Wasserstein contractions associated with the curvature-dimension condition

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1. Introduction

Framework

M: cpl. Riem. mfd., $\dim \geq 2$, $\partial M = \emptyset$
$P_t = e^{t\Delta}$: heat semigroup on M, $P_t 1 \equiv 1$

Goal

Characterize

$$\mathrm{Ric} \geq K \ \& \ \dim M \leq N$$

in terms of behavior of Wasserstein distances between heat distributions $P_t^* \mu$ ($\mu \in \mathcal{P}(M)$, $t > 0$)

lower Ricci bound on metric meas. sp.

Recent developments:

Generalization of "$\mathrm{Ric} \geq K$"

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  or only "metric and measure"

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Generalization of "$\mathrm{Ric} \geq K$"

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  or only "metric and measure"
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  $\Rightarrow$ e.g. "Stable" sufficient cond.

  for Lipschitz regularity of $P_t f$

1. Introduction

2. Known results for lower Ricci bounds

3. Curvature-dimension conditions

4. Proofs & extensions

4.1 Duality

4.2 Coupling methods

4.3 Questions

lower Ricci curv. bound

For $K \in \mathbb{R}$, TFAE ([von Renesse & Sturm '05] etc.):

(i) $\mathrm{Ric} \geq K$

(ii) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$,

(iii) $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2)$

(iv) $\frac{1}{2}(\Delta |\nabla f|^2 - 2\langle \nabla f, \nabla \Delta f \rangle) \geq K |\nabla f|^2$

(v) Ent is K-convex w.r.t. W_2

How important?

- **(iii)(iv)** has rich applications
  in functional ineq. & differential geometry,
  e.g. quantitative Lipschitz regularization of P_t
  [Bakry & Émery etc.]

$\Rightarrow$ More applications if "$\mathrm{dim}\, M$" is involved
(e.g. Harnack inequalities)

(iii) $|\nabla P_t f|^2 \le e^{-2Kt} P_t(|\nabla f|^2)$

(iv) $\frac{1}{2}(\Delta|\nabla f|^2 - 2\langle \nabla f, \nabla \Delta f\rangle) \ge K|\nabla f|^2$

How important?

(v) Ent: K-convex w.r.t. W_2

- **(v)** makes sense even on met. meas. sp.'s & stable (e.g., under Gromov-Hausdorff conv.)

  [Sturm '06, Lott & Villani '09]

  $\Rightarrow$ extension of **(ii)(iii)(iv)** to singular spaces

  [Ambrosio, Gigli & Savaré] etc.

Implications

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$\Updownarrow$ $\leftsquigarrow$ Bochner-Weitzenböck formula

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(ii) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

Implications

On non-smooth sp.:

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Implications

On non-smooth sp.:

(v) Ent is K-convex w.r.t. W_2

- Identification of $P_t^* \mu$ with the gradient flow of Ent in $(\mathcal{P}(M), W_2)$

$\Downarrow$ $\leftrightsquigarrow$

- Linearity of heat flow w.r.t. initial data

[Gigli, K. & Ohta]: cpt. Alexandrov sp.

[Ambrosio, Gigli & Savaré]: met. meas. sp.

[Koskela & Zhou]

(ii) $W_2(P_t^* \mu_0, P_t^* \mu_1) \le e^{-Kt} W_2(\mu_0, \mu_1)$

Implications

On non-smooth sp.:

† TFAE [Ambrosio, Gigli & Savaré]

(v)* Ent is K-convex w.r.t.W_2 & linearity of heat flow

(vi) ${}^\forall\mu_0$, ${}^\exists$sol $(\mu_t)_{t\ge 0}$ to **EVI**$_K$ of Ent: for ${}^\forall\nu$,

$$\frac{d}{dt}\left(\frac{W_2(\mu_t,\nu)^2}{2}\right)+\frac{K}{2}W_2(\mu_t,\nu)^2 + \mathrm{Ent}(\mu_t)\le \mathrm{Ent}(\nu)$$

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★ **(iii)** ⇒ **(v)** under a suitable ass.(incl. linearity)

[Ambrosio, Gigli & Savaré]

Summary of implications (when the heat flow is linear)

$$\begin{array}{ccccc} \textbf{(vi)}\ \text{EVI} & \Longleftrightarrow & \textbf{(v)}\ \text{Ent} & & \\ \Downarrow & & \Uparrow & & \\ \textbf{(ii)}\ \text{Wass.} & \Longleftrightarrow & \textbf{(iii)}\ \boldsymbol{\nabla}\text{-est.} & \Longleftrightarrow & \textbf{(iv)}\ \boldsymbol{\Gamma_2} \end{array}$$

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$$

What we did for $\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$:

- Formulate a missing condition corresponding to **(ii)**
- Extension of the implication **(ii)** $\Leftrightarrow$ **(iii)** (even in an abstract setting)
- Another approach based on a coupling method

Remarks

(1) On a Riem. mfd., stronger results follow:

