Applications of Hopf-Lax formulae to analysis of heat distributions

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1. Introduction

Hopf-Lax semigroup

(X, d): metric space

For $f \in C_b(X)$, $t > 0$,

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + tL\left(\frac{d(x,y)}{t}\right) \right]$$

$$L : [0,\infty) \to [0,\infty) : \text{ convex, increasing, } L(0) = 0$$

(Typically $L(s) = \dfrac{1}{p} s^p$ with $p \in (1, \infty)$)

Hopf-Lax semigroup

Hamilton-Jacobi equation

When $X = \mathbb{R}^m$, for $f \in C_b(X)$,

$$\partial_t Q_t f = -L^*(|\nabla Q_t f|),$$
$$\lim_{t \downarrow 0} Q_t f = f,$$

where L^*: Legendre conjugate of L:

$$L^*(u) := \sup_{v \geq 0} [uv - L(v)]$$

$\star$ $L(s) = p^{-1} s^p \Rightarrow L^*(u) = q^{-1} u^q$
(we only consider this case & the case $p = 2$)

Recent developments

- Connection with optimal transportation
- Application to functional inequalities
- Extension on metric measure spaces

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- Connection with optimal transportation

† W_p: L^p-Wasserstein distance on $\mathcal{P}(X)$

$$W_p(\mu_0, \mu_1) := \inf_{\pi \in \Pi(\mu_0, \mu_1)} \|d\|_{L^p(\pi)}$$

$(\Pi(\mu_0 \mu_1) \subset \mathcal{P}(X^2)$: coupling of μ_0 & $\mu_1)$

† Kantorovich duality

$$\frac{W_p(\mu_0, \mu_1)^p}{p} = \sup_{f \in C_b(X)} \left[\int_X Q_1 f \, d\mu_1 - \int_X f \, d\mu_0 \right]$$

- Application to functional inequalities
- Extension on metric measure spaces

Recent developments

- Connection with optimal transportation
- Application to functional inequalities

  (
  e.g. log-Sobolev ineq./transport-entropy ineq.
  [Otto & Villani '00]
  [Bobkov, Gentil & Ledoux '01]
  [Golzan, Roberto & Samson] etc.
)

- Extension on metric measure spaces

Recent developments

- Connection with optimal transportation
- Application to functional inequalities
- Extension on metric measure spaces

  † [Lott & Villani '07]
  Under volume doubling & Poincaré ineq.

  † [Balogh, Engoulatov, Hunziker & Maasalo '12]
  general Lagrangian L

  † [Ambrosio, Gigli & Savaré]
  No (VD) or (PI). $\dim X = \infty$ is allowed

  † [Gozlan, Roberto & Samson]
  General L. Typically $\dim X < \infty$

Purpose of the talk

Studying functional inequalities

involving $\begin{cases} \text{the heat semigroup} \\ \text{a lower Ricci curvature bound} \end{cases}$

by means of Hopf-Lax semigroup

Outline of the talk

1. Introduction

2. Bakry-Émery type gradient estimate

3. Gradient flow of the relative entropy

3.1 Identification of heat flows

3.2 Applications of the identification

4. Curvature-dimension conditions

General scheme (heuristic)

$(\mu_t)_{t\in[0,1]} \subset \mathcal{P}(X)$: 1-parameter family

$$\frac{W_p(\mu_0,\mu_1)^p}{p} = \sup_{f\in C_b^{\mathrm{Lip}}(X)} \left[\int_X Q_1 f\, d\mu_1 - \int_X Q_0 f\, d\mu_0\right]$$

$$\begin{aligned}[\quad \cdots \quad] &\le \int_0^1 \partial_t \left(\int_X Q_t f\, d\mu_t\right) dt \\ &= \int_0^1 \left\{\int_X (\partial_t Q_t f)\, d\mu_t\right\} dt \\ &\qquad + \int_0^1 \left\{\int_X Q_t f\, d(\partial_t \mu_t)\right\} dt \\ &\le (\text{indep. of } f)\end{aligned}$$

Framework

(X, d): Polish sp., d: length distance

- $(P(x,\cdot))_{x\in X} \subset \mathcal{P}(X)$: Markov kernel
  (e.g. $P(x,dy) = p_t(x,dy)$: heat semigroup)
- $$|\nabla_d f|(x) := \limsup_{y\to x} \left| \frac{f(x) - f(y)}{d(x,y)} \right|$$
  (local Lipschitz const.)

