Optimal transportation costs of heat distributions in stochastic geometric analysis

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UK-Japan Mathematical Forum

on Geometry, Probability and their Applications

(Jul. 15–20, 2012, at Keio Univ.)

1. Introduction

† (M, g): compl. Riem. mfd, $\dim M = n \geq 2$ (Riem. met. g may depend on time), $\partial M = \emptyset$

† $X^x(t)$: Brownian motion on M, $X^x(0) = x$ (diffusion process generated by Δ)

Ass $X^x(\cdot)$ has infinite lifetime

Interest: Relation between

- (Lower bound of) Ricci curvature
- Behavior of (a coupling of) Brownian motions

$(X_0(t), X_1(t))$: a coupling of $X^{x_0}(t)$ & $X^{x_1}(t)$

$\overset{\text{def}}{\Leftrightarrow} (X_i(t))_{t \geq 0} \overset{d}{=} (X^{x_i}(t))_{t \geq 0}$ $(i = 0, 1)$

<u>Example</u>

★ BM is invariant under an isometry $\varphi : M \to M$

$\Rightarrow (X^x(t), \varphi(X^x(t)))$: coupling of BMs starting from $(x, \varphi(x))$

Optimal transportation cost

$c : M \times M \to \mathbb{R}$: cost function

$(c(x,y)$: cost of bringing a unit mass from x to $y)$

For $\mu, \nu \in \mathcal{P}(M)$,

$$\Pi(\mu,\nu) := \left\{ \pi \;\middle|\; \begin{array}{l} \pi(E \times M) = \mu(E), \\ \pi(M \times E) = \nu(E) \end{array} \right\}$$

(set of all couplings between μ and ν),

$$\mathcal{T}_c(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{M \times M} c(x,y)\pi(dxdy)$$

Why transportation costs instead of coupling?

★ $\mu_i(t) := \mathbb{P} \circ X^{x_i}(t)^{-1}$: distribution of $X^{x_i}(t)$

$(X_0(t), X_1(t))$: coupling of $X^{x_0}(t)$ & $X^{x_1}(t)$

$$\Downarrow$$

$$\boxed{\mathcal{T}_c(\mu_0(t), \mu_1(t)) \le \mathbb{E}\left[c(X_0(t), X_1(t))\right]}$$

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- Reflects the geometry of M well
  (when c is a (fn. of) distance)
- Stable under perturbation
- Nice properties (e.g. an alternative variational expression (Kantorovich duality))

Relation with Ric

$P_t := \mathrm{e}^{t\Delta}$: the heat semigroup

Thm 1.1 [von Renesse & Sturm '05, etc.]

For $K \in \mathbb{R}$, the following are equivalent:

(i) $\mathrm{Ric} \geq K$

(ii) $\exists/\forall p \in [1, \infty]$,

$$\mathcal{T}_{(\mathrm{e}^{Kt} d)^p}(P_t^* \mu_0, P_t^* \mu_1) \searrow$$

(iii) $\exists/\forall q \in [1, \infty]$,

$$|\nabla P_t f|(x)^q \leq \mathrm{e}^{-qKt} P_t(|\nabla f|^q)(x)$$

Related results

- Formulation of (i) via optimal transportation [Sturm '06, Lott & Villani '09]
  $\Rightarrow$ extension to metric measure spaces
- (ii) $\Leftrightarrow$ (iii) for $p, q \in [1, \infty]$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ [K. '10]
- "(i)" $\Rightarrow$ (ii) with $p = 2$ on singular sp.'s ([Savaré '07], [Ohta '09]) & [Gigli, K. & Ohta], [Ambrosio, Gigli & Savaré]

Contents of the talk

- (i) $\Rightarrow$ (ii) via coupling by parallel transport
- Ricci flow, Perel'man's $\mathcal{L}$-distance
- Coupling by reflection
- Curvature-dimension condition

2. Coupling by parallel transport

Example ($M = \mathbb{R}^n$)

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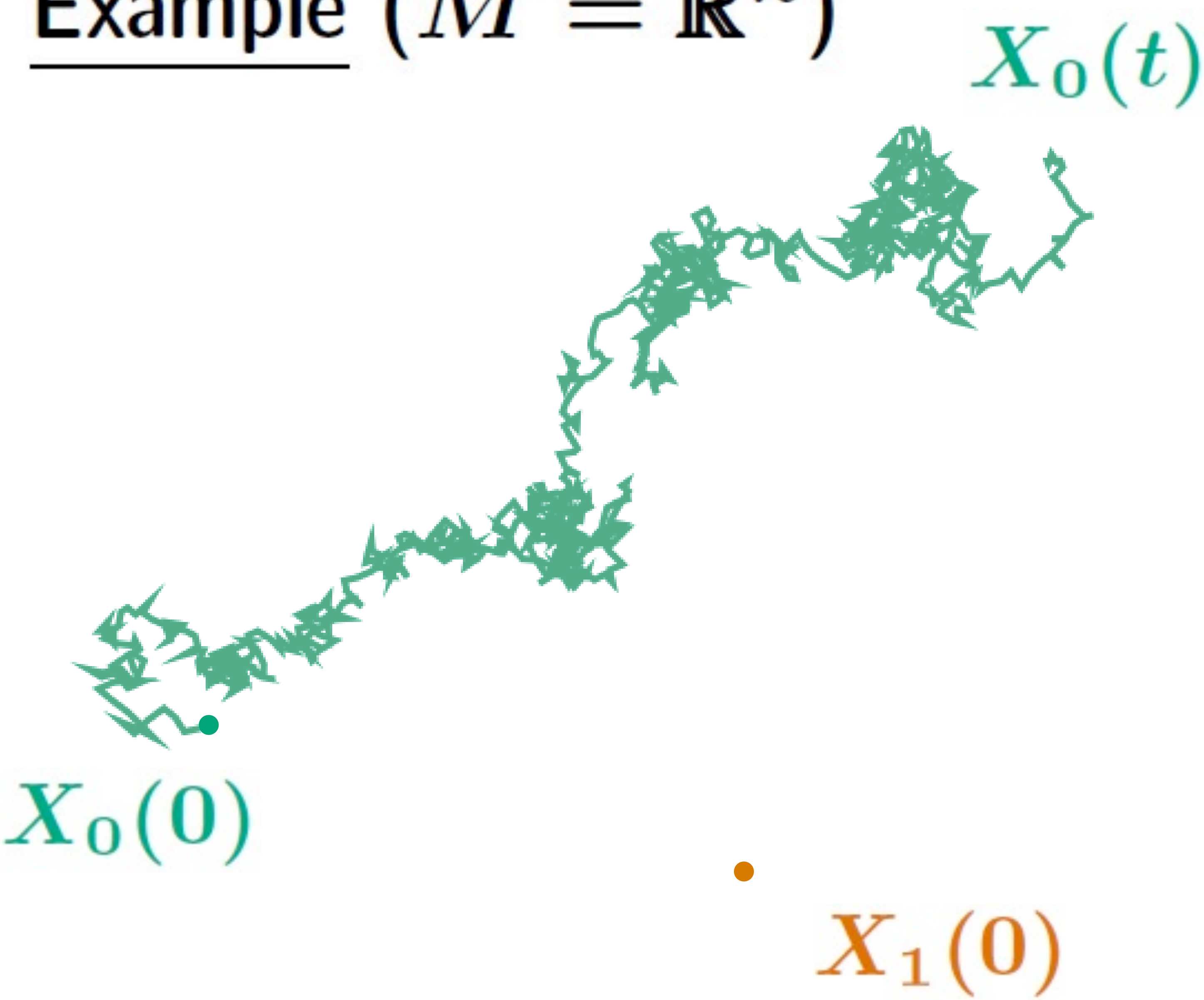

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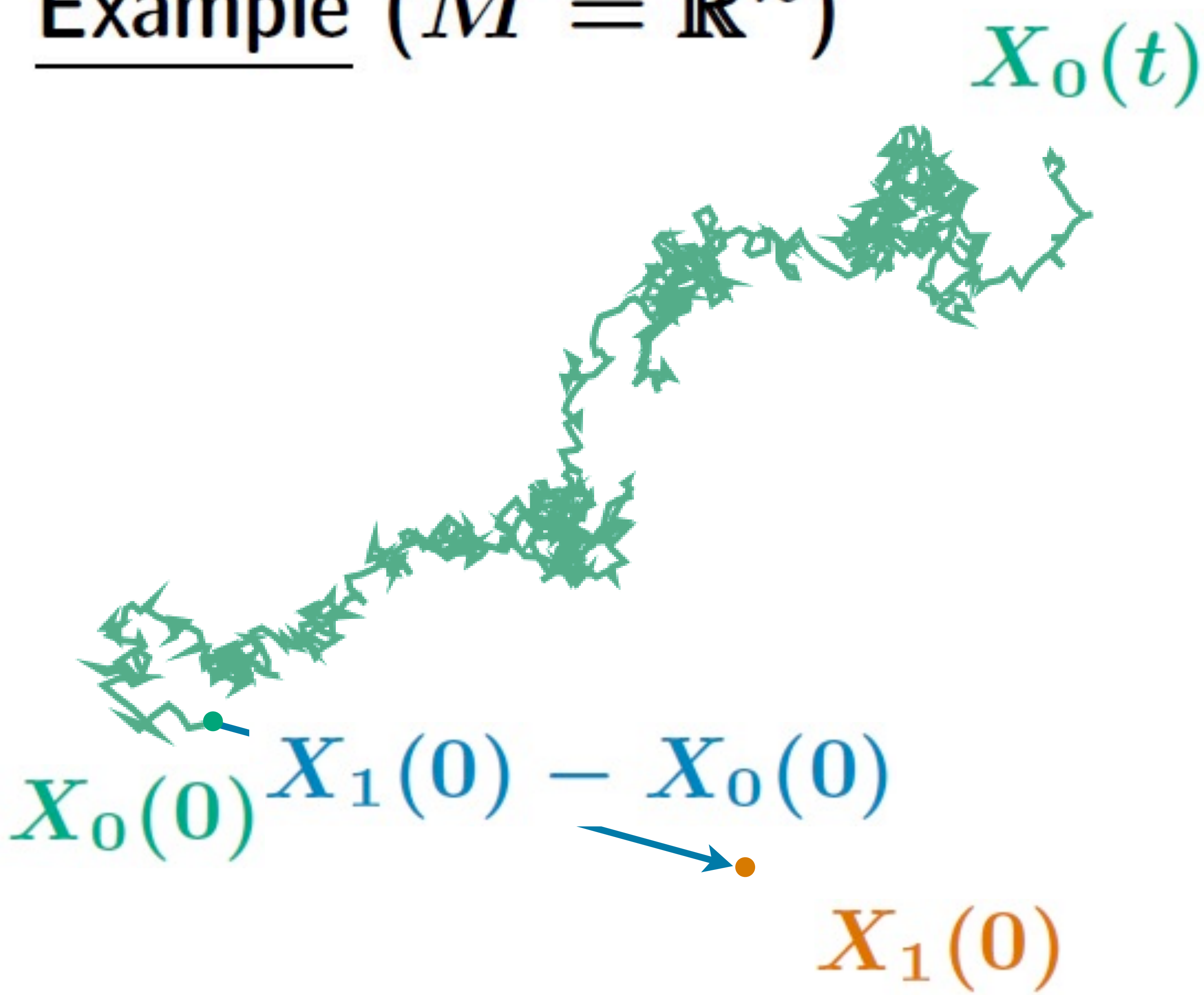