- Constructing a coupling of Brownian motions yields

  (i) $\Rightarrow W_\infty(P_t^* \mu_0, P_t^* \mu_1) \le e^{-Kt} W_\infty(\mu_0, \mu_1)$

- Γ_2-calculus yields

  (i) $\Rightarrow |\nabla P_t f| \le e^{-Kt} P_t(|\nabla f|)$

(2) All the results of this talk on Riem. mfd. hold for

$$P_t = e^{t\mathcal{L}} \text{ with } \mathcal{L} = \Delta + Z$$

under $\mathrm{Ric}_N^Z := \mathrm{Ric} - \nabla Z - \frac{1}{N-n} Z \otimes Z \ge K$

instead of $\mathrm{Ric} \ge K$ & $\dim M \le N$

Known conditions

(i) $\mathrm{Ric} \geq K$

$\Updownarrow$

(iv) $\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$

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(iv)' $$\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K |\nabla f|^2 + \frac{1}{N}(\Delta f)^2$$

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$\Updownarrow \quad \leftsquigarrow$ [F.-Y. Wang '11]

(iii)' $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) - \frac{1 - e^{-2Kt}}{NK}(\Delta P_t f)^2$

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(i)' $\Leftrightarrow$ **(v)'**: **CD**(K, N) [Sturm '06 / Lott & Villani '09]

Theorem 1 ([K.])

For $K \in \mathbb{R}$ *and* $N \in [2, \infty)$,
(iii)' *is equivalent to the following* **(ii)'**:

(ii)' $W_2(P_s^* \mu_0, P_t^* \mu_1)^2$

$$\leq \left(\int_s^t e^{2Kr} \xi(dr) \right)^{-1} W_2(\mu_0, \mu_1)^2 + \frac{N}{2} \xi([s, t])^2$$

where $\xi(dr) = \left(\dfrac{2K}{1 - e^{-2Kr}} \right)^{-1/2} dr$

The case $K = 0$

Corollary 2 ([K.])

For $N \in [2, \infty)$, TFAE:

(i)' $\mathrm{Ric} \geq 0 \ \&\ \dim M \leq N$

(ii)' $$W_2(P_s^* \mu_0, P_t^* \mu_1)^2 \leq W_2(\mu_0, \mu_1)^2 + 2N(\sqrt{t} - \sqrt{s})^2$$

(iii)' $$|\nabla P_t f|^2 \leq P_t(|\nabla f|^2) - \frac{2}{N}(\Delta P_t f)^2$$

The case $K = 0$

(ii)' $W_2(P_s^* \mu_0, P_t^* \mu_1)^2$
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$\Downarrow$ $\mu_0 = \delta_{x_0}$, $\mu_1 = \delta_{x_1}$, $s = 0$

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$$\Downarrow \mu_0 = \delta_{x_0},\ \mu_1 = \delta_{x_1},\ s = 0$$

$$P_t(d(x_0, \cdot)^2)(x_1) \leq d(x_0, x_1)^2 + 2Nt$$

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$\Downarrow \mu_0 = \delta_{x_0},\ \mu_1 = \delta_{x_1},\ s = 0$

$$P_t(d(x_0, \cdot)^2)(x_1) \leq d(x_0, x_1)^2 + 2Nt$$
$$\Rightarrow \Delta(d(x_0, \cdot)^2)(x_1) \leq 2N$$

(sharp Laplacian comparison)

Idea of the proof

(ii)' $\Rightarrow$ **(iii)'**: Differentiation

(iii)' $\Rightarrow$ **(ii)'**: Kantorovich duality
& analysis of the Hopf-Lax semigroup
(cf. [K. '10 / K.] when $N = \infty$)

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(i)' $\Rightarrow$ **(ii)'**: Coupling of Brownian motions with
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$\Rightarrow$ Extension to more general setting

(i)' $\Rightarrow$ **(ii)'**: Coupling of Brownian motions with different speed
$\Rightarrow$ (possibly) sharper estimate

Sketch of proof: (ii)' $\Rightarrow$ (iii)'

(ii)' $W_2(P_s^* \mu_0, P_t^* \mu_1)^2$

$$\leq \left(\int_s^t e^{2Kr} \xi(dr) \right)^{-1} W_2(\mu_0, \mu_1)^2 + \frac{N}{2} \xi([s,t])^2$$

(iii)' $\dfrac{2\Psi(t)}{N} (\Delta P_t f)^2 + |\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2)$

where $\xi(dr) = \left(\dfrac{2K}{1 - e^{-2Kr}} \right)^{-1/2} dr =: \dfrac{dr}{\sqrt{\Psi(r)}}$

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For π: coupling of $P_t^* \delta_x$ and $P_s^* \delta_y$,

$$P_t f(x) - P_s f(y) = \int (f(z) - f(w)) \pi(dz dw)$$

Sketch of proof: (ii)' $\Rightarrow$ (iii)'