Theorem 1 (cf. [K.'10])

For $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$ and $C > 0$, TFAE:

(i) *For $\mu_0, \mu_1 \in \mathcal{P}(X)$,*

$$W_p(P^*\mu_0, P^*\mu_1) \le C W_p(\mu_0, \mu_1) \qquad (W_p)$$

(ii) *For $f \in C_b^{\mathsf{Lip}}(X)$,*

$$|\nabla_d P f|(x) \le C P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

(When $q = \infty$, (RHS of (G_q)) will be $\||\nabla_d f|\|_\infty$)

Motivation

X: cpl. Riem. mfd, P_t: heat semigr.

$\Rightarrow$ TFAE for $K \in \mathbb{R}$ ([von Renesse & Sturm '05] etc.)

(a) $\mathrm{Ric} \geq K$

(b) $W_p(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_p(\mu_0, \mu_1)$

for some $p \in [1, \infty]$

(c) $|\nabla P_t f|(x) \leq e^{-Kt} P_t(|\nabla f|^q)(x)^{1/q}$

for some $q \in [1, \infty]$ (Bakry-Émery's gradient est.)

(d) $\dfrac{1}{2} \Delta |\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K |\nabla f|^2$

(e) Relative entropy Ent is K-convex on $(\mathcal{P}(X), W_2)$

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- No direct proof of "(c) $\Rightarrow$ (b)" before Theorem 1

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- For "canonical" diffusions on sub-Riem. mfds,

$$|\nabla P_t f|(x) \le C(t) P_t(|\nabla f|^q)(x)^{1/q}$$

where $C(t)$: disconti. at 0
[Driver & Melcher '05 / H.-Q. Li '06 / Melcher '08 / Bakry et al '08 / Eldredge '09 etc.]

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- "(e) $\Rightarrow$ (W_2)" even on singular spaces
[Savaré '07 / Ohta '09 / Gigli & Ohta '10 / Gigli, K. & Ohta / Koskela & Zhou / Ambrosio, Gigli & Savaré] etc.

Sketch of the proof of $(G_q) \Rightarrow (W_p)$

$\star$ For simplicity, $p \in (1,\infty)$ & $\mu_i = \delta_{x_i}$ $(i = 0,1)$

$\gamma : [0,1] \to X$: d-geod. with $\gamma(i) = x_i$ $(i = 0,1)$

$\mu_t := P(\gamma(t), \cdot\,) = P^* \delta_{\gamma(t)}$

$$\Rightarrow \frac{W_p(P^*\mu_0, P^*\mu_1)^p}{p} \le \sup_f \left[\quad \cdots \quad \right]$$

$$[\,\cdots\,] = \int_0^1 dt \left(\int_X (\partial_t Q_t f)\, d\mu_t + \int_X Q_t f \, d(\partial_t \mu_t) \right)$$

$$\le \int_0^1 dt \Bigg(- \frac{P(|\nabla_d Q_t f|^q)}{q}(\gamma(t)) + d(x_0, x_1) |\nabla_{\tilde{d}} P Q_t f|(\gamma(t)) \Bigg)$$