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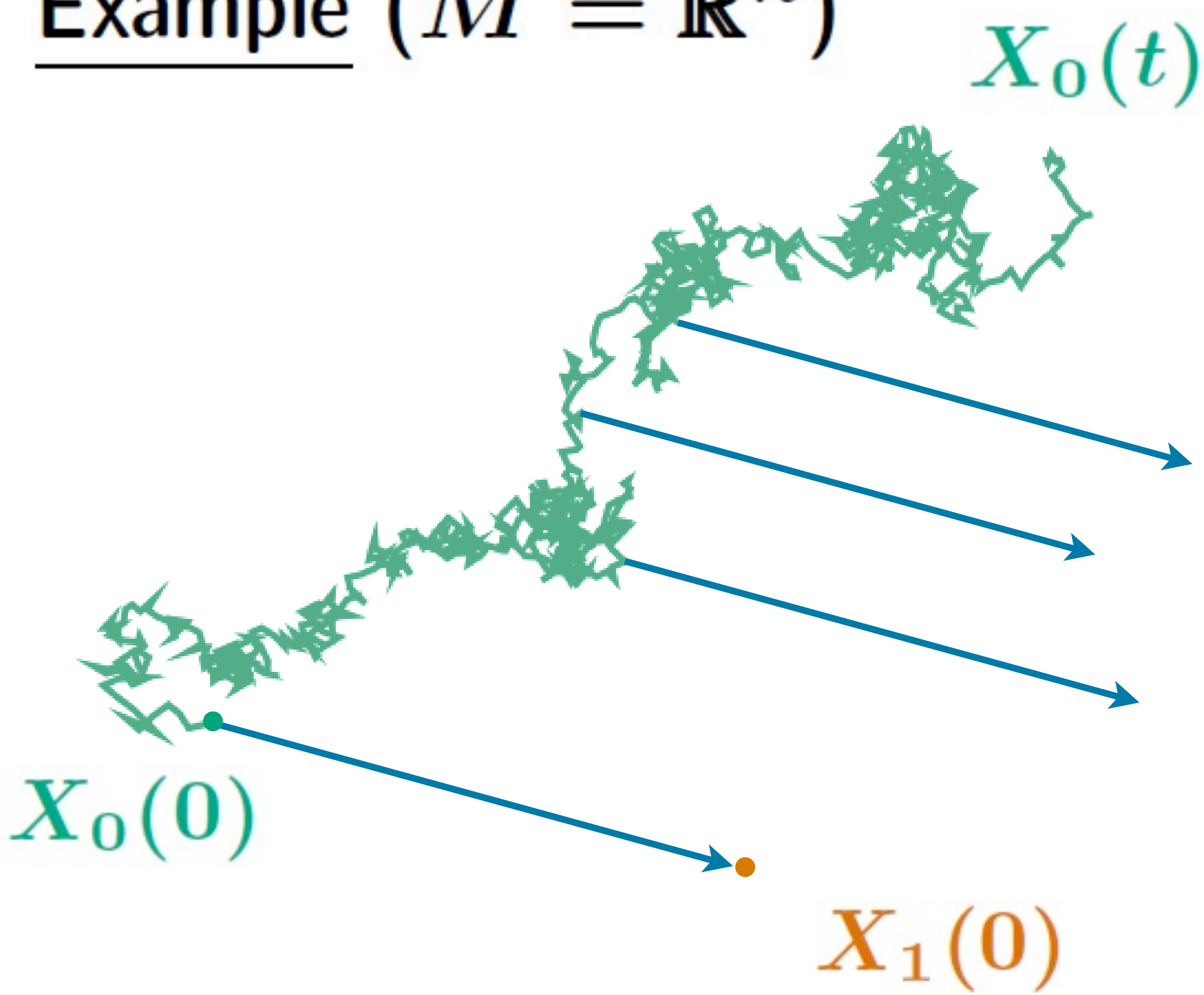

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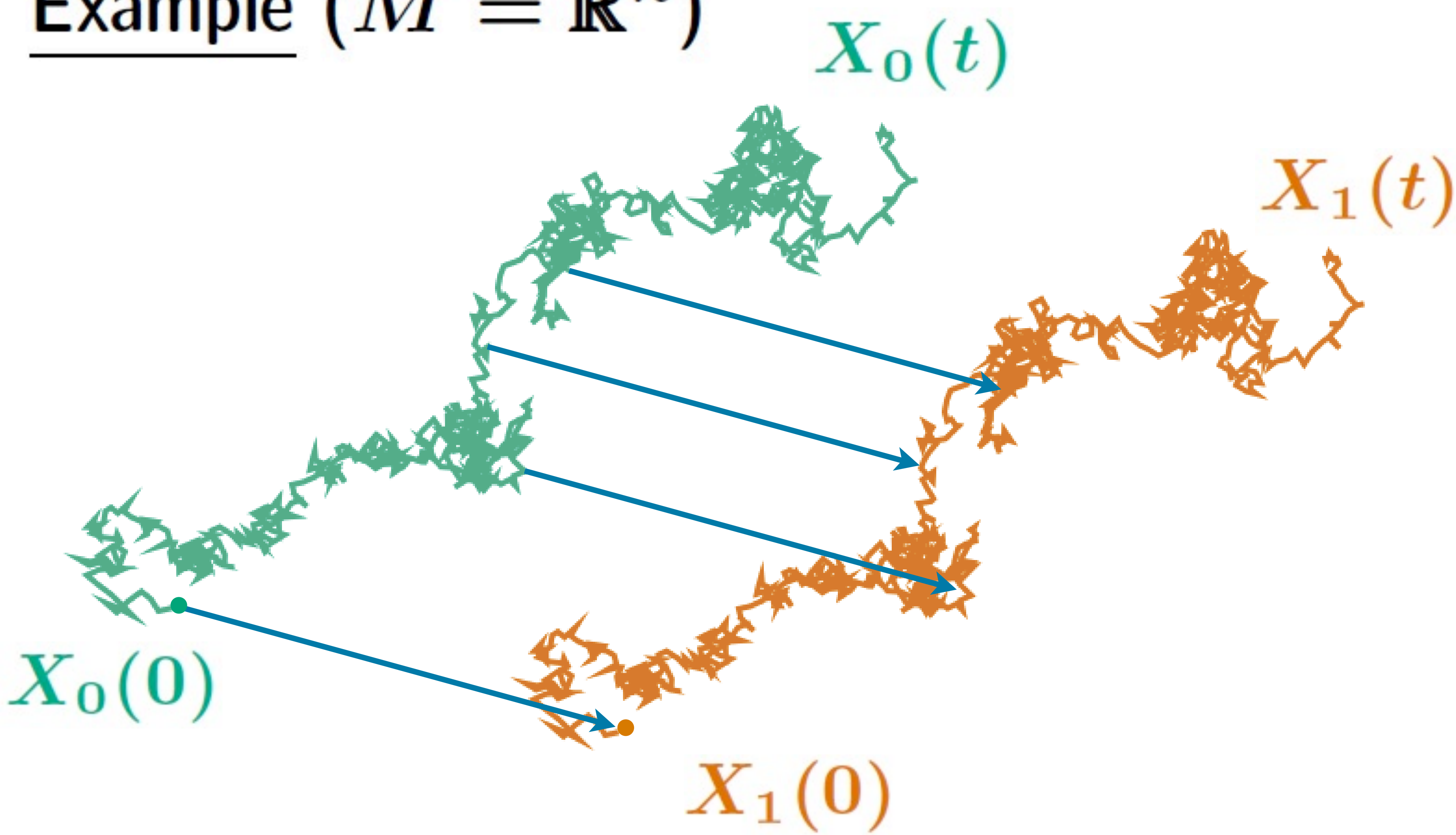

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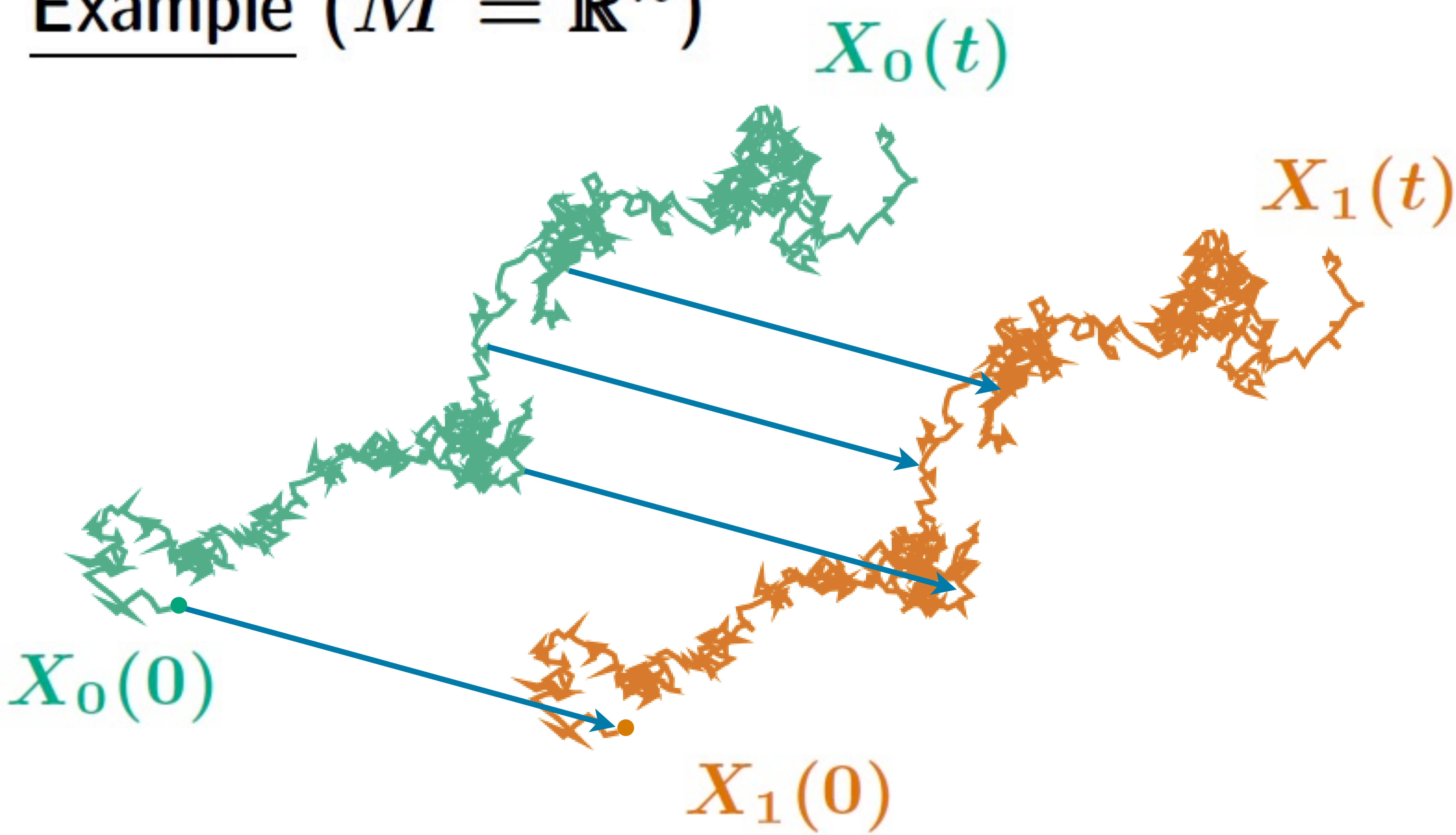

$$\Rightarrow d(X_0(t), X_1(t)) = d(X_0(0), X_1(0))$$

$$\Rightarrow \mathcal{T}_{d^p}(P_t^* \mu_0, P_t^* \mu_1) \searrow$$

Idea

- Couple two BMs to keep their distance as much as possible
- Construct the coupling by integrating a coupled infinitesimal motions

$\Rightarrow$ coupling of infin. motions by parallel transport

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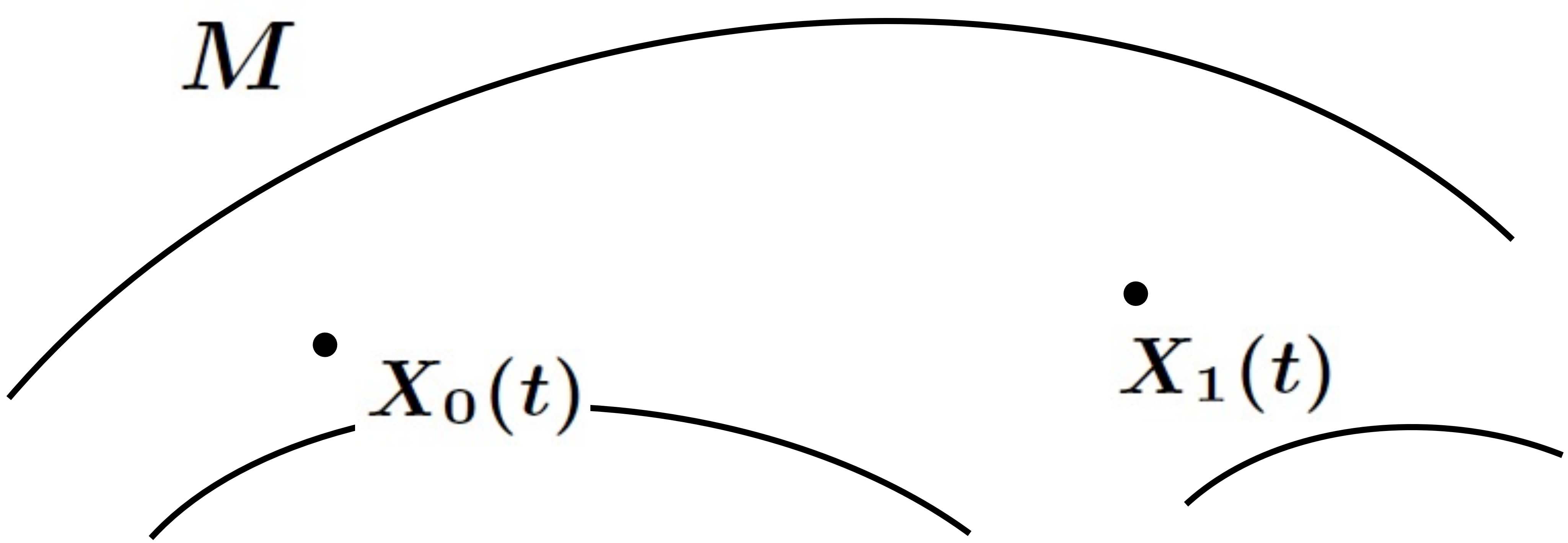

