{\sqrt{\kappa}}, \quad \sigma_\kappa^{(t)}(\theta) := \frac{\mathfrak{s}_\kappa(t\theta)}{\mathfrak{s}_\kappa(\theta)}.$$

We call a function V on a metric space (Y, d_Y) *(K, N)-convex* if for each $x, y \in Y$ there is a constant speed geodesic $\gamma : [0, 1] \to Y$ from x to y such that the following holds:

$$V_N(\gamma_t) \geq \sigma_{K/N}^{(1-t)}(d_Y(x, y))V_N(\gamma_0) + \sigma_{K/N}^{(t)}(d_Y(x, y))V_N(\gamma_1), \quad \text{where } V_N := \exp\left(-\frac{1}{N}V\right).$$

We call V *strongly (K, N)-convex* if the last inequality holds for each (and at least one) geodesic γ. This is an integral formulation of the following inequality in the distributional sense:

$$\partial_t^2 V_N(\gamma_t) \leq -\frac{K}{N} d(x, y)^2 V_N(\gamma_t).$$

If V is C^2-function on a Riemannian manifold, then V is (K, N)-convex if and only if

$$\operatorname{Hess} V - \frac{1}{N}\nabla V \otimes \nabla V \geq K.$$

Let $\mathscr{P}_2(X)$ be the L^2-Wasserstein space, consisting of probability measures on X with finite second moments, equipped with the *L^2-Wasserstein distance* W_2 given by

$$W_2(\mu, \nu) := \inf \left\{ \|d\|_{L^2(\pi)} \mid \pi\text{: a coupling of } \mu \text{ and } \nu \right\}.$$

Note that $(\mathscr{P}_2(X), W_2)$ is also a Polish geodesic metric space. We denote the *relative entropy* by Ent: For $\mu \in \mathscr{P}(X)$,

$$\operatorname{Ent}(\mu) := \begin{cases} \displaystyle\int_X \rho \log \rho \, dm & \text{if } \mu = \rho m \text{ with } (\rho \log \rho)_+ \in L^1(X, m), \\ \infty & \text{otherwise.} \end{cases}$$

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We say that (X,d,m) satisfies the (strong) *entropic curvature-dimension condition* $\mathsf{CD}^e(K,N)$ if Ent is (strongly) (K,N)-convex on $\mathscr{P}_2(X)$ respectively.

Let Ch be *Cheeger's L^2-energy functional* given by a relaxation of the energy functional associated with local Lipschitz constants. It can be written as an energy integral in terms of the *weak upper gradient* $|\nabla f|_w$, i.e.

$$\mathsf{Ch}(f) := \frac{1}{2}\int_X |\nabla f|_w^2 \, dm$$

(see [3]). We say (X,d,m) *infinitesimally Hilbertian* if Ch coincides with a closed symmetric bilinear form $\mathcal{E}$: $2\mathsf{Ch}(f) = \mathcal{E}(f,f)$. In this case $\mathcal{E}(f,g)$ has a density denoted by $\langle \nabla f, \nabla g\rangle$ and in particular $|\nabla f|_w^2 = \langle \nabla f, \nabla f\rangle$ m-a.e. (see [4]). Let Δ be the associated generator of $\mathcal{E}$ and T_t a Markov semigroup generated by Δ.

Example

Let (X,d,m) be an N-dimensional complete Riemannian manifold, $\partial X = \emptyset$, equipped with the Riemannian measure m. Suppose $\mathrm{Ric} \ge K$. Let V be a (K',N')-convex function on (X,d). Then $(X,d,\mathrm{e}^{-V}m)$ satisfies $\mathsf{CD}^e(K+K',N+N')$. In this framework, Ch coincides with the usual Dirichlet energy and hence $(X,d,\mathrm{e}^{-V}m)$ is infinitesimally Hilbertian.

Theorem A

The following are equivalent:

(i) (X,d,m) is infinitesimally Hilbertian and satisfies the *reduced curvature-dimension condition* $\mathsf{CD}^*(K,N)$ introduced by Bacher and Sturm [5]. That is, for $\mu_0 = \rho_0 m, \mu_1 = \rho_1 m \in \mathscr{P}(X)$ with bounded supports, there exists an optimal coupling q of them and a geodesic $\mu_t = \rho_t m \in \mathscr{P}_2(X)$ with bounded supports such that for all $t \in [0,1]$ and $N' \ge N$:

$$\int_X \rho_t^{-1/N'} d\mu_t \ge \int_{X\times X} \big[\sigma_{K/N'}^{(1-t)}(d(x_0,x_1))\rho_0(x_0)^{-1/N'} + \sigma_{K/N'}^{(t)}(d(x_0,x_1))\rho_1(x_1)^{-1/N'}\big] q(dx_0,dx_1).$$

(ii) (X,d,m) is infinitesimally Hilbertian and satisfies $\mathsf{CD}^e(K,N)$.