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Take $t - s = a\, d(x, y)$ for "suitable" $a \in \mathbb{R}$:

$$\Rightarrow \frac{\text{(LHS)}}{d(x, y)} \to a \Delta P_t f(x) + |\nabla P_t f|(x) \text{ as } s \to t$$

Sketch of proof: (ii)' ⇒ (iii)'

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$$\Rightarrow \text{(RHS)} = \int \frac{f(z) - f(w)}{d(z, w)} d(z, w) \pi(dz dw)$$

$$\text{"}\le\text{"}\ P_t(|\nabla f|^2)(x)^{1/2} W_2(P_t^* \delta_x, P_s^* \delta_y) \cdots \square$$

Sketch of proof: (iii)' $\Rightarrow$ (ii)'

Ingredients

- Kantorovich duality:

$$\frac{W_2(\nu,\mu)^2}{2} = \sup_f \left[\int Q_1 f \, d\mu - \int f \, d\nu\right]$$

- Hopf-Lax semigroup:

$$Q_r f(x) := \inf_{y \in M}\left[f(y) + \frac{d(x,y)^2}{2r}\right]$$

$\star$ $\partial_r Q_r f = -\frac{1}{2}|\nabla Q_r f|^2$ (Hamilton-Jacobi eq.)

Sketch of proof: (iii)' ⇒ (ii)'

(ii)' $W_2(P_s^*\mu_0, P_t^*\mu_1)^2$

$$\leq \left(\int_s^t e^{2Kr}\xi(dr)\right)^{-1} W_2(\mu_0,\mu_1)^2 + \frac{N}{2}\xi([s,t])^2$$

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For simplicity, $\mu_0 = \delta_{x_0}$, $\mu_1 = \delta_{x_1}$

$$\frac{W_2(P_s^* \delta_{x_0}, P_t^* \delta_{x_1})^2}{2} = \sup_f \left[P_t Q_1 f(x_1) - P_s f(x_0) \right]$$

<u>Idea</u>: give an upper bound of $[\cdots]$ being uniform in f

Sketch of proof: (iii)' $\Rightarrow$ (ii)'

(ii)' $W_2(P_s^*\mu_0, P_t^*\mu_1)^2$

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$\gamma : [0,1] \to M$: geod. joining x_0 & x_1

$\alpha : [0,1] \to [s,t]$, $\eta : [0,1] \to [0,1]$: $\nearrow$, surj.
(suitably chosen)

$$\Rightarrow P_t Q_1 f(x_1) - P_s f(x_0)$$

$$= P_{\alpha(1)} Q_1 f(\gamma(\eta(1))) - P_{\alpha(0)} Q_0 f(\gamma(\eta(0)))$$

$$= \int_0^1 \partial_r P_{\alpha(r)} Q_r f(\gamma(\eta(r))) dr$$

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―――――― ○ ―――――― ○ ――――――

$$\begin{aligned} \partial_r P_{\alpha(r)} Q_r f(\gamma(\eta(r))) & \\ \le \alpha'(r) \Delta P_{\alpha(r)} Q_r f(\gamma(\eta(r))) & \\ - \frac{1}{2} P_{\alpha(r)}(|\nabla Q_r f|^2)(\gamma(\eta(r))) & \\ + \eta'(r) |\nabla P_{\alpha(r)} Q_r f|(\gamma(\eta(r))) & \\ \le \cdots & \end{aligned}$$

□

Remarks

(ii) / (ii)' Wass. contr.	differentiation $\Rightarrow$ $\Leftarrow$ integration	(iii) / (iii)' gradient est.

- If an est. like (ii)' is "infinitesimally sharp", then it implies (iii)'

  $\Rightarrow$ An weaker est. than (ii)' can be equiv. to (iii)'

  $\Rightarrow$ Self-improvements in Wass. contr.'s

Extended duality

Theorem 3 ([K.])

M: Polish geod. sp., $P_t = e^{t\mathcal{L}}$: (str) Feller semigr.

Then for $a, b : [0, \infty) \to (0, \infty)$, TFAE:

(A) $$W_p(P_s^*\mu_0, P_t^*\mu_1)^2 \le \left(\int_s^t \frac{\xi(dr)}{a(r)} \right)^{-1} W_p(\mu_0, \mu_1)^2 + \xi([s,t])^2$$

(B) $$|\nabla P_t f|^2 \le a(t) \left[P_t(|\nabla f|^q)^{2/q} - b(t)(\mathcal{L} P_t f)^2 \right]$$

where $p^{-1} + q^{-1} = 1$, $\xi(dr) := b(r)^{-1/2} dr$,
$|\nabla f|(x)$: loc. Lip. const. of f at x

Coupling by parallel transport

$(X_0(t), X_1(t))$: coupling of BMs with different speeds

Driving noise $dB_1(t)$ of $X_1(t)$
$= $ parallel transport of $dB_0(t)$ along a geod.
& scaling