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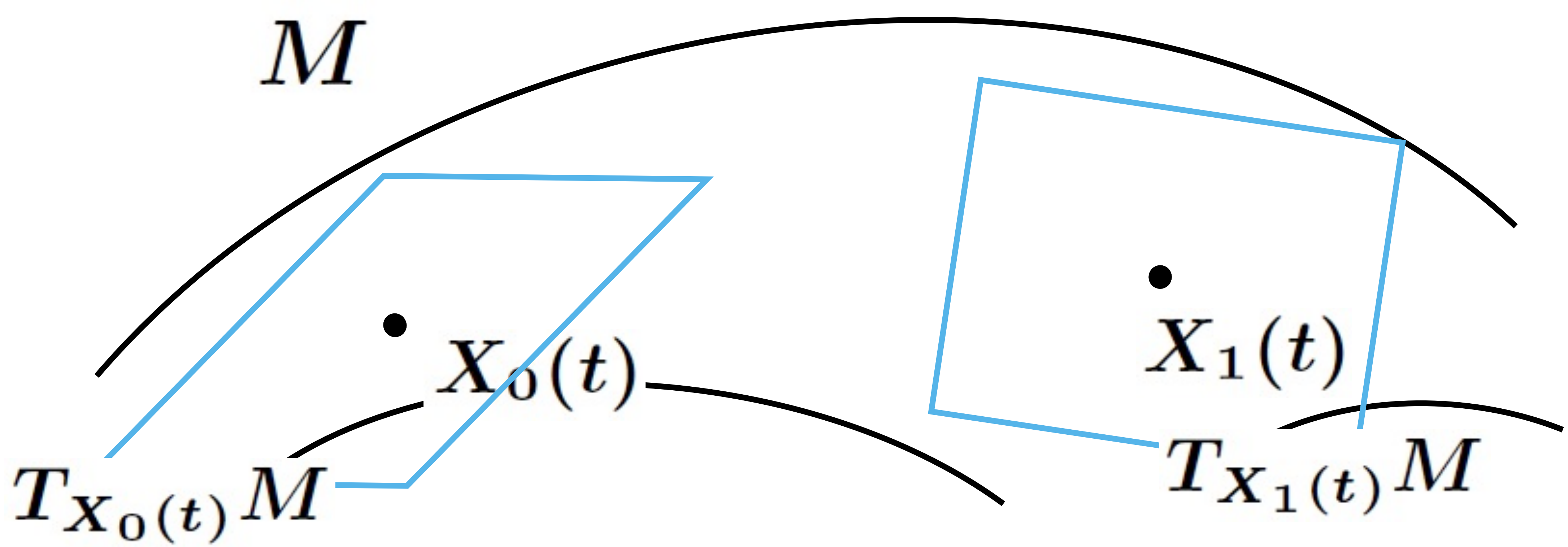

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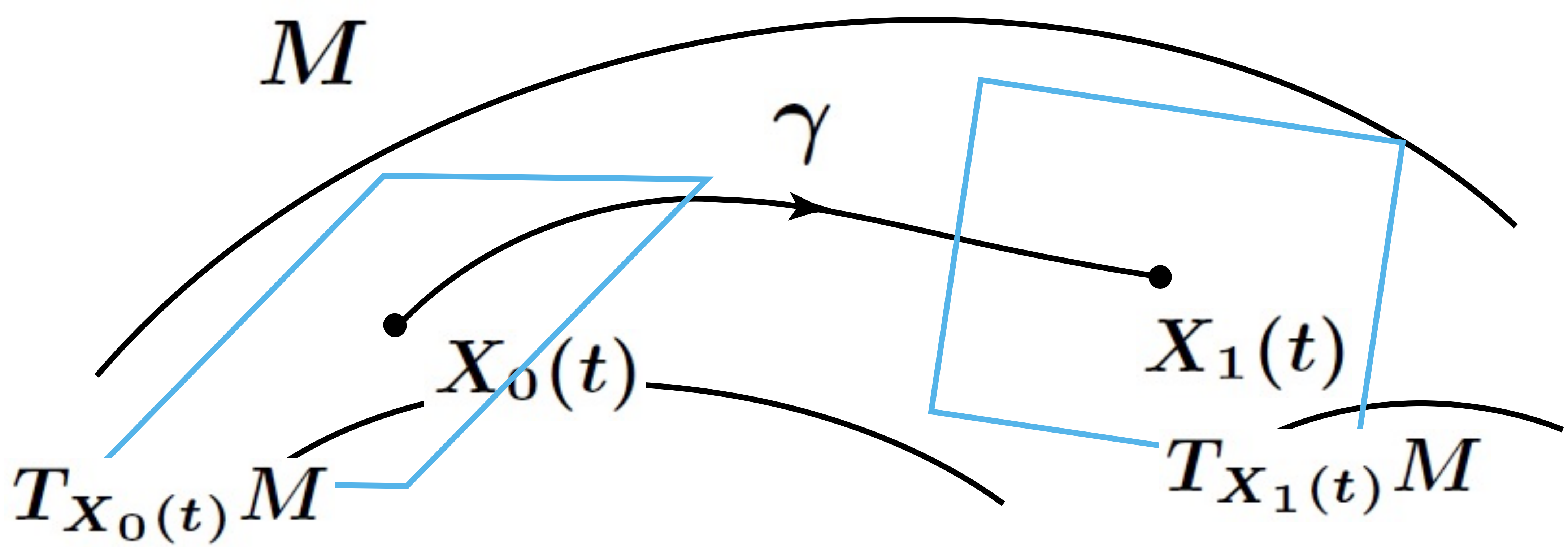

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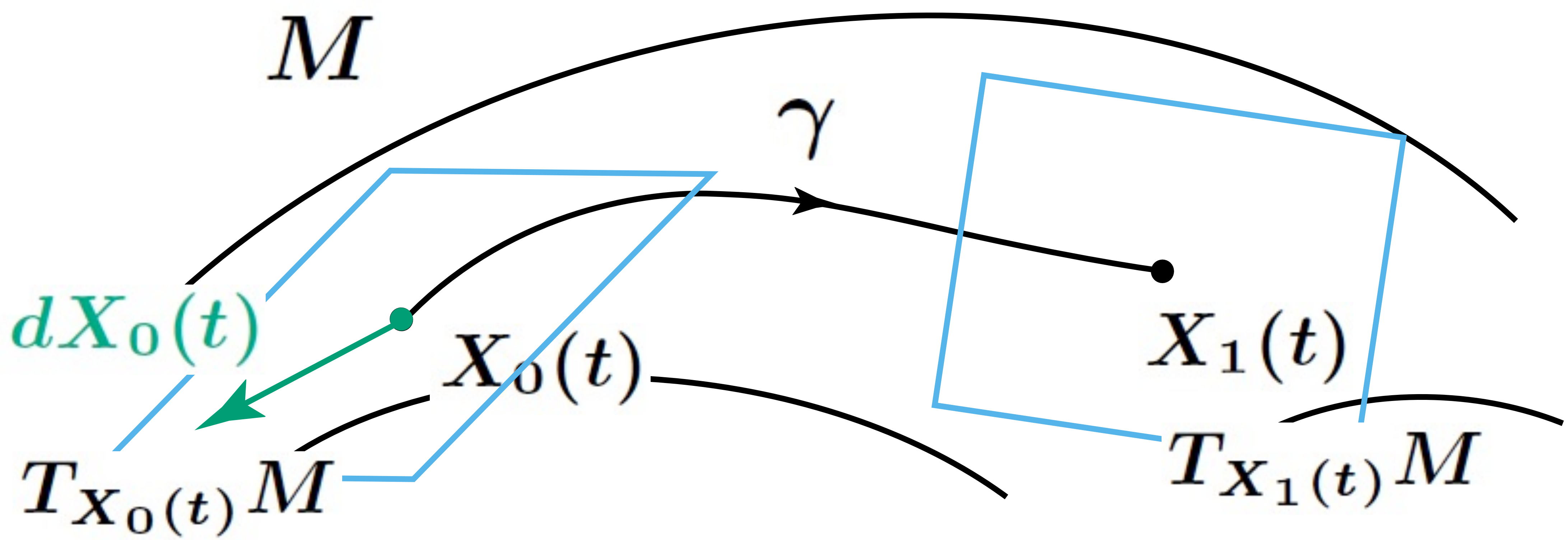

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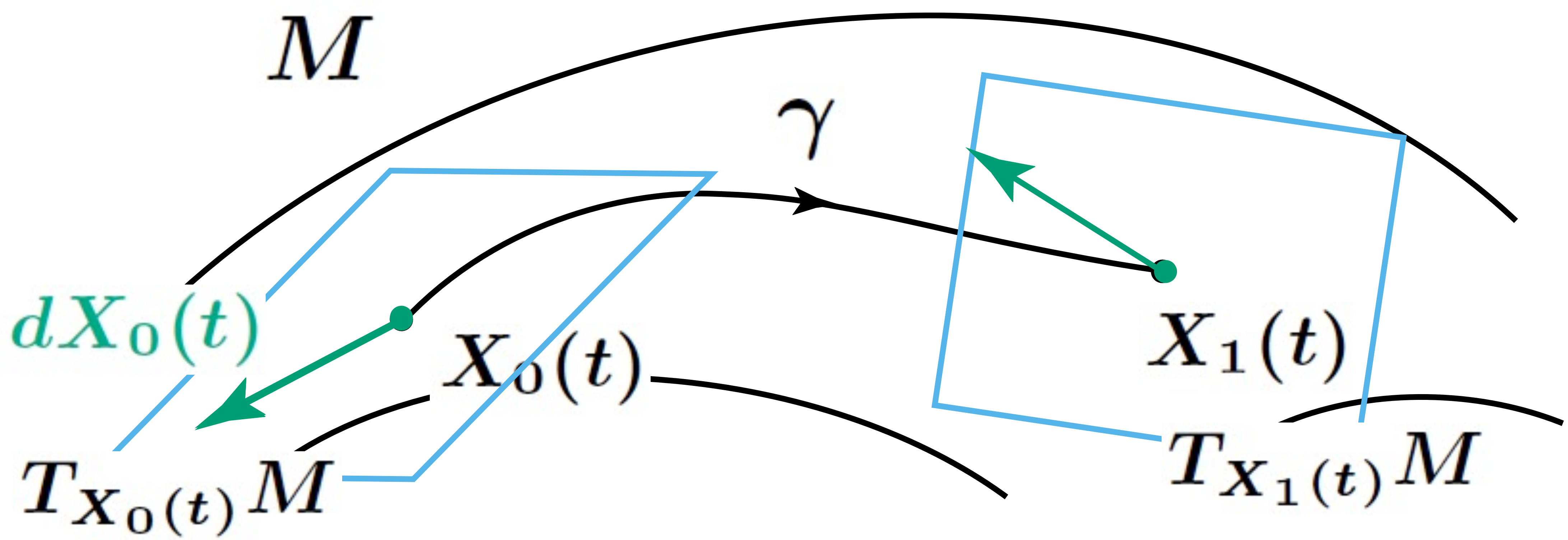