Sketch of the proof of $(G_q) \Rightarrow (W_p)$

Let $\sigma(t) := P(|\nabla_d Q_t f|^q)(\gamma(t))^{1/q}$

$\Rightarrow (G_q)$ implies

$$-\frac{P(|\nabla_d Q_t f|^q)}{q}(\gamma(t)) + d(x_0, x_1)|\nabla_d P Q_t f|(\gamma(t))$$

$$\leq C d(x_0, x_1)\sigma(t) - \frac{1}{q}\sigma(t)^q \leq \frac{(C d(x_0, x_1))^p}{p}$$

$$\Rightarrow \frac{W_p(P^*\delta_0, P^*\delta_1)^p}{p} \leq \int_0^1 dt \left(\frac{(C d(x_0, x_1))^p}{p} \right)$$

$$= \frac{(C W_p(\delta_{x_0}, \delta_{x_1}))^p}{p} \qquad \square$$

Two formulations of heat flow

$(X, d, \mathfrak{m})$: metric measure space

(1) Gradient flow in $L^2(\mathfrak{m})$ of the Dirichlet energy

$$\mathcal{E}(f) := \frac{1}{2} \int_X |\nabla f|^2 \, d\mathfrak{m}$$

$\rightsquigarrow$ Dirichlet form, heat semigroup

(2) Gradient flow in $(\mathcal{P}(X), W_2)$ of the rel. entropy

$$\mathrm{Ent}(\mu) := \int_X \rho \log \rho \, d\mathfrak{m} \quad (\text{if } \mu = \rho \mathfrak{m})$$

([Jordan, Kinderlehrer & Otto '98], ...)

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Q. When (1) = (2) ?

Ans. $\mathbf{Ent}$: $^{\exists}\boldsymbol{K}$-convex w.r.t. $\boldsymbol{W_2} \Rightarrow (1) = (2)$

Ans. **Ent**: $^{\exists}K$-convex w.r.t. $W_2 \Rightarrow (1) = (2)$

Known results

- $\mathbb{R}^m$ [Jordan, Kinderlehrer & Otto '98]
- cpl. Riem. mfd [Erbar '10]
- Finsler mfd [Ohta & Sturm '09] (nonlinear)
- Wiener sp. [Fang, Shao & Sturm '09] (∞-dim)
- Heisenberg gr. [Juillet] (sub-elliptic)

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- cpt. Alexandrov sp. [Gigli, K. & Ohta] (singular)
- metric measure sp. [Koskela & Zhou]
  [Ambrosio, Gigli & Savaré]

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- finite set [Maas '11] (discrete Markov chain)
- jump process [Erbar] (nonlocal generator)

Ans. $\mathbf{Ent}$: $^{\exists}\boldsymbol{K}$-convex w.r.t. $\boldsymbol{W_2} \Rightarrow (1) = (2)$

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Def. of GF (of Ent) on $(\mathcal{P}(X), W_2)$

$(\mu_t)_{t\ge 0}$: abs. conti., $\mathrm{Ent}(\mu_t) < \infty$,

$$\partial_t \mathrm{Ent}(\mu_t) = -\frac{1}{2}|\dot{\mu}_t|_{W_2}^2 - \frac{1}{2}|\nabla_- \mathrm{Ent}(\mu_t)|^2,$$

where

$$|\nabla_- \mathrm{Ent}(\mu)| := \limsup_{\nu\to\mu} \frac{[\mathrm{Ent}(\mu) - \mathrm{Ent}(\nu)]_+}{W_2(\mu,\nu)}$$

$$|\dot{\mu}_t|_{W_2}^2 := \limsup_{s\downarrow 0} \frac{W_2(\mu_{t+s},\mu_t)^2}{s^2}$$

[Ambrosio, Gigli & Savaré '05]

Def. of GF (of Ent) on $(\mathcal{P}(X), W_2)$

$$\partial_t \,\mathrm{Ent}(\mu_t) = -\frac{1}{2}|\dot{\mu}_t|^2_{W_2} - \frac{1}{2}|\nabla_- \mathrm{Ent}(\mu_t)|^2$$

Heuristics: Why does this definition work?