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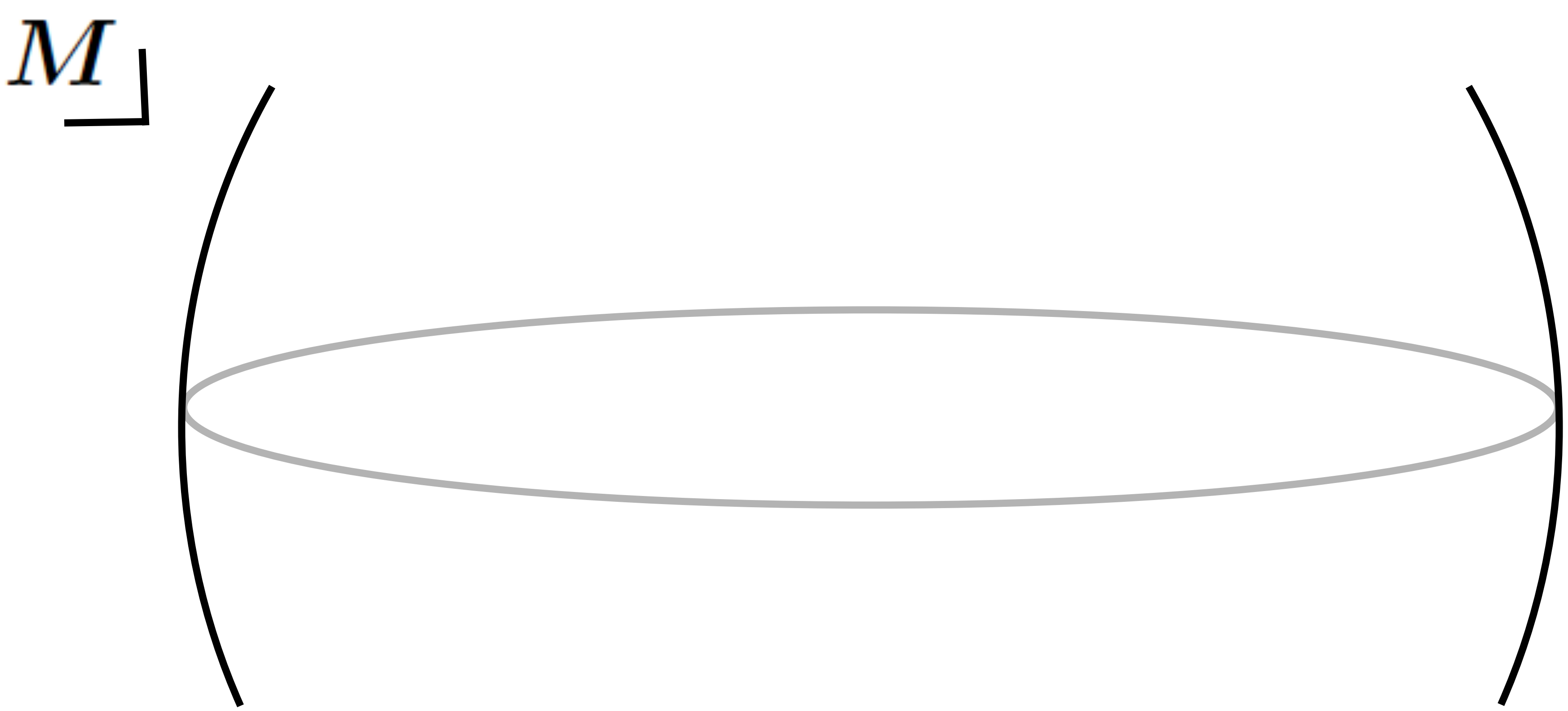