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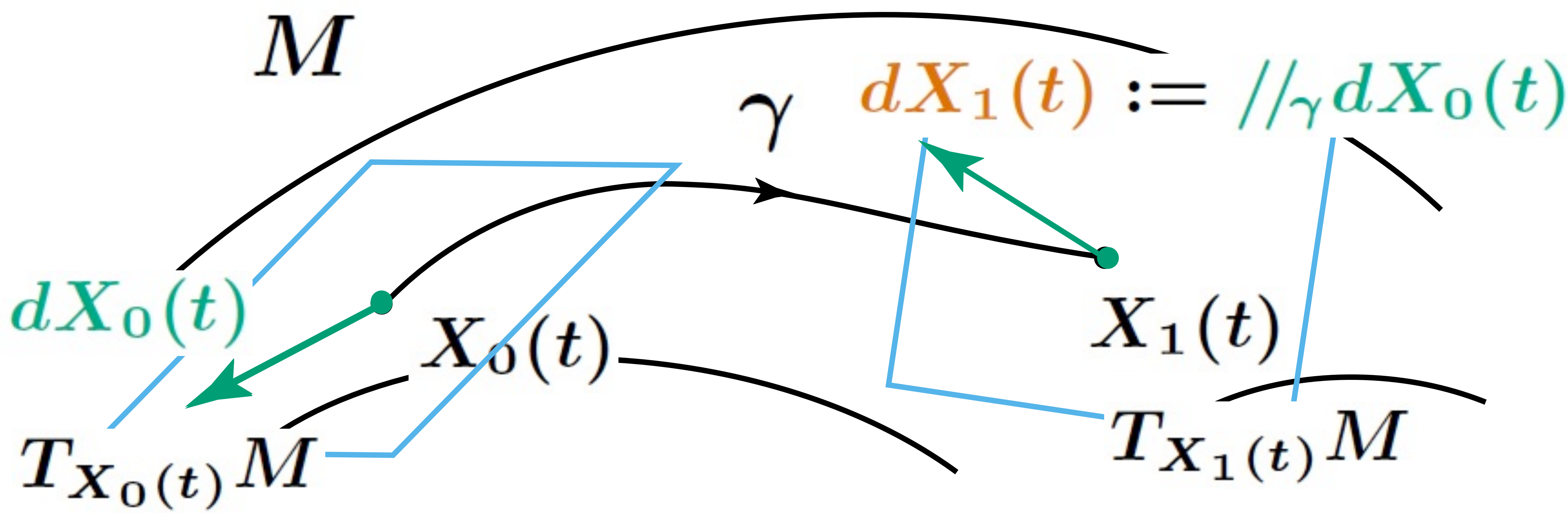

Heuristics:

$\rho(t) := d(X_0(t), X_1(t))$

$\Rightarrow$ By the Itô formula, under $\mathrm{Ric} \geq K$,

$$\boxed{d\rho(t) \leq K\rho(t)dt}$$

- Coupling by parallel transport
  $\Rightarrow \nabla d \cdot d(X_0(t), X_1(t)) = 0$
- Comparison theorem for 2nd variation
  $\Rightarrow (\nabla^2 d) \cdot d(X_0(t), X_1(t))^{\otimes 2} \leq K\rho(t)$

$\Rightarrow \mathbb{E}[\mathrm{e}^{pKt}\rho(t)^p] \searrow \quad \Rightarrow$ Conclusion $\quad \square$

Time-dependent metric $g(t)$

$P_t = P_{0\to t}$: transition semigroup of a $g(t)$-BM
(diffusion process generated by $\Delta_{g(t)}$)

Thm 2.1

Suppose $\mathrm{Ric}_{g(t)} \geq \frac{1}{2}\partial_t g(t) + K \quad (*)$

$\Rightarrow$ for any $\varphi : \mathbb{R}_+ \to \mathbb{R}$, $\nearrow$,

$$\mathcal{T}_{\varphi(\mathrm{e}^{Kt} d_{g(t)})}(P_t^* \mu_0, P_t^* \mu_1) \searrow$$

(e.g. $\varphi(u) = u^p$)

★ $K = 0$ & "$=$" in $(*)$ $\Leftrightarrow$ backward Ricci flow

Difficulty: singularity at cutlocus

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History

Time-homogeneous case

- [F.-Y.Wang '97]
- [von Renesse '04, K. '10] Approximation by RWs

Time-inhomogeneous case

- [McCann & Topping '10] via optimal transport
- [Arnaudon, Coulibaly & Thalmaier ' 09] Coupling of particles constituting a string
- [K.] Approximation by RWs

3. Coupling by spacetime parallel transport and Perel'man's $\mathcal{L}$-distance

(Metric g depends on $t \in [0, T]$)

Perel'man's $\mathcal{L}$-distance:

For $\gamma : [\tau_0, \tau_1] \to M$ (curve in spacetime),

$$\mathcal{L}(\gamma) := \int_{\tau_0}^{\tau_1} \sqrt{\tau} \left(|\dot{\gamma}(\tau)|^2_{g(\tau)} + R_{g(\tau)}(\gamma(\tau)) \right) d\tau$$

$$L(\tau_0, x_0; \tau_1, x_1) := \inf_{\substack{\gamma(\tau_i)=x_i \\ i=0,1}} \mathcal{L}(\gamma)$$

(Normalization): Given $0 \leq \bar{\tau}_0 < \bar{\tau}_1 \leq T$,

$\Theta_t(x_0, x_1) :=$

$$2(\sqrt{\bar{\tau}_1 t} - \sqrt{\bar{\tau}_0 t}) L(\bar{\tau}_0 t, x_0; \bar{\tau}_1 t, x_1) - 2n(\sqrt{\bar{\tau}_1 t} - \sqrt{\bar{\tau}_0 t})^2$$

Thm 3.1 [K. & Philipowski '11]

Suppose
$$\begin{cases} \partial_t g(t) = 2\,\mathrm{Ric}_{g(t)}, \\ \inf\limits_{\substack{V \in TM \\ t \in [0,T]}} \dfrac{\mathrm{Ric}_{g(t)}(V,V)}{g(t)(V,V)} > -\infty \end{cases}$$

$\Rightarrow \exists (X_0(\tau), X_1(\tau))$: coupling of $g(\tau)$-BMs

s.t. $(\Theta_t(X_0(\bar{\tau}_0 t), X_1(\bar{\tau}_1 t)))_{t \in [1, T/\bar{\tau}_1]}$

is a supermartingale

Cor 3.2 [K. & Philipowski '11]

$\forall \varphi$: $\nearrow$, concave & $\forall \mu_i(t)$: heat distributions,
$\mathcal{T}_{\varphi(\Theta_t)}(\mu_0(\bar{\tau}_0 t), \mu_1(\bar{\tau}_1 t)) \searrow$

- [Topping '09]: $\mathcal{T}_{\Theta_t}(\mu_0(\bar{\tau}_0 t), \mu_1(\bar{\tau}_1 t)) \searrow$ when M: cpt, via optimal transport techniques
  ($\Rightarrow$ Monotonicity of Perel'man's $\mathcal{W}$-entropy)

Strategy of the Proof

- Properties of $\mathcal{L}$-distance being analogous to the Riem. distance
  - $\mathcal{L}$-geodesic, 1st & 2nd variation of $\mathcal{L}$-distance, $\mathcal{L}$-index lemma, $\mathcal{L}$-cut locus
- Approximation by RWs $(\boldsymbol{X_0^{\varepsilon}(t)}, \boldsymbol{X_1^{\varepsilon}(t)})$
- Coupling of $dX_0^{\varepsilon}(\bar{\tau}_0 \cdot)(t)$ and $dX_1^{\varepsilon}(\bar{\tau}_1 \cdot)(t)$ by spacetime-parallel transport along $\mathcal{L}$-geodesic

Why does the martingale part survive?

- For $\mathcal{L}$-geodesic γ, $\sqrt{u}\dot{\gamma}_u$ is NOT spacetime parallel to γ

- "speed" of $X_0(\bar{\tau}_0 t)$ and $X_1(\bar{\tau}_1 t)$ is different

Why does the martingale part survive?

Spacetime parallel transport

For $\gamma : [s, t] \to M$ & V: vector field along γ,

V: spacetime parallel

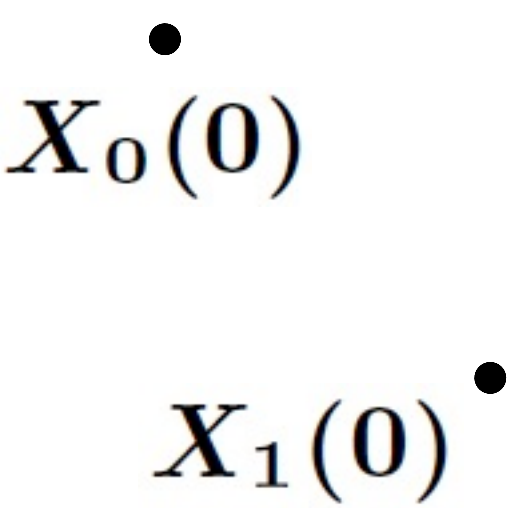
$$\overset{\text{def}}{\Leftrightarrow} \nabla^{g(u)}_{\dot\gamma(u)} V(u) = -\frac{1}{2}\partial_u g(u)^\sharp V(u)$$

- For $\mathcal{L}$-geodesic γ, $\sqrt{u}\dot\gamma_u$ is NOT spacetime parallel to γ
- "speed" of $X_0(\bar\tau_0 t)$ and $X_1(\bar\tau_1 t)$ is different

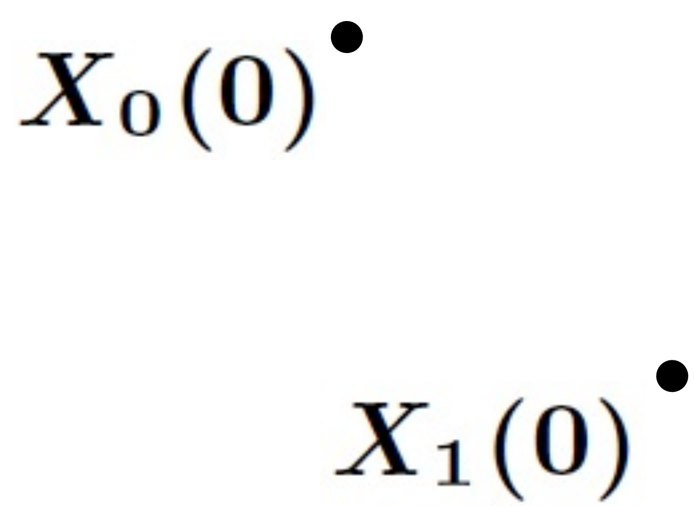
4. Coupling by reflection

Example ($M = \mathbb{R}^n$)

parallel transport



reflection



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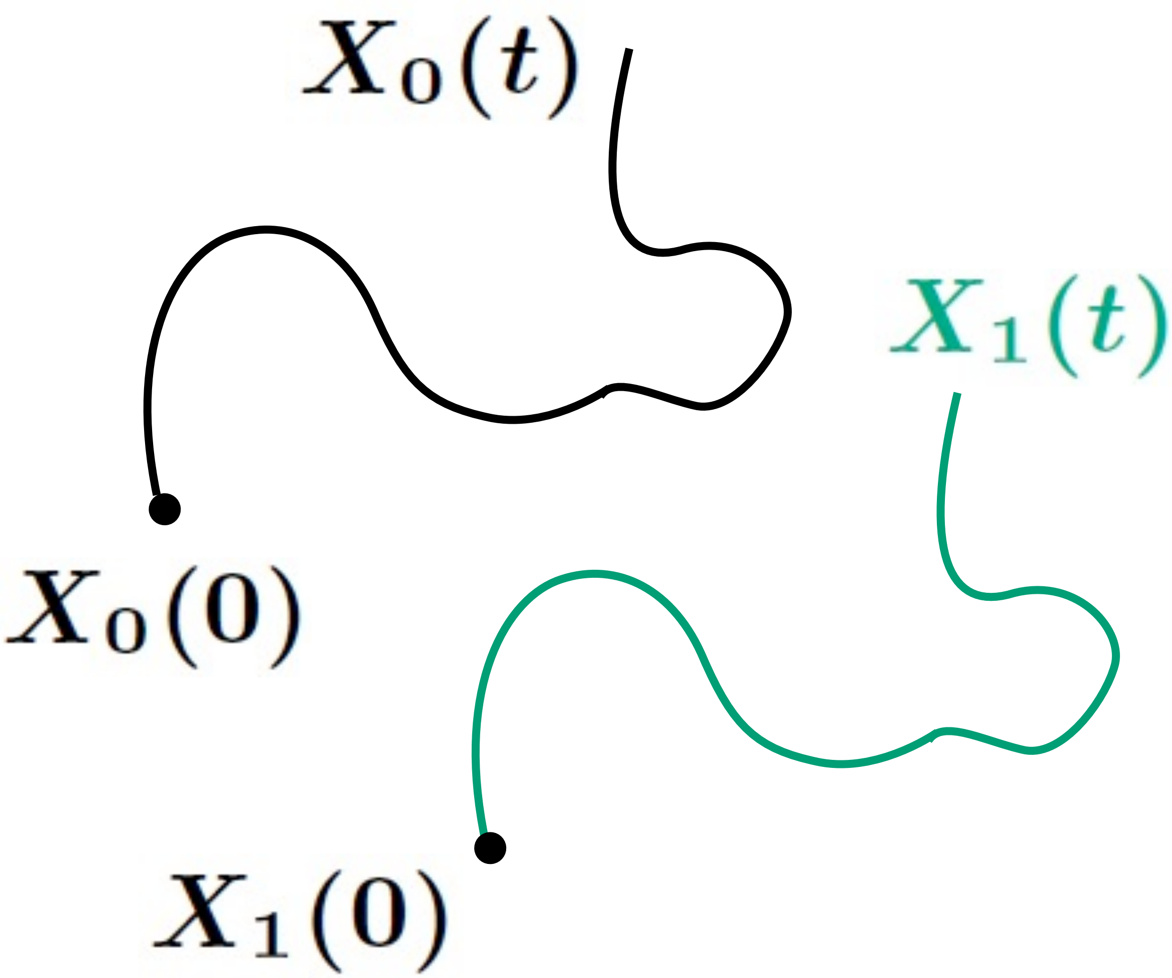