$$\begin{aligned}
&\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_s) \\
&\quad \text{“}=\text{”} \int_s^t \langle \dot{\mu}_r, \nabla \mathrm{Ent}(\mu_r)\rangle dr \\
&\quad \geq -\frac{1}{2}\int_s^t |\dot{\mu}_r|^2 dr - \frac{1}{2}\int_s^t |\nabla \mathrm{Ent}|^2(\mu_r) dr
\end{aligned}$$

$$\left(\because \langle u, v\rangle \geq -\frac{1}{2}\left(|u|^2 + |v|^2\right)\right)$$

and “=” holds iff $\dot{\mu}_r = -\nabla \mathrm{Ent}(\mu_r)$

Strategy of the identification

Let $\mu_t := (P_t\varphi)\,\mathfrak{m} = (e^{t\Delta}\varphi)\,\mathfrak{m} \in \mathcal{P}(X)$

Q. $\partial_t \operatorname{Ent}(\mu_t) \le -\frac{1}{2}|\dot{\mu}_t|^2_{W_2} - \frac{1}{2}|\nabla_- \operatorname{Ent}(\mu_t)|^2$?

(A) $\partial_t \operatorname{Ent}(\mu_t) = -\int_X \frac{|\nabla P_t\varphi|^2}{P_t\varphi} d\mathfrak{m}$

(B) $|\nabla_- \operatorname{Ent}(\rho\mathfrak{m})|^2 \le \int_X \frac{|\nabla\rho|^2}{\rho} d\mathfrak{m}$

(C) $|\dot{\mu}_t|^2_{W_2} \le \int_X \frac{|\nabla P_t\varphi|^2}{P_t\varphi} d\mathfrak{m}$

Strategy of the identification

- Integration by parts + chain rule $\Rightarrow$ (A)
- $\mathbf{Ent}$ is $\boldsymbol{K}$-convex ($\Leftrightarrow$ $\mathbf{Ric} \geq \boldsymbol{K}$)

  $\Rightarrow$ uniqueness of GF of $\mathbf{Ent}$ [Gigli '10]

  $\Rightarrow$ (B) (naive)

  [Villani '09, Ambrosio, Gigli & Savaré, ...]

  $\leadsto$ notion of "upper gradients" as a gradient

  $\leadsto$ requires "identification of gradients"
- Lemma 2 below $\Rightarrow$ (C)

An estimate of W_2-speed

Lemma 2 (e.g. [Gigli, K. & Ohta])

X*: cpl., stoch. cpl. Riem. mfd.*
P_t*: the heat semigroup on* X
$\varphi : X \to [0,\infty)$ *with* $\|\varphi\|_{L^1} = 1$, $\mu_t := P_t\varphi \, \mathrm{vol}$.
Then

$$|\dot{\mu}_t|_{W_2}^2 := \limsup_{s \downarrow 0} \frac{W_2(\mu_{t+s}, \mu_t)^2}{s^2} = \int_X \frac{|\nabla P_t \varphi|^2}{P_t \varphi} \, d\,\mathrm{vol}$$

An estimate of W_2-speed

Sketch of the proof

Let $\rho_r := P_{t+sr}\varphi$ ($r \in [0,1]$), $\mu_r := \rho_r \,\mathrm{vol}$

$$\Rightarrow \frac{W_2(\mu_0,\mu_1)^2}{2} \le \sup_f \left[\quad \cdots \quad \right]$$

$$\left[\cdots \right] = \int_0^1 dt \left(\int_X (\partial_t Q_t f)\, d\mu_t + \int_X Q_t f \, d(\partial_t \mu_t) \right)$$

