The entropic curvature dimension condition and Bochner's inequality

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There are several different ways to characterize "$\mathrm{Ric} \geq K$ & $\dim X \leq N$" for a Riemannian manifold X, where $K \in \mathbb{R}$ and $N \in (0, \infty)$. Among them, the curvature-dimension condition introduced by Sturm, Lott and Villani works well even in the framework of metric measure spaces. It is described in terms of optimal transportation and it possesses many nice geometric stability properties. On the other hand, Bochner's inequality introduced by Bakry and Émery is formulated for an abstract diffusion generator. As the Bochner formula has played a significant role in Riemannian geometry, the Bochner inequality provides enormous important functional inequalities in geometric analysis. The purpose of this talk is to unify these two concepts by introducing new conditions equivalent to either (and hence both) of them on an abstract metric measure space. Such a program is essentially finished by Ambrosio, Gigli, Savaré and their collaborators when $N = \infty$ and our main focus is on the case $N < \infty$.

Let (X, d, m) be a Polish geodesic metric measure space, where the measure m is locally finite, σ-finite and $\operatorname{supp} m = X$. Let $\mathfrak{s}_\kappa(\theta) := \kappa^{-1/2} \sin(\sqrt{\kappa}\theta)$ and $\sigma_\kappa^{(t)}(\theta) := \mathfrak{s}_\kappa(t\theta)/\mathfrak{s}_\kappa(\theta)$ for $\kappa \in \mathbb{R}$ and $\kappa\theta^2 \leq \pi^2$. We call a function V on X (K,N)-convex if for each $x, y \in X$ there is a constant speed geodesic $\gamma : [0,1] \to X$ from x to y such that $V_N := \exp(-N^{-1}V)$ satisfies the following:

$$V_N(\gamma_t) \geq \sigma_{K/N}^{(1-t)}(d(x,y)) V_N(\gamma_0) + \sigma_{K/N}^{(t)}(d(x,y)) V_N(\gamma_1).$$

We call V strongly (K,N)-convex if the last inequality holds for all geodesics γ. This is an integral formulation of $\partial_t^2 V_N(\gamma_t) \leq -\frac{K}{N} d(x,y)^2 V_N(\gamma_t)$ in the distributional sense. In particular, if X is Riemannian manifold and $V \in C^2(X)$, then (K,N)-convexity of V is equivalent to $\operatorname{Hess} V - N^{-1} \nabla V \otimes \nabla V \geq K$.

Let $\mathscr{P}_2(X,d)$ be the L^2-Wasserstein space, consisting of probability measures on X with finite second moments, equipped with L^2-Wasserstein distance W_2. Note that $(\mathscr{P}_2(X,d), W_2)$ is also a Polish geodesic metric space. We denote the relative entropy by Ent: For $\mu \in \mathscr{P}(X)$, $\mathrm{Ent}(\mu) := \int_X \rho \log \rho \, dm$ if $\mu = \rho m$ with $(\rho \log \rho)_+ \in L^1(X, m)$ and $\mathrm{Ent}(\mu) := \infty$ otherwise. We say that (X, d, m) satisfies the (strong) *entropic curvature-dimension condition* $\mathsf{CD}^e(K,N)$ if Ent is (strongly) (K,N)-convex on $\mathscr{P}_2(X,d)$. For instance, if (X,d,m) is N-dim. complete Riemannian manifold with $\partial X = \emptyset$ equipped with the Riemannian measure, $\mathrm{Ric} \geq K$ and V is (K', N') convex, then $(X, d, \mathrm{e}^{-V} m)$ satisfies $\mathsf{CD}^e(K + K', N + N')$.

We say (X,d,m) *infinitesimally Hilbertian* if the Cheeger's L^2-energy functional Ch, associated with the weak upper gradient $|\nabla f|_w$ i.e. $\mathsf{Ch}(f) := \frac{1}{2} \int_X |\nabla f|_w^2 dm$, coincides with a closed symmetric bilinear form $\mathcal{E}$: $2\mathsf{Ch}(f) = \mathcal{E}(f,f)$. In this case $\mathcal{E}(f,g)$ has a density denoted by $\langle \nabla f, \nabla g \rangle$ and in particular $|\nabla f|_w^2 = \langle \nabla f, \nabla f \rangle$ m-a.e. Let Δ be the associated generator of $\mathcal{E}$ and H_t a Markov semigroup generated by Δ.

In the sequel, we sometimes require the following regularity assumptions.

Assumption

(a) There exists $c > 0$ such that $\displaystyle\int_X \exp\left(-c d(x, x_0)^2\right) dm < \infty$ for some $x_0 \in X$.