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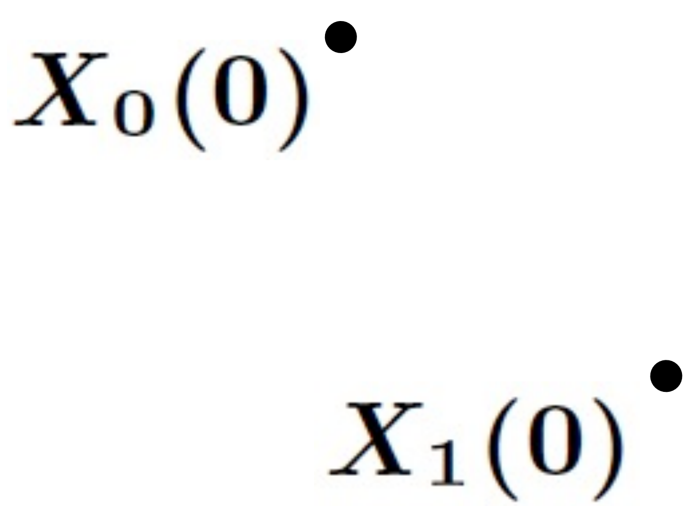

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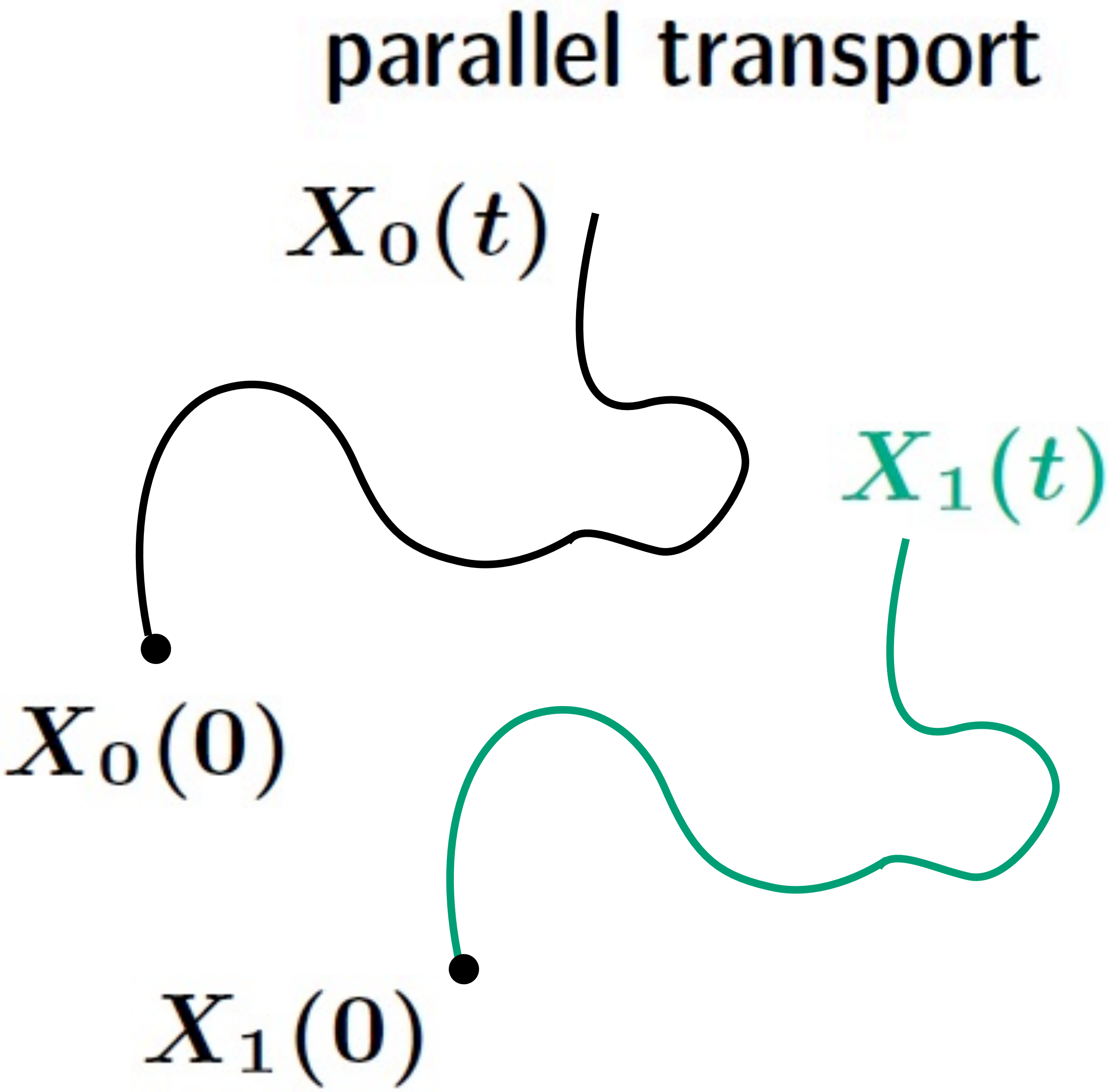

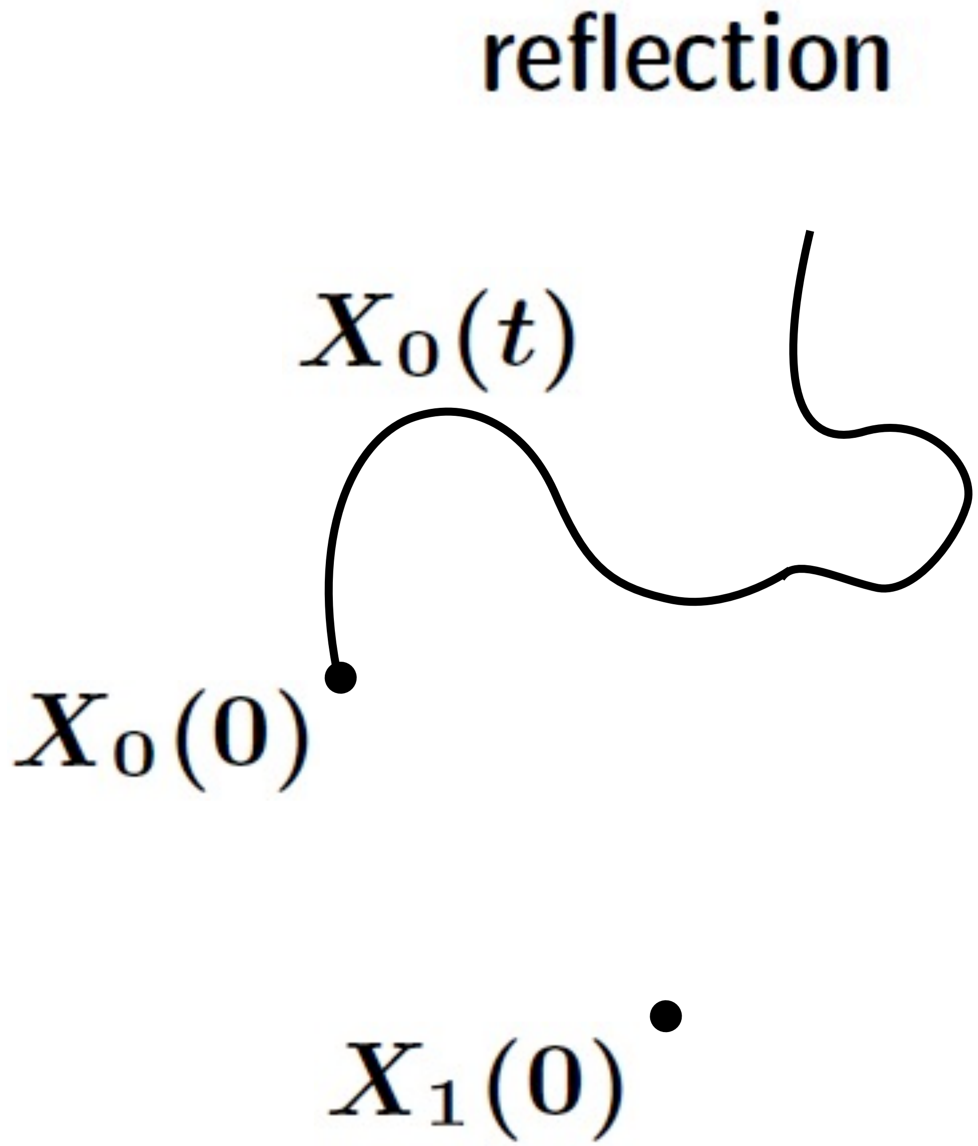

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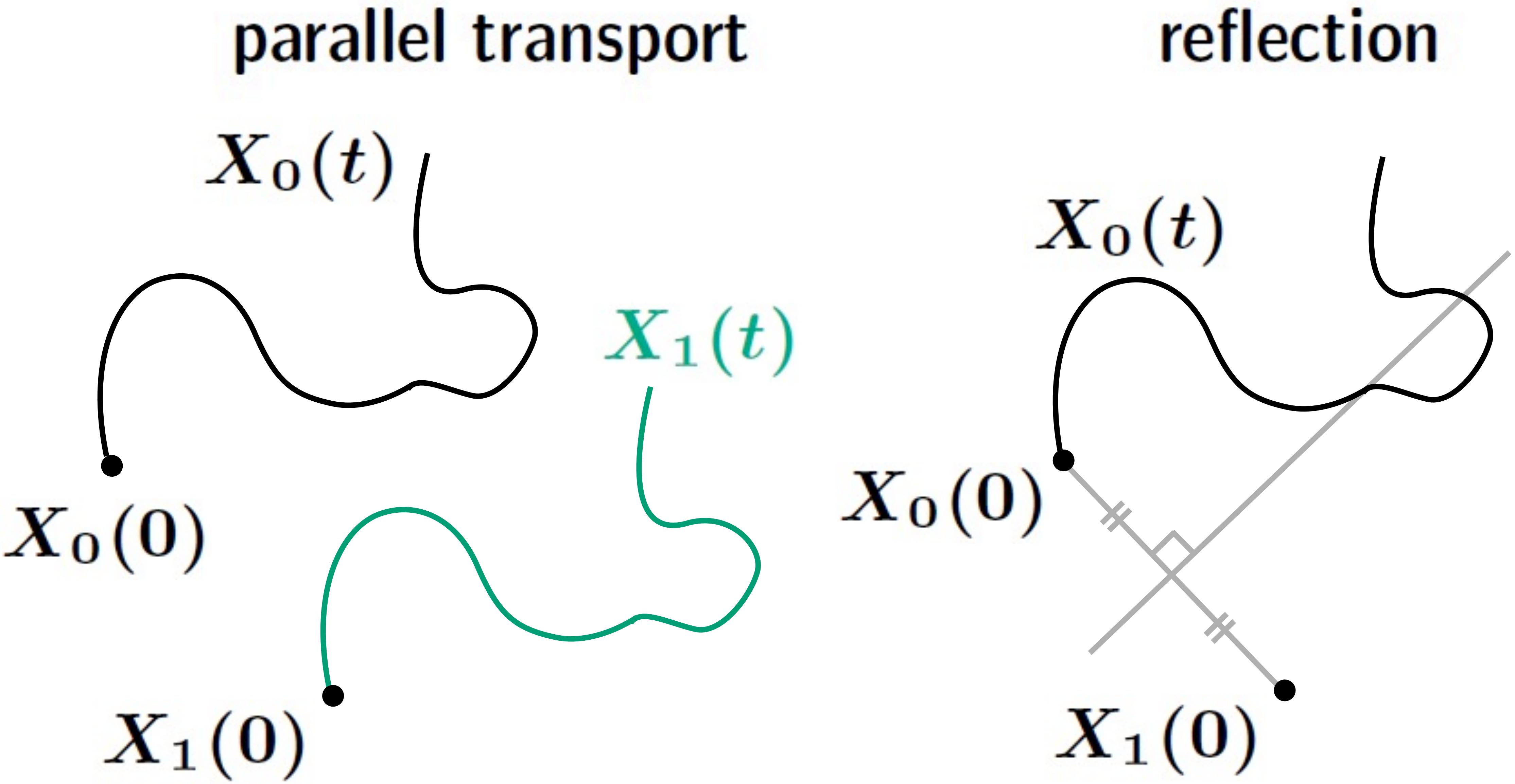