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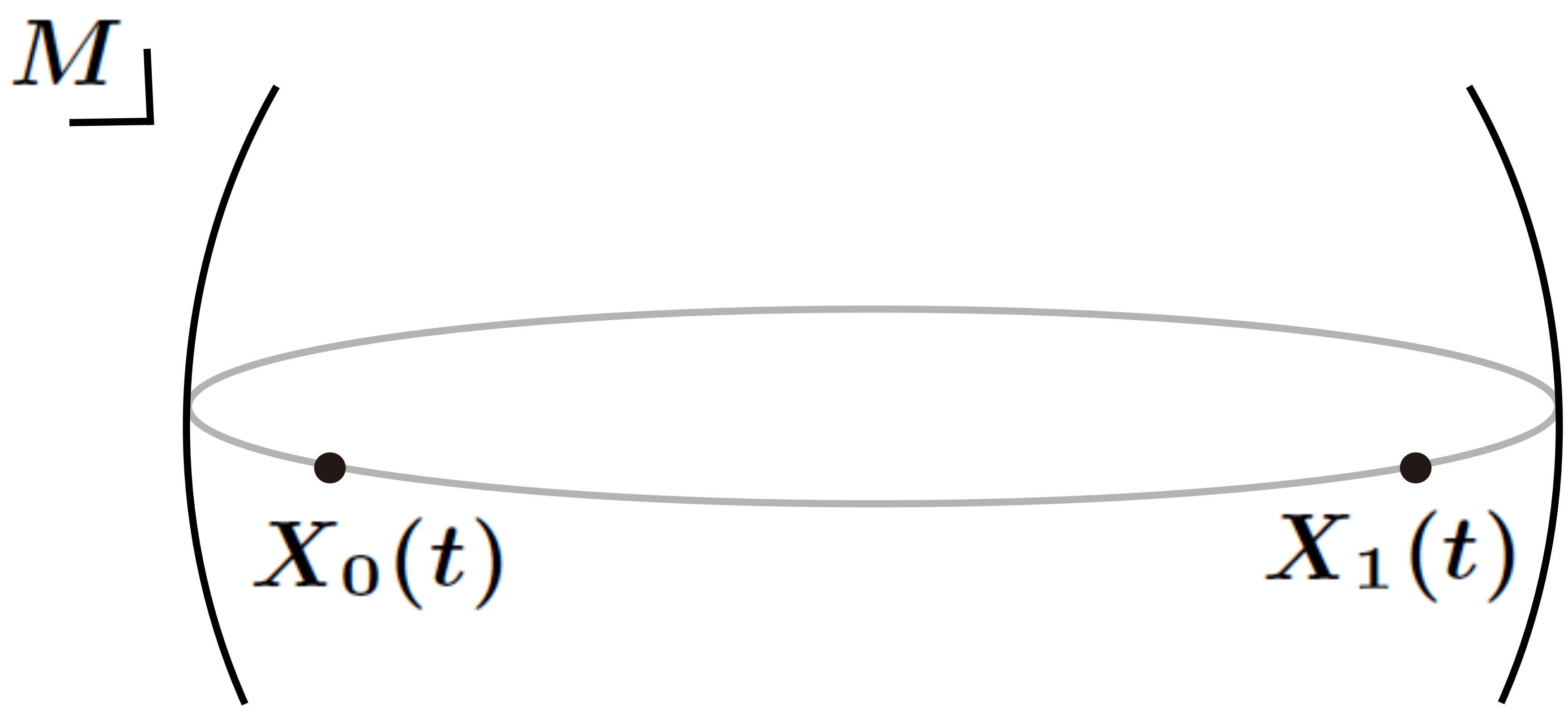

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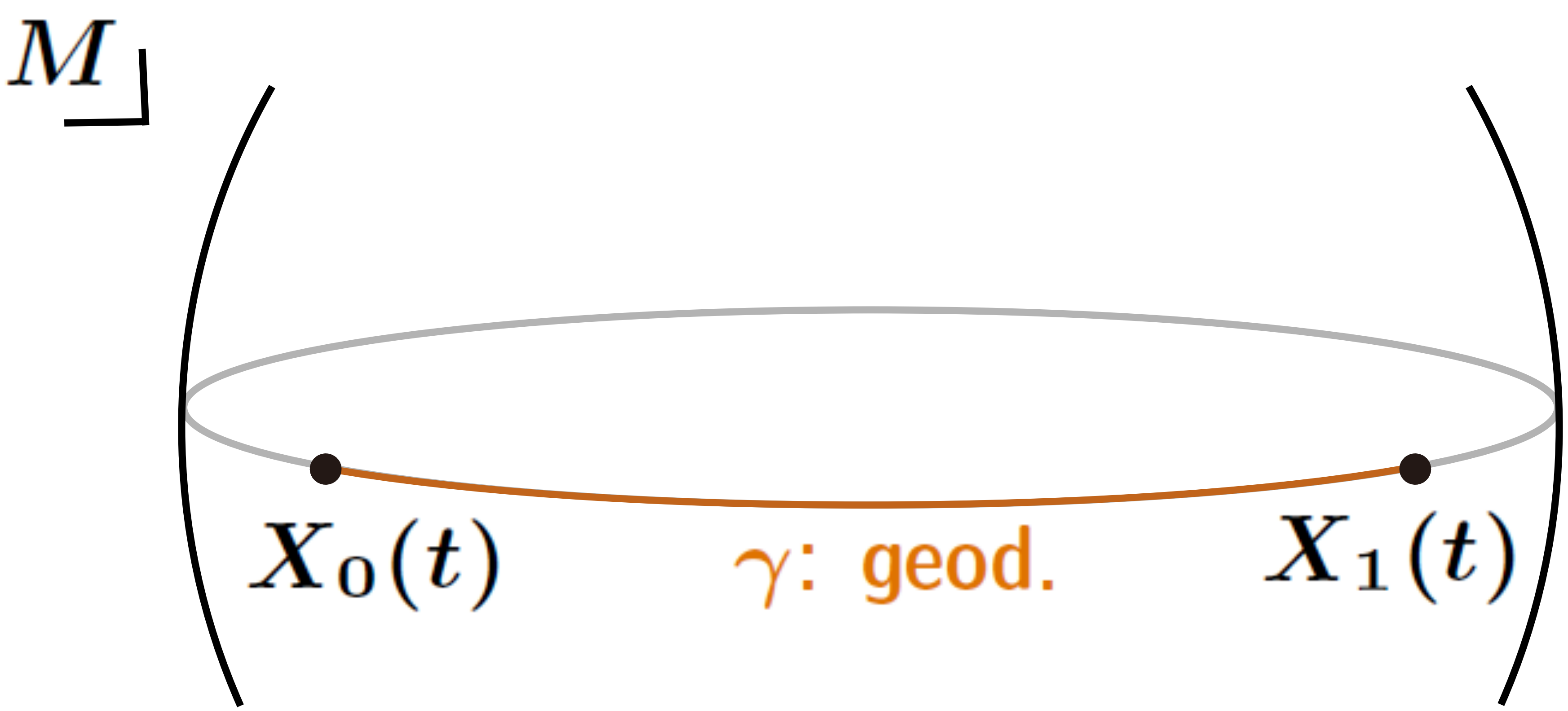