$$\le \int_0^1 dr \left(- \int_X \frac{|\nabla Q_r f|^2}{2} d\mu_r \right.$$

$$\left. + s \int_X Q_r f \Delta \rho_r \, d\,\mathrm{vol} \right)$$

An estimate of W_2-speed

$$
\begin{aligned}
&- \int_X \frac{|\nabla Q_r f|^2}{2} d\mu_r + s \int_X Q_r f \Delta \rho_r \, d\,\mathrm{vol} \\
&\quad = \int_X d\mu_r \left(-\frac{|\nabla Q_r f|^2}{2} - s \langle \nabla Q_r f, \frac{\nabla \rho_r}{\rho_r} \rangle \right) \\
&\quad \leq \frac{s^2}{2} \int_X \frac{|\nabla \rho_r|^2}{\rho_r} d\,\mathrm{vol}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \frac{W_2(\mu_0, \mu_1)^2}{2} &= \frac{W_2(P_t \varphi \,\mathrm{vol}, P_{t+s} \varphi \,\mathrm{vol})^2}{2} \\
&\leq \frac{s^2}{2} \int_0^1 \left(\int_X \frac{|\nabla P_{t+sr} \varphi|^2}{P_{t+sr} \varphi} d\,\mathrm{vol} \right) dr \quad \square
\end{aligned}
$$

Remark

- There appear "two gradients"

  † The one as a local Lipschitz const. (in Hamilton-Jacobi)

  † The other from integration by parts

$$\int_X f \Delta g \, d\,\mathrm{vol} = - \int_X \langle \nabla f, \nabla g \rangle \, d\,\mathrm{vol}$$

- This argument works even on more singular spaces

  $\rightsquigarrow$ We need to take care on the def. of "$|\nabla f|$"

  ([Koskela & Zhou / Ambrosio, Gigli & Savaré])

W_2-contraction

Claim

For heat distributions $\mu_0(t), \mu_1(t)$,

$$W_2(\mu_0(t), \mu_1(t)) \leq e^{-Kt} W_2(\mu_0(0), \mu_1(0)) \qquad (\star)$$

- Heuristically,

$$\boxed{\mathrm{Ent}:\ K\text{-convex} \ \&\ \dot{\mu}_t = -\nabla \,\mathrm{Ent}} \Rightarrow (\star)$$

W_2-contraction

Claim

For heat distributions $\mu_0(t), \mu_1(t)$,

$$W_2(\mu_0(t), \mu_1(t)) \le e^{-Kt} W_2(\mu_0(0), \mu_1(0)) \quad (\star)$$

- On general metric measure space,

  Linearity of heat flow,
  Ent: K-convex & $\dot{\mu}_t = -\nabla \mathrm{Ent}$ $\Rightarrow (\star)$

  [Koskela & Zhou / Ambrosio, Gigli & Savaré]

- $^\exists$Counterexample when heat flow is nonlinear

  [Ohta & Sturm '12]

Bakry-Émery's gradient estimate

Under the linearity of the heat flow,
by Theorem 1,

$$\begin{aligned} W_2(\mu_0(t), \mu_1(t)) &\le e^{-Kt} W_2(\mu_0(0), \mu_1(0)) \\ &\Downarrow \\ |\nabla P_t f|(x)^2 &\le e^{-2Kt} P_t(|\nabla f|^2)(x) \qquad (\star\star) \end{aligned}$$

Bakry-Émery's gradient estimate

$$|\nabla P_t f|(x)^2 \leq e^{-2Kt} P_t(|\nabla f|^2)(x) \qquad (\star\star)$$

- $(\star\star) \Rightarrow$ Lipschitz continuity
  of $P_t f$/heat kernel/eigenfunction
- $(\star\star) \Leftrightarrow$ (weak) Bakry-Émery's Γ_2-condition:
  $$\frac{1}{2}\Delta\langle\nabla f, \nabla f\rangle - \langle\nabla f, \nabla\Delta f\rangle \geq K\langle\nabla f, \nabla f\rangle$$

Results on Alexandrov spaces

X: n-dim. cpt. Alex. sp. of curv. $\geq k$

- [Kuwae, Machigashira & Shioya '01]: Study of Dirichlet energy and associated heat semigroup
  $\Rightarrow$ Linearity
- (Modulus of) Sobolev gradient $=$ loc. Lip. const
- [Petrunin '11]: Ent is $(n-1)k$-convex

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- [Petrunin '03]: Another proof of Lip. conti. of e.fn.
- [Zhang & Zhu '10] etc.: Similar results under stronger assumption on "lower Ricci curv. bound"