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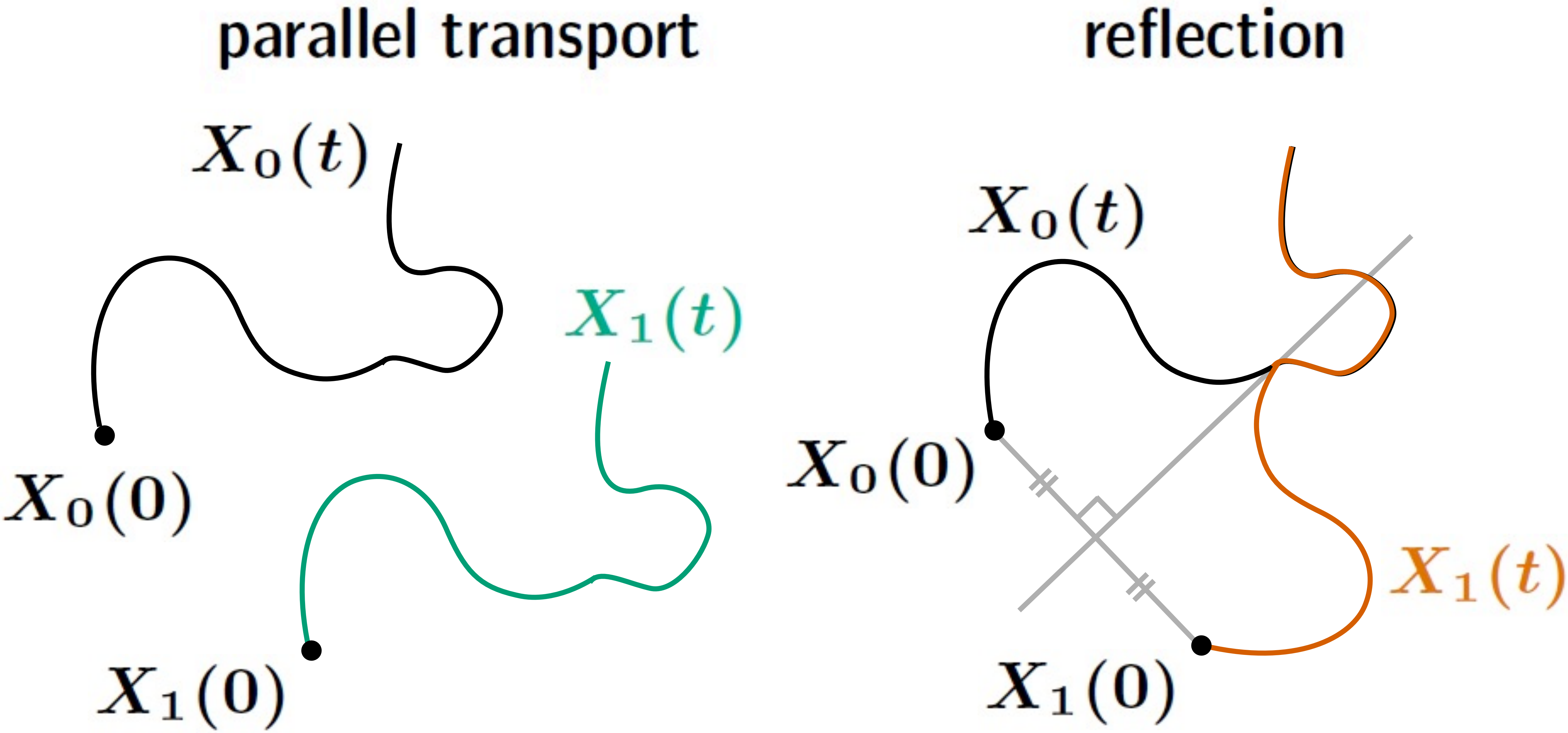

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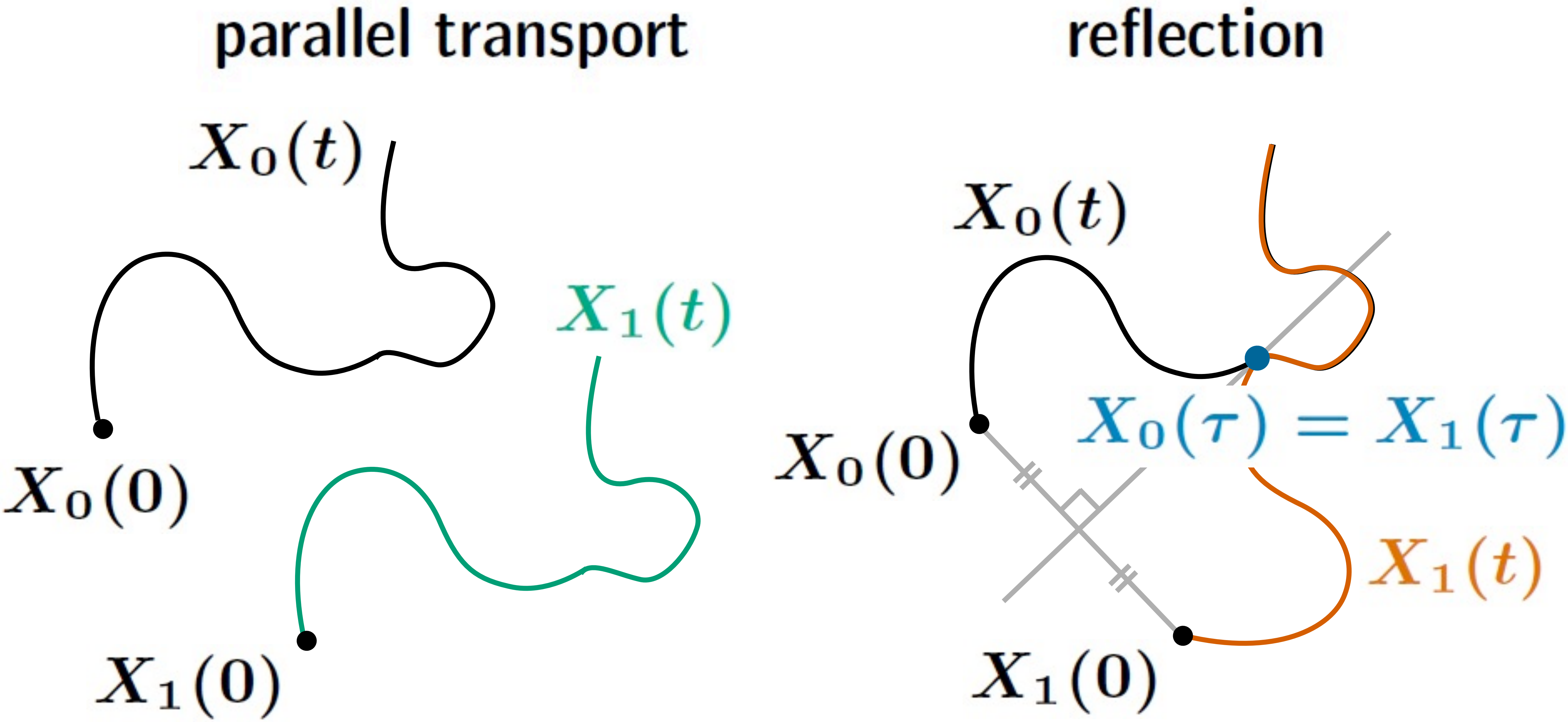
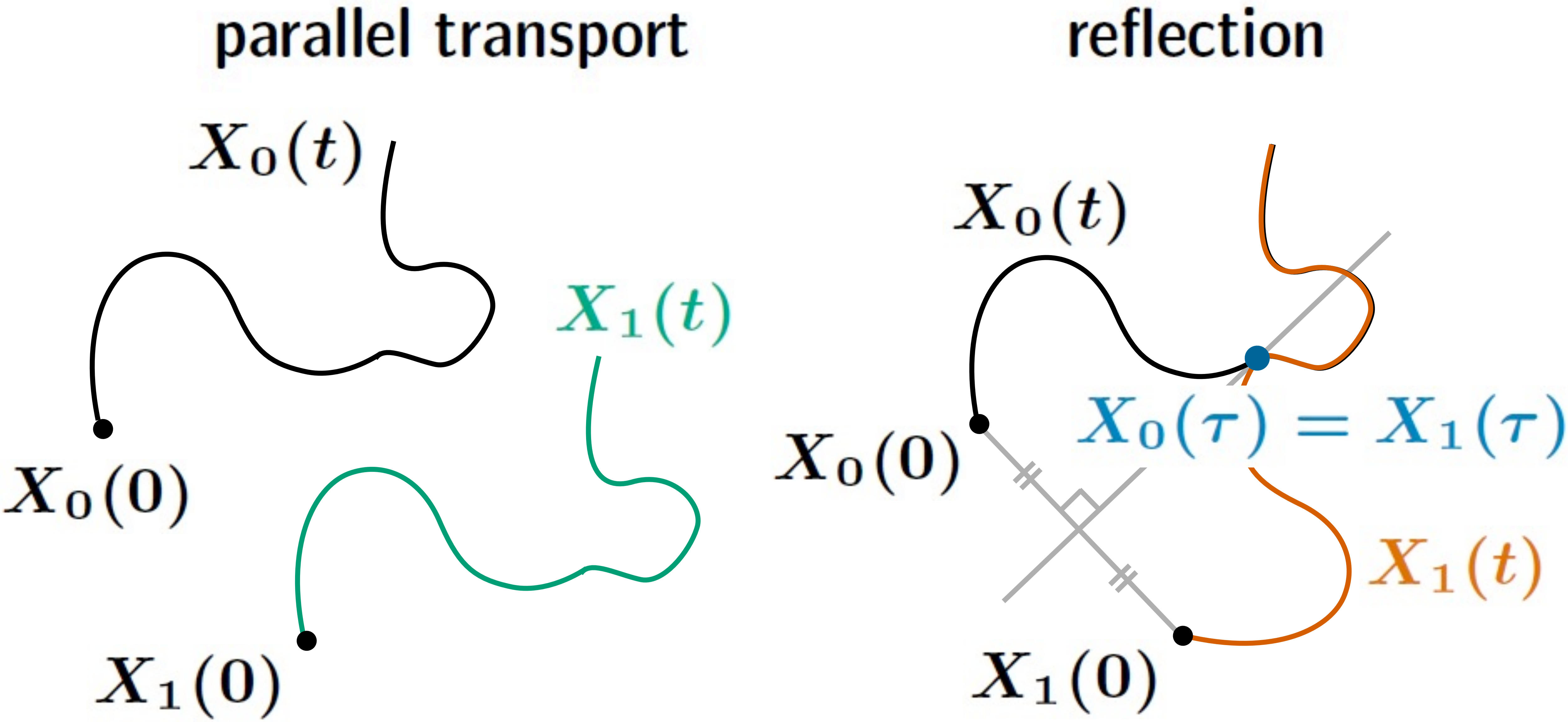