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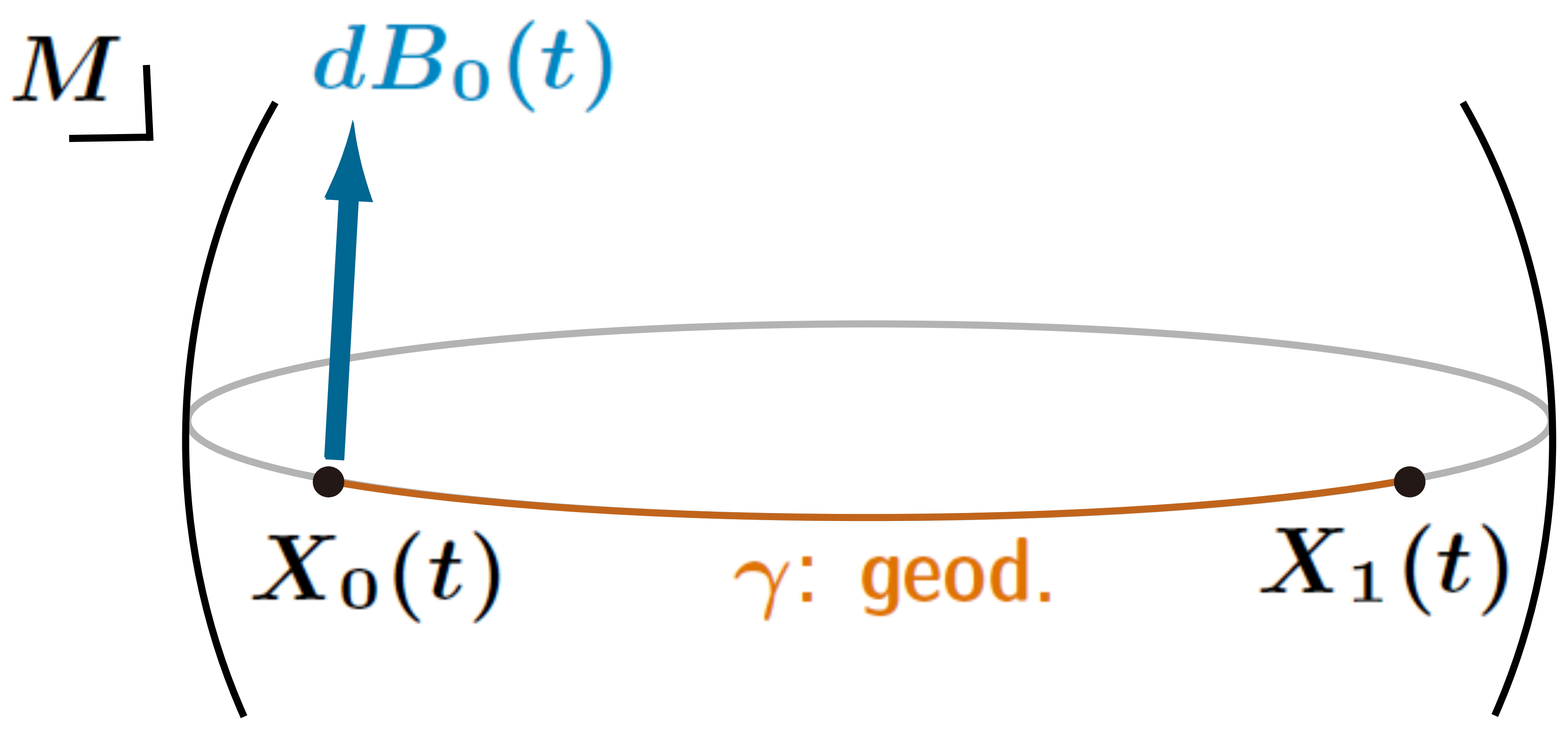