Curvature-dimension inequality

M: cpl. Riem. mfd, P_t: heat semigroup

$$\mathrm{Ric} \geq K$$

$$\Leftrightarrow \frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K |\nabla f|^2$$

[Bakry & Émery '84]

$$\Leftrightarrow |\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2)$$

Curvature-dimension inequality

M: cpl. Riem. mfd, P_t: heat semigroup

$\mathrm{Ric} \geq K$ & $\dim M \leq N$

$$\Leftrightarrow \frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2$$

[Bakry & Émery '84]

Curvature-dimension inequality

M: cpl. Riem. mfd, P_t: heat semigroup

$\mathrm{Ric} \geq K$ & $\dim M \leq N$

$$\Leftrightarrow \frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2$$

[Bakry & Émery '84]

$$\Leftrightarrow |\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) + \frac{1 - e^{-2Kt}}{NK}(\Delta P_t f)^2$$

[F.-Y. Wang '11]

Extended duality

Theorem 3 ([K.])

M: Polish length sp., $P_t = e^{t\mathcal{L}}$: Feller semigr.

Then, for $a, b : [0, \infty) \to (0, \infty)$, TFAE:

(1) $$W_2(P_s^* \mu_0, P_t^* \mu_1)^2 \le \left(\int_s^t \frac{\xi(dr)}{a(r)} \right)^{-1} W_2(\mu_0, \mu_1)^2 + \xi([s, t])^2$$

(2) $$|\nabla_d P_t f|^2 \le a(t) \left[P_t(|\nabla_d f|^2) + b(t)(\mathcal{L} P_t f)^2 \right],$$

where $\xi(dr) := \dfrac{dr}{\sqrt{b(r)}}$

Extended duality

Idea of the proof

(1) $\Rightarrow$ (2): Coupling argument (differentiation)

(2) $\Rightarrow$ (1): Suppose $\mu_i = \delta_{x_i}$, $\gamma : [0,1] \to M$ geod.

$$\Downarrow$$

$$W_2(P_s^* \delta_{x_0}, P_t^* \delta_{x_1})^2 = \sup_f \int_0^1 \partial_r \left[P_{\xi(r)} Q_r f(\gamma(\eta(r))) \right] dr$$

with $\forall \xi : [0,1] \to [s,t]$, $\forall \eta : [0,1] \to [0,1] \nearrow$, surj.

$\star$ Choose ξ, η nicely $\Rightarrow$ Conclusion $\square$

Corollary 4 ([K.])

M: cpl., stoch. cpl. Riem. mfd of $\dim \geq 2$

Then, for $K \in \mathbb{R}$ and $N \in [2, \infty]$, TFAE:

(1) $\mathrm{Ric} \geq K$ & $\dim M \leq N$

(2) $W_2(P_s^* \mu_0, P_t^* \mu_1)^2$

$$\leq \left(\fint_s^t \frac{\xi(dr)}{e^{-2Kr}} \right)^{-1} W_2(\mu_0, \mu_1)^2 + \frac{N}{2} \xi([s,t])^2$$

(3) $|\nabla P_t f|^2 \leq e^{-2Kt} \left[P_t(|\nabla f|^2) + \frac{2\Psi(t)}{N} (\Delta P_t f)^2 \right]$,

where $\Psi(t) := \dfrac{e^{2Kt} - 1}{2K}$ & $\xi(dr) := \dfrac{dr}{\sqrt{\Psi(r)}}$

Questions and remarks

- Connection with Sturm-Lott-Villani's $\mathbf{CD}(K, N)$?
- How sharp?

  ∘ $K = 0 \Rightarrow$ sharp Laplacian comparison

  † When $K \neq 0$?

  † Similarity with a monotonicity of $\mathcal{L}$-transportation cost between heat distributions under Ricci flow

  $\leftrightsquigarrow$ time-inhomogeneous analog of $\mathrm{Ric} = 0$

  [Topping '09, K.-Philipowski '11]