(τ: first time to meet)

Example ($M = \mathbb{R}^n$)



$\Rightarrow$ Est. of $\mathbb{P}[\tau \geq t]$ (τ: first time to meet)

$\Rightarrow$ Est. of total variations between distributions

On M: compl. Riem. mfd:

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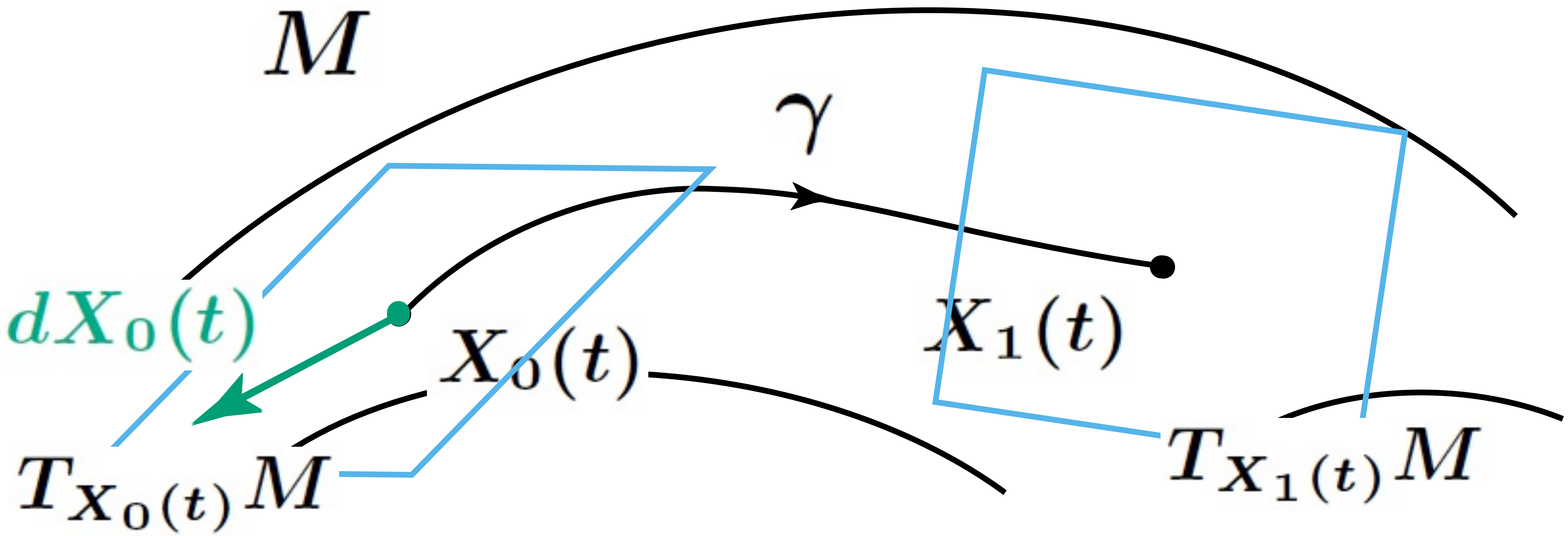

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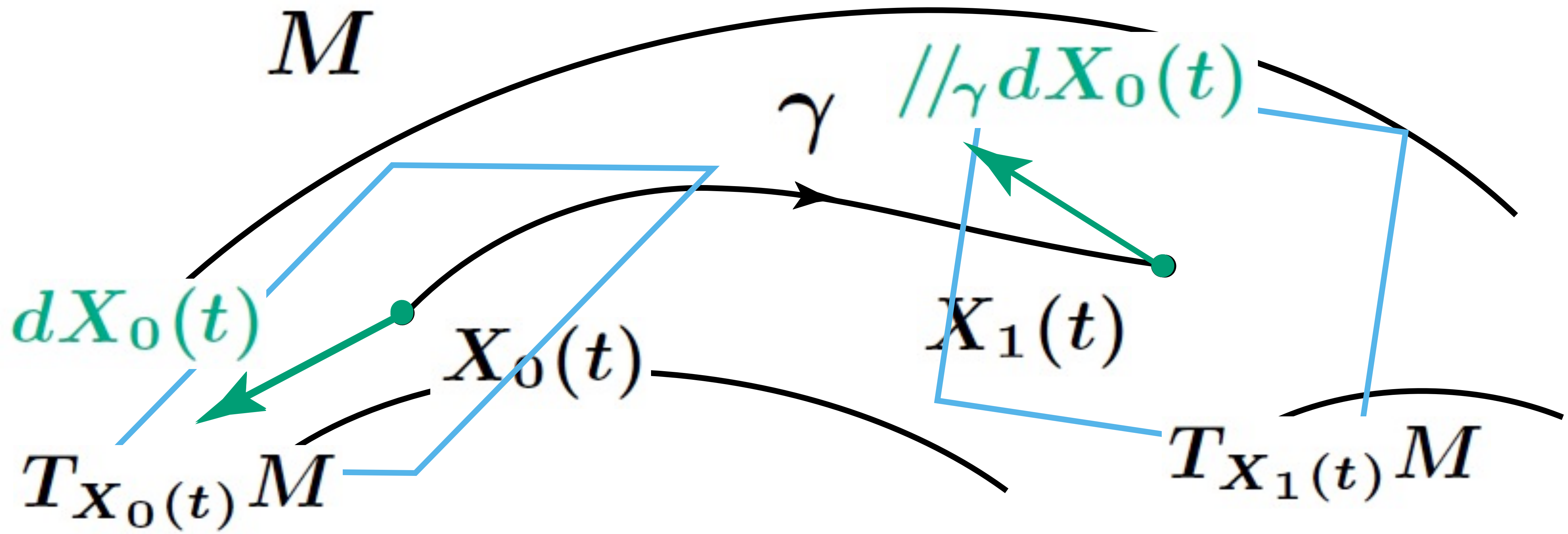

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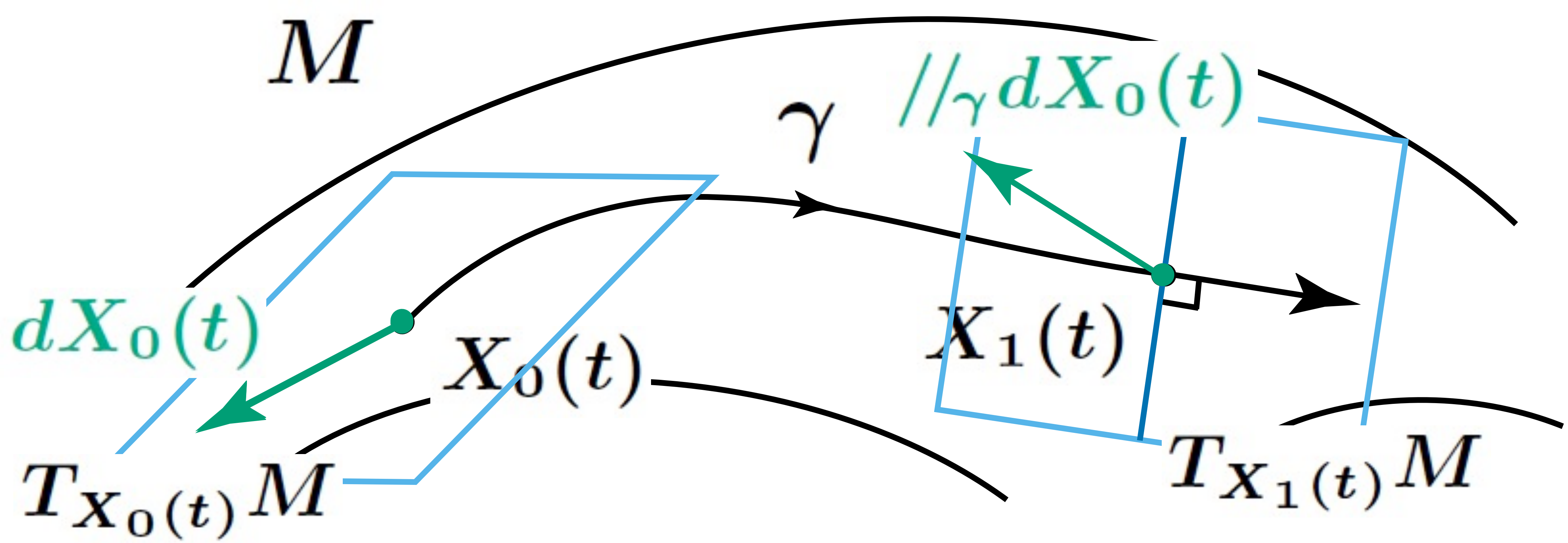

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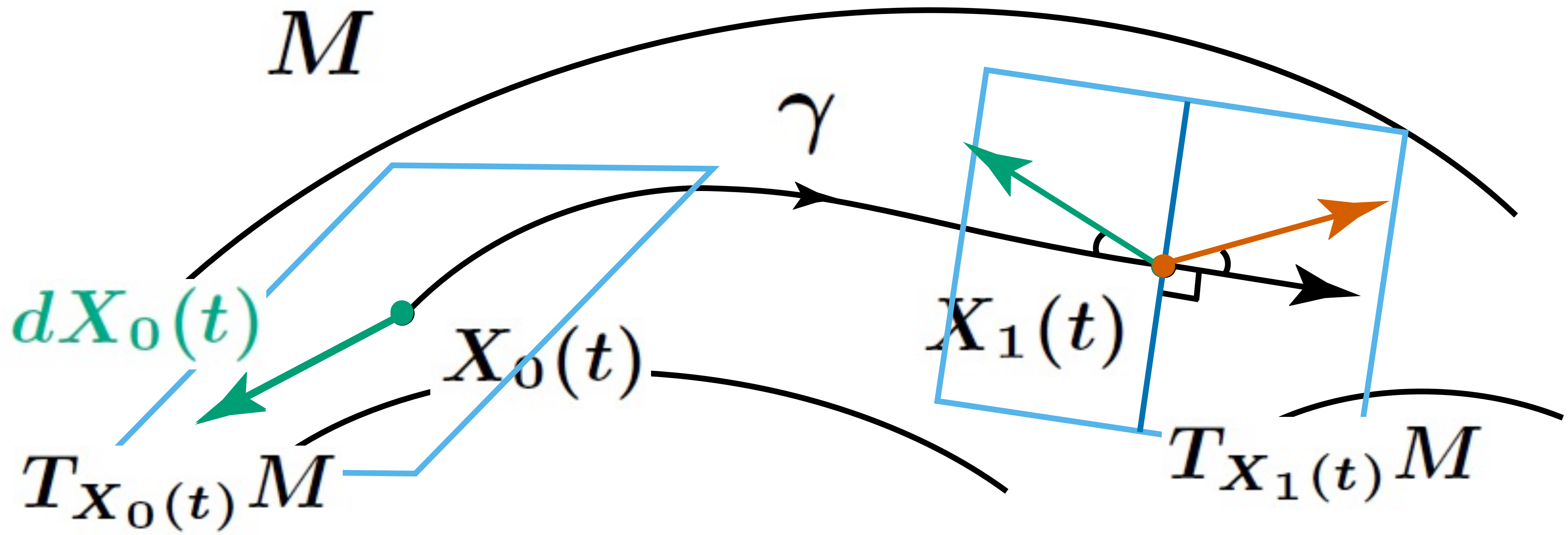

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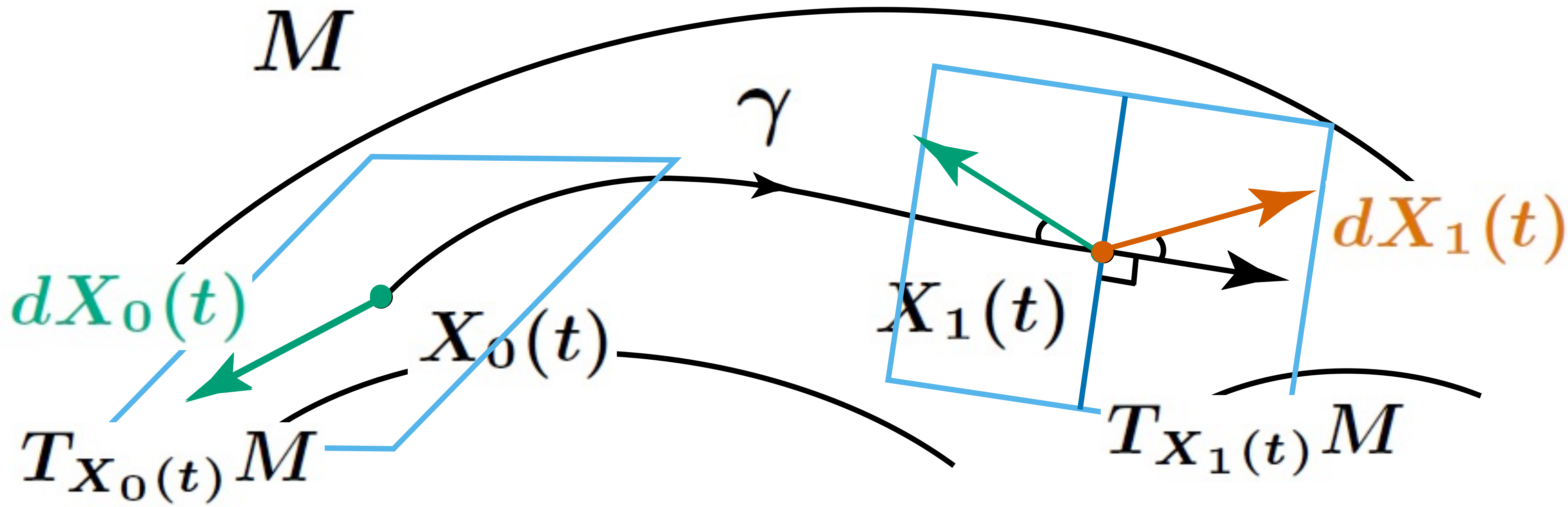

History

[Kendall '86, Cranston '91, F.-Y.Wang '97, '05, von Renesse '04, K. '10, K. . . .]