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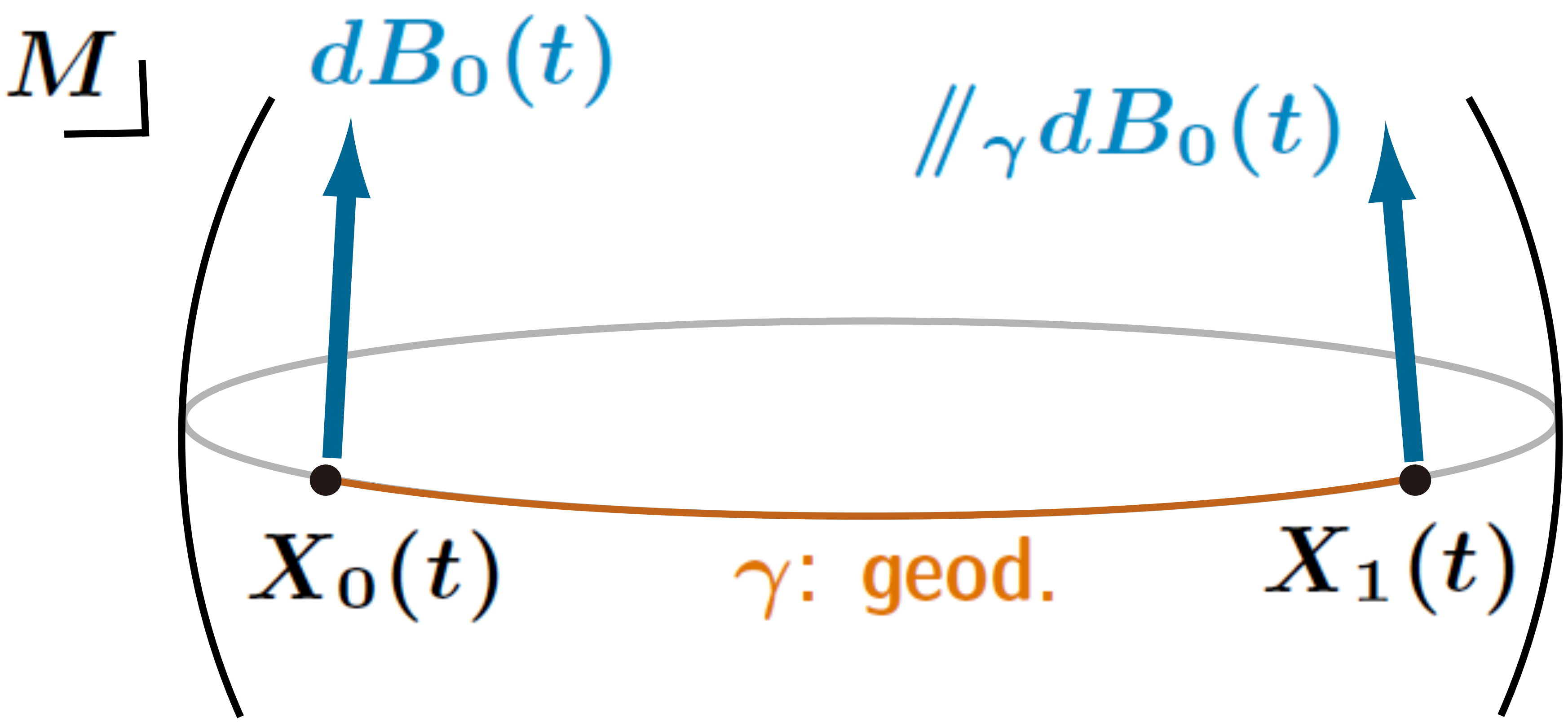