(iii) Assumption (a) holds, and for each $\mu \in D(\mathrm{Ent})$ there is a locally absolutely continuous curve $\mu_t \in \mathscr{P}_2(X)$ with $\mu_0 = \mu$ satisfying the following: For each $\sigma \in \mathscr{P}_2(X)$,

$$\frac{d}{dt}\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_t,\sigma)}{2}\right) + K\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_t,\sigma)}{2}\right) \le \frac{N}{2}\left(1 - \exp\left(-\frac{1}{N}(\mathrm{Ent}(\sigma) - \mathrm{Ent}(\mu_t))\right)\right)$$

(the (K,N)-*evolution variational inequality* ($\mathsf{EVI}_{K,N}$)).

In the condition (iii), the solution μ_t to $\mathsf{EVI}_{K,N}$ can be regarded as a gradient curve of Ent (in a stronger sense). This was heuristically known that μ_t coincides with the heat distribution. We can verify it in this framework (see [3]) and this fact connects $\mathsf{CD}^e(K,N)$ with analysis of the heat semigroup T_t. This connection was hidden in $\mathsf{CD}^*(K,N)$ since there appears no Ent.

Assumption

(a) There exists $c > 0$ such that $\displaystyle\int_X \exp\left(-cd(x,x_0)^2\right) dm < \infty$ for some $x_0 \in X$.

(b) Every $f \in D(\mathsf{Ch})$ with $|\nabla f|_w \le 1$ m-a.e. has a 1-Lipschitz representative.

Note that $\mathsf{CD}^e(K,N)$ implies (a). In addition, the condition (ii) implies (b).

Theorem B

If one of (i)–(iii) holds, then ((X,d,m) is infinitesimally Hilbertian and) the following holds:

(iv) [*Space-time W_2-control*] For $\mu_0,\mu_1 \in \mathscr{P}_2(X)$ and $t,s \geq 0$,

$$\mathfrak{s}_{K/N}^2\left(\frac{W_2(T_t\mu_0,T_s\mu_1)}{2}\right) \leq \mathrm{e}^{-K(s+t)}\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu_0,\mu_1)}{2}\right) + \frac{N}{2}\frac{1-\mathrm{e}^{-K(s+t)}}{K(s+t)}\left(\sqrt{t}-\sqrt{s}\right)^2.$$

(v) [*Bakry-Ledoux gradient estimate*] For $f \in D(\mathsf{Ch})$ and $t > 0$,

$$|\nabla T_t f|_w^2 + \frac{2tC(t)}{N}|\Delta T_t f|^2 \leq \mathrm{e}^{-2Kt}T_t(|\nabla f|_w^2) \quad m\text{-a.e.},$$

where $C(t) > 0$ is a function satisfying $C(t) = 1 + O(t)$ as $t \to 0$.

(vi) [(a weak form of) *Bochner's inequality*] For $f \in D(\Delta)$ with $\Delta f \in D(\mathsf{Ch})$ and all $g \in D(\Delta) \cap L^\infty(X,m)$ with $g \geq 0$ and $\Delta g \in L^\infty(X,m)$,

$$\frac{1}{2}\int_X \Delta g |\nabla f|_w^2 \, dm - \int_X g\langle \nabla f, \nabla \Delta f\rangle \, dm \geq K\int_X g|\nabla f|_w^2 \, dm + \frac{1}{N}\int_X g(\Delta f)^2 \, dm.$$

Conversely, if Assumptions (a) and (b) holds and (X,d,m) is infinitesimally Hilbertian, then one of (iv)–(vi) implies (i)–(iii) and hence (i)–(vi) are all equivalent.

Applications and related results will be mentioned in the talk.

References

[1] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala. Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure. To appear in Trans. Amer. Math. Soc., 2012.

[2] L. Ambrosio, N. Gigli, and G. Savaré. Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds. Preprint. Available at: arXiv: 1209.5786.

[3] L. Ambrosio, N. Gigli, and G. Savaré. Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. To appear in Invent. Math. Available at: arXiv:1106.2090.

[4] L. Ambrosio, N. Gigli, and G. Savaré. Metric measure spaces with Riemannian Ricci curvature bounded from below. Preprint. Available at: arXiv:1109.0222.

[5] K. Bacher and K.-T. Sturm. Localization and tensorization properties of the curvature-dimension condition for metric measure spaces. *J. Funct. Anal.*, 259(1):28–56, July 2010.

[6] M. Erbar, K. Kuwada, and K.-T. Sturm. On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces. Preprint. Available at: arXiv:1303.4382.

[7] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. Math.*, 169(3):903–991, 2009.

[8] K.-Th. Sturm. On the geometry of metric measure spaces. I and II. *Acta. Math.*, 196(1):65–177, 2006.