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(b) Every $f \in D(\mathsf{Ch})$ with $|\nabla f|_w \le 1$ m-a.e. has a 1-Lipschitz representative.

Note that $\mathsf{CD}^e(K,N)$ implies (a). In addition, it also implies (b) if (X,d,m) is infinitesimally Hilbertian.

Theorem The following are equivalent:

(i) (X,d,m) is infinitesimally Hilbertian and satisfies the *reduced curvature-dimension condition* $\mathsf{CD}^*(K,N)$ introduced by Bacher and Sturm. That is, for $\mu_0 = \rho_0 m, \mu_1 = \rho_1 m \in \mathscr{P}(X)$ with bounded supports, there exists an optimal coupling q of them and a geodesic $\mu_t = \rho_t m \in \mathscr{P}_2(X,d)$ with bounded supports such that for all $t \in [0,1]$ and $N' \ge N$:

$$\int_X \rho_t^{-1/N'} d\mu_t \ge \int_{X\times X} \big[\sigma_{K/N'}^{(1-t)}(d(x_0,x_1))\rho_0(x_0)^{-1/N'} + \sigma_{K/N'}^{(t)}(d(x_0,x_1))\rho_1(x_1)^{-1/N'}\big] q(dx_0,dx_1).$$

(ii) (X,d,m) is infinitesimally Hilbertian and satisfies $\mathsf{CD}^e(K,N)$.

(iii) Assumption (a) holds, and for each $\mu \in D(\mathrm{Ent})$ there is a locally absolutely continuous curve $\mu_t \in \mathscr{P}_2(X,d)$ with $\mu_0 = \mu$ satisfying the following: For each $\sigma \in \mathscr{P}_2(X,d)$,

$$\frac{d}{dt}\mathfrak{s}_{K/N}^2\left(\frac{1}{2}d(\mu_t,\sigma)\right) + K\mathfrak{s}_{K/N}^2\left(\frac{1}{2}d(\mu_t,\sigma)\right) \le \frac{N}{2}\left(1-\exp\left(-\frac{1}{N}(\mathrm{Ent}(\sigma)-\mathrm{Ent}(\mu_t))\right)\right)$$

(the (K,N)-*evolution variational inequality* $\mathsf{EVI}_{K,N}$).

If one of them holds then ((X,d,m) is infinitesimally Hilbertian and) the following holds:

(iv) For $f,g \in L^2(X,m)$ with $fm, gm \in \mathscr{P}_2(X,d)$ and $t,s \ge 0$,

$$\mathfrak{s}_{K/N}^2\left(\frac{1}{2}W_2(H_t fm, H_s gm)\right) \le \mathrm{e}^{-K(s+t)}\mathfrak{s}_{K/N}^2\left(\frac{1}{2}W_2(fm,gm)\right) + \frac{N}{2}\frac{1-\mathrm{e}^{-K(s+t)}}{K(s+t)}\left(\sqrt{t}-\sqrt{s}\right)^2$$

(the *space-time* W_2*-expansion bound*).

(v) For $f \in D(\mathsf{Ch})$ and $t > 0$,

$$|\nabla H_t f|_w^2 + \frac{2tC(t)}{N}|\Delta H_t f|^2 \le \mathrm{e}^{-2Kt} H_t(|\nabla f|_w^2) \quad m\text{-a.e.},$$

where $C(t) > 0$ is a function satisfying $C(t) = 1 + O(t)$ as $t \to 0$ (the *Bakry-Ledoux gradient estimate*).

(vi) For $f \in D(\Delta)$ with $\Delta f \in D(\mathsf{Ch})$ and all $g \in D(\Delta) \cap L^\infty(X,m)$ with $g \ge 0$ and $\Delta g \in L^\infty(X,m)$,

$$\frac{1}{2}\int_X \Delta g |\nabla f|_w^2\, dm - \int_X g\langle \nabla f, \nabla \Delta f\rangle\, dm \ge K\int_X g|\nabla f|_w^2\, dm + \frac{1}{N}\int_X g(\Delta f)^2\, dm$$

(a weak form of *Bochner inequality*).

Conversely, if Assumption (a)(b) holds and (X,d,m) is infinitesimally Hilbertian, then one of (iv)–(vi) implies (i)–(iii) and hence (i)–(vi) are all equivalent.

Note that $\mathsf{CD}^*(K,N)$ enjoys a local-to-global property and it is equivalent to the local version of curvature-dimension condition $\mathsf{CD}(K,N)$ introduced by Sturm (and Lott & Villani).