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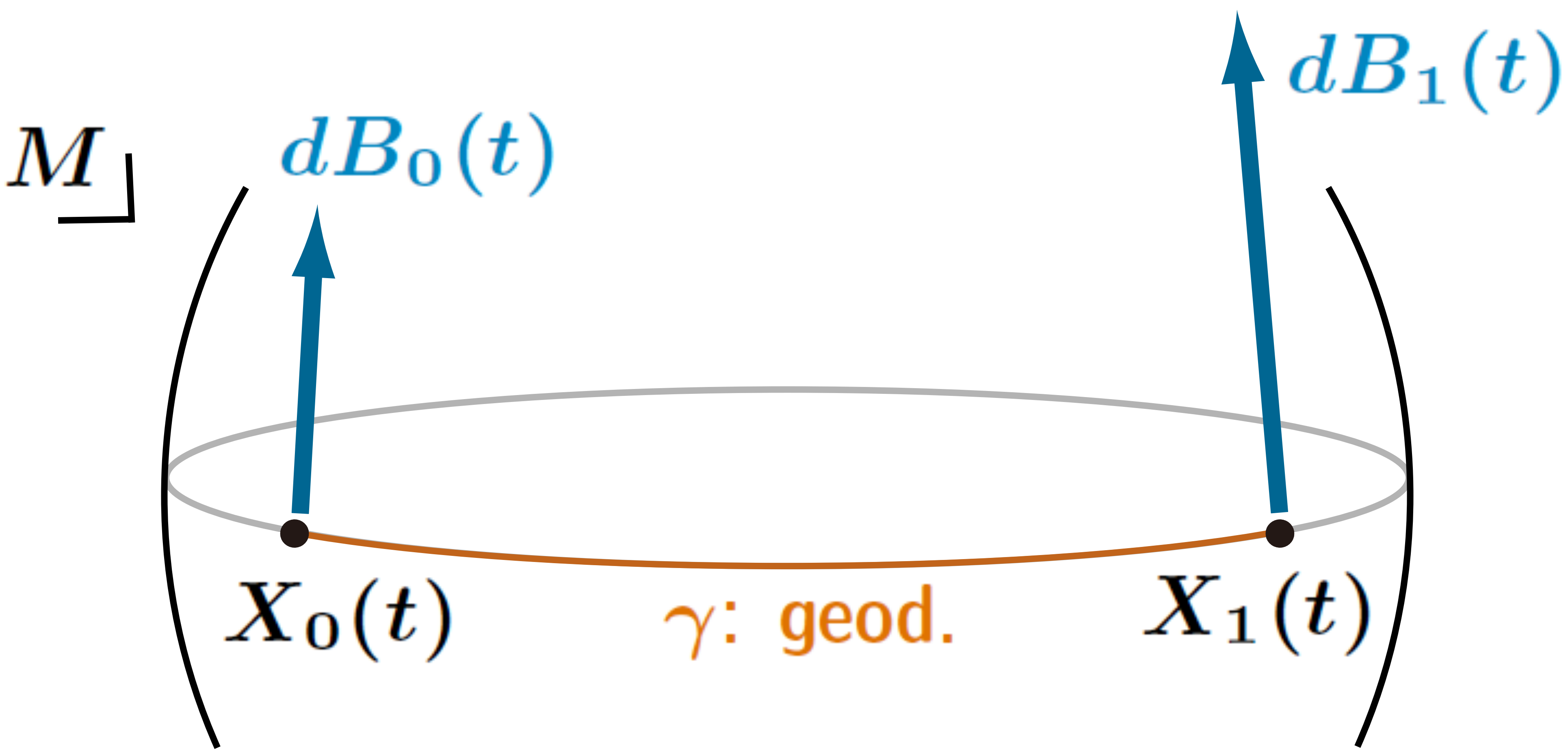
$=$ parallel transport of $dB_0(t)$ along a geod.
& scaling



Coupling by parallel transport

$s < t$ fixed, $(\mu_r)_{r \in [0,1]}$: W_2-geod. in $\mathcal{P}(X)$
$\alpha : [0,1] \to [s,t]$, $\eta : [0,1] \to [0,1]$: $\nearrow$, surj.

$(X_r(t), X_{r'}(t))_{t \in [0,1]}$: coupling by parallel transport of $(X(\alpha(r)t), \mathbb{P}_{\mu_{\eta(r)}})$ and $(X(\alpha(r')t), \mathbb{P}_{\mu_{\eta(r')}})$

Coupling by parallel transport

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$$\Rightarrow W_2(P^*_{\alpha(r)}\mu_r, P^*_{\alpha(r')}\mu_{r'})^2 \le \mathbb{E}\left[d(X_r(1), X_{r'}(1))^2\right] \le \cdots$$

Coupling by parallel transport

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$$\Rightarrow W_2(P^*_{\alpha(r)}\mu_r, P^*_{\alpha(r')}\mu_{r'})^2 \le \mathbb{E}\left[d(X_r(1), X_{r'}(1))^2\right] \le \cdots$$

$$\Rightarrow W_2(P^*_s\mu_0, P^*_t\mu_1)^2 \le \int_0^1 |\dot{P^*_{\alpha(r)}\mu_r}|^2_{W_2} dr \le \cdots,$$

where $|\dot{P^*_{\alpha(r)}\mu_r}|_{W_2} = \varlimsup_{r' \downarrow r} \dfrac{W_2(P^*_{\alpha(r)}\mu_r, P^*_{\alpha(r')}\mu_{r'})}{r' - r}$ $\square$

L^p-extension

Theorem 4 ([K.])

When $\mathrm{Ric} \geq K$ *and* $\dim M \leq N$, *for* $p \in [2, \infty)$,

$$\begin{aligned} W_p(P_s^* \mu_0, P_t^* \mu_1)^2 &\leq \left(\fint_s^t e^{2Kr} \xi(dr) \right)^{-1} W_p(\mu_0, \mu_1)^2 \\ &\quad + \frac{N + p - 2}{2} \xi([s,t])^2 \end{aligned}$$

1. Introduction

2. Known results for lower Ricci bounds

3. Curvature-dimension conditions

4. Proofs & extensions

4.1 Duality

4.2 Coupling methods

4.3 Questions

Questions

- $\mathrm{CD}(K,N) \Rightarrow$ **(ii)'** ?
  Are there an alternative of EVI?
- How sharp **(ii)'** is?
  - ★ Seems to be sharp when $K = 0$
    (Laplacian comparison)
- Connection with the monotonicity of normalized $\mathcal{L}$-transp. cost under backward Ricci flow?
  [cf. Topping '09, K.-Philipowski '11]