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Q.

Can we formulate properties of coupling by reflection in terms of transportation costs?

Framework

† Z: C^1-vector field

† $X^x(t)$: diffusion process generated by $\Delta + Z$
($X(t)$: BM $\Leftrightarrow$ $Z = 0$)

Bakry-Émery Ricci tensor

For $N \in [n, \infty]$,

$$\mathrm{Ric}^{Z,N} := \mathrm{Ric} - (\nabla Z)^{\mathrm{sym}} - \frac{1}{N-n} Z \otimes Z$$

Ass

$\mathrm{Ric}^{Z,N} \geq K$ for some $K \in \mathbb{R}$ & $N \in [n, \infty]$

Remarks

- When $Z = 0$,

$$\boxed{\text{Ass} \Leftrightarrow \mathrm{Ric} \geq K \text{ \& } n \leq N}$$

- The Riem. metric g and Z CAN depend on t:

$$\mathrm{Ric}_{g(t)}^{Z(t),\infty} \geq \frac{1}{2}\partial_t g(t) + K$$

- $K > 0$ & $N < \infty \Rightarrow$ max. diam. thm. [K.]

Thm 4.1 [K. & Sturm]

$(X_1(t), X_2(t))$: a coupling by refl. of two BMs.

$\Rightarrow$ For $\varphi_t = \varphi_t^{N,K} \; : \; [0,\infty) \to [0,1]$ given below,

$$\mathbb{E}[\varphi_{t-s}(d(X_1(s), X_2(s)))] \searrow$$

in $s \in [0, t]$

Cor 4.2 [ibid.]

For $t > 0$, $\mu_1, \mu_2 \in \mathcal{P}(M)$,

$$\mathcal{T}_{\varphi_{t-s}(d)}(P_s^* \mu_1, P_s^* \mu_2) \searrow \text{ in } s \in [0, t]$$

Definition of $\varphi_t^{K,N}(a)$ (for $N \in \mathbb{N}$)

$$\varphi_t^{K,N}(a) := \frac{1}{2} \left\| \tilde{P}_t^* \delta_{\tilde{x}} - \tilde{P}_t^* \delta_{\tilde{y}} \right\|_{\mathrm{TV}}$$

- $\tilde{P}_t$: heat semigr. on the spaceform $\mathbb{M}_{K,N}$ ($\mathbb{M}_{K,N}$: sphere, Eucl. sp. or hyp. sp.)
- $d(\tilde{x}, \tilde{y}) = a$

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———— ○ ———— ○ ————

★ $\varphi_t^{K,N}(a)$ can be described in terms of a sol. to SDE which $d(X_0(t), X_1(t))$ on $\mathbb{M}_{K,N}$ solves

$\Rightarrow$ Definition for $N \notin \mathbb{N}$

Properties of φ_t

- $\varphi_t \nearrow$, concave, $\varphi_t(0) = 0$ ($\Rightarrow \varphi_t(d)$: dist.)
- $\varphi_\cdot(a) \searrow$
- $\varphi_0 = 1_{(0,\infty)}$
  $\left(\Rightarrow \mathcal{T}_{\varphi_0(d)}(\mu_0, \mu_1) = \frac{1}{2}\|\mu_0 - \mu_1\|_{\mathrm{TV}}\right)$
- $\left.\begin{array}{c} N \le N' \\ K \ge K' \end{array}\right\} \Rightarrow \varphi_t^{K,N}(a) \le \varphi_t^{K',N'}(a)$

Sketch of the proof (when $N \in \mathbb{N}$)

$(\tilde{X}_0(t), \tilde{X}_1(t))$: coupling by refl. on $\mathbb{M}_{\boldsymbol{K},\boldsymbol{N}}$

- If $d(X_0(0), X_1(0)) = d(\tilde{X}_0(0), \tilde{X}_1(0))$, then

  $$\text{“}d(X_0(t), X_1(t)) \le d(\tilde{X}_0(t), \tilde{X}_1(t))\text{”}$$

  (under a suitable realization)

- $\mathbb{E}[\varphi_{t-s}(d(\tilde{X}_0(s), \tilde{X}_1(s)))]$: const. in s $\square$

Applications 1: Comparison thm for total variations

$$\mathcal{T}_{\varphi_0(d)}(P_t^*\delta_x, P_t^*\delta_y) \le \mathcal{T}_{\varphi_t(d)}(\delta_x, \delta_y)$$

$$\Downarrow$$

Cor 4.3 [ibid.]

$$\left\| P_t^*\delta_x - P_t^*\delta_y \right\|_{\mathrm{TV}} \le 2\varphi_t(d(x,y))$$

When $N \in \mathbb{N}$,

$$\text{(RHS)} = \left\| \tilde{P}_t^*\delta_{\tilde{x}} - \tilde{P}_t^*\delta_{\tilde{y}} \right\|_{\mathrm{TV}}$$

with $d_{\mathbf{M}_{\boldsymbol{K},\boldsymbol{N}}}(\tilde{x},\tilde{y}) = d(x,y)$

Applications 2: New monotonicity (when $K < 0$)

$$\exists \lim_{t\to\infty} \varphi_t^{K,N}(a) =: \Phi^{K,N}(a) \quad (> 0 \text{ iff } a > 0)$$

$$\Downarrow$$

Cor 4.4 [ibid.]

$$\mathcal{T}_{\Phi^{K,N}(d)}(P_t^* \mu_0, P_t^* \mu_1) \searrow \text{ in } t \geq 0$$

Recall:

$$\mathrm{Ric} \geq K \Rightarrow \mathcal{T}_{\mathrm{e}^{pKt} d^p}(P_t^* \mu_0, P_t^* \mu_1) \searrow$$

$$\chi(r) := \mu([-r,r]),\ \mu \sim N(0,1)$$

- $\Phi^{K,\infty}(a) = \chi\left(\frac{a\sqrt{-K}}{2}\right)$
- $\Phi^{K,N}(a)$

$$= \int_0^\infty \chi\left(\sqrt{\frac{u}{2}}\sinh\left(\frac{a}{2}\sqrt{\frac{-K}{N-1}}\right)\right)\nu(du),$$

where ν: Gamma distr. of param. $\dfrac{N-1}{2}$,

i.e. $\nu(dx) = \Gamma\left(\frac{N-1}{2}\right)^{-1} x^{(N-3)/2}\mathrm{e}^{-x}dx$

Further results

- $\exists$ an integral expression of $\varphi_t(a)$
  ($\Rightarrow$ concavity of φ_t)

- There's an explicit expression of $\varphi_t(a)$
  (but it's complicated when $N < \infty$ & $K \neq 0$)

- The conclusion of Cor 4.2 is stable under Gromov-Hausdorff convergence with a uniform curvature-dimension bound

5. Curvature-dimension conditions

(Metric g is time-independent)

Interest: Characterization of

$$\boxed{\mathrm{Ric} \geq K \ \&\ \dim M \leq N}$$

in terms of transportation cost

(For $\mathrm{Ric} \geq K$ only, given in Thm 1.1 (ii))

—————— ∘ —————— ∘ ——————

(1) ∃ Analytic characterization [Bakry & Émery '84]

(2) ∃ Characterization by the heat semigroup [F.-Y. Wang '11]

(3) ∃ Characterization via optimal transportation [Sturm '06, Lott & Villani '09]

$$\mathcal{L} := \Delta + Z,\ P_t := \mathrm{e}^{t\mathcal{L}}$$

Thm 5.1 ([Bakry & Émery '84, F.-Y. Wang '11])

For $N \in [n, \infty]$ & $K \in \mathbb{R}$, TFAE:

(0) $\mathrm{Ric}^{Z,N} \geq K$

(1) $$\frac{1}{2}\mathcal{L}(|\nabla f|^2) - \langle \nabla f, \nabla \mathcal{L} f \rangle \geq K|\nabla f|^2 + \frac{1}{N}(\mathcal{L}f)^2$$

(2) $$|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt} P_t(|\nabla f|^2) + \frac{1 - \mathrm{e}^{-2Kt}}{NK}(\mathcal{L}P_t f)^2$$

Thm 5.2 [K.]

Let $p \in [2, \infty)$. Thm 5.1 (0) yields the following:

(4) $\mathcal{T}_{d^p}(P_s^* \mu_0, P_t^* \mu_1)^{2/p}$

$$\leq \frac{e^{-2Kt} - e^{-2Ks}}{2K(s-t)} \mathcal{T}_{d^p}(\mu_0, \mu_1)^{2/p} + \frac{(N+p-2)}{2}(s-t)\log\left(\frac{1-e^{-2Ks}}{1-e^{-2Kt}}\right)$$

★ By letting $s \longrightarrow t$, (4) yields Thm 1.1 (ii)

Thm 5.3 [K.]

For $q \in (1,2]$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

Thm 5.2 (4) is equivalent to the following:

(2)' $$|\nabla P_t f|^2 \le e^{-2Kt} P_t(|\nabla f|^q)^{2/q} + \frac{1 - e^{-2Kt}}{(N + p - 2)K} (\mathcal{L} P_t f)^2$$

In particular, (4) is equivalent to (0) when $p = 2$

★ When $N = \infty$, Thm 5.3 is known [K. '10]

Rough sketch of the proof

- Thm 5.3 $\cdots$ extension of the argument
  when $N = \infty$
- Thm 5.2 $\cdots$ coupling method for BMs
  with different speed
  & analysis of transportation costs

Application ($p = 2$)

(4) $\mathcal{T}_{d^2}(P_s^* \mu_0, P_t^* \mu_1)$

$$\leq \frac{\mathrm{e}^{-2Kt} - \mathrm{e}^{-2Ks}}{2K(s-t)} \mathcal{T}_{d^2}(\mu_0, \mu_1) + \frac{N}{2}(s-t) \log\left(\frac{1 - \mathrm{e}^{-2Ks}}{1 - \mathrm{e}^{-2Kt}} \right)$$

$\Downarrow$ $\mu_0 = \mu_1 = \mu$,

dividing by $(s-t)^2$ & $s \to t$

Since $\dfrac{\mathcal{T}_{d^2}(P_s^*\mu, P_t^*\mu)}{(s-t)^2} \to F(P_t\mu)$

(Fisher information),

$$\int_M \frac{|\nabla \rho_t|^2}{\rho_t}\, d\,\mathrm{vol}_g \le \frac{NK}{\mathrm{e}^{2Kt}-1},$$

where $\dfrac{dP_t\mu}{d\,\mathrm{vol}_g} = \rho_t$

★ (RHS) = Fisher info. of Ornstein-Uhlenbeck process on $\mathbb{R}^N$

Questions

- (Direct) relation with Sturm-Lott-Villani's curvature-dimension condition
- Relation between inequalities with different p
- Further applications (Laplacian comparison? Sobolev inequalities?)
- Different formulations?