熱分布の結合法と次元曲率条件

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(お茶の水女子大学)

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1. Introduction

Riemannian mfd $\leftrightarrow$ Brownian motion

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Δ_g: Laplace-Beltrami op.

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Δ_g: Laplace-Beltrami op.

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$B(t)$: Brownian motion on M
(diffusion process generated by Δ)

$P_t = e^{t\Delta}$: heat semigroup

Brownian motion $\leftrightarrow$ curvature

Riem. met. $\boldsymbol{g}$ $\quad \boldsymbol{\leftrightarrow} \quad$ "Shape of M"

Brownian motion ↔ curvature

$$\begin{tikzcd}
\text{Riem. met. } g \arrow[rr, leftrightarrow] \arrow[dr, squiggly] & & \text{“Shape of } M\text{”} \\
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(2nd order analysis)

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  “$\leftrightsquigarrow$” averaged (in direction) curvature
  (Ricci curvature)

Q. $B(t)$ or $P_t \leftrightsquigarrow$ Ricci curvature?

lower Ricci bound on metric meas. sp.

Recent developments:

Generalization of "$\mathrm{Ric} \geq K$" on met. meas. sp.

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$\rightsquigarrow$ Geometry/Analysis on non-smooth sp.

$\rightsquigarrow$ Different viewpoints even on smooth sp.

The aim of this talk

- Review of these developments
- (Partial) extension involving both $\mathbf{Ric}$ and $\mathbf{dim}$

Outline of the talk

(# in []: the corresponding section in the abstract)

1. Introduction

2. Lower Ricci curvature bounds

2.1 Bakry-Émery's theory [3.3]
2.2 Coupling method [3.1, 3.3]
2.3 Digression: Wasserstein distance [2.1]
2.4 Optimal transportation [3.3]

3. Implications between “$\mathrm{Ric} \geq K$” [3.3, 2.2, 3.2]

4. Curvature-dimension conditions [4]

5. Concluding remarks

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Γ_2-criterion

Bochner-Weitzenböck formula

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle = \|\operatorname{Hess} f\|_{\mathsf{HS}}^2 + \operatorname{Ric}(\nabla f, \nabla f)$$

$$\Downarrow$$

Bakry-Émery's Γ_2-criterion [Bakry & Émery '84]

$$\operatorname{Ric} \geq K \Leftrightarrow \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$$

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Gradient estimate

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$$\frac{\partial}{\partial s}\left[P_s(|\nabla P_{t-s} f|^2)\right] \geq 2K P_s(|\nabla P_{t-s} f|^2)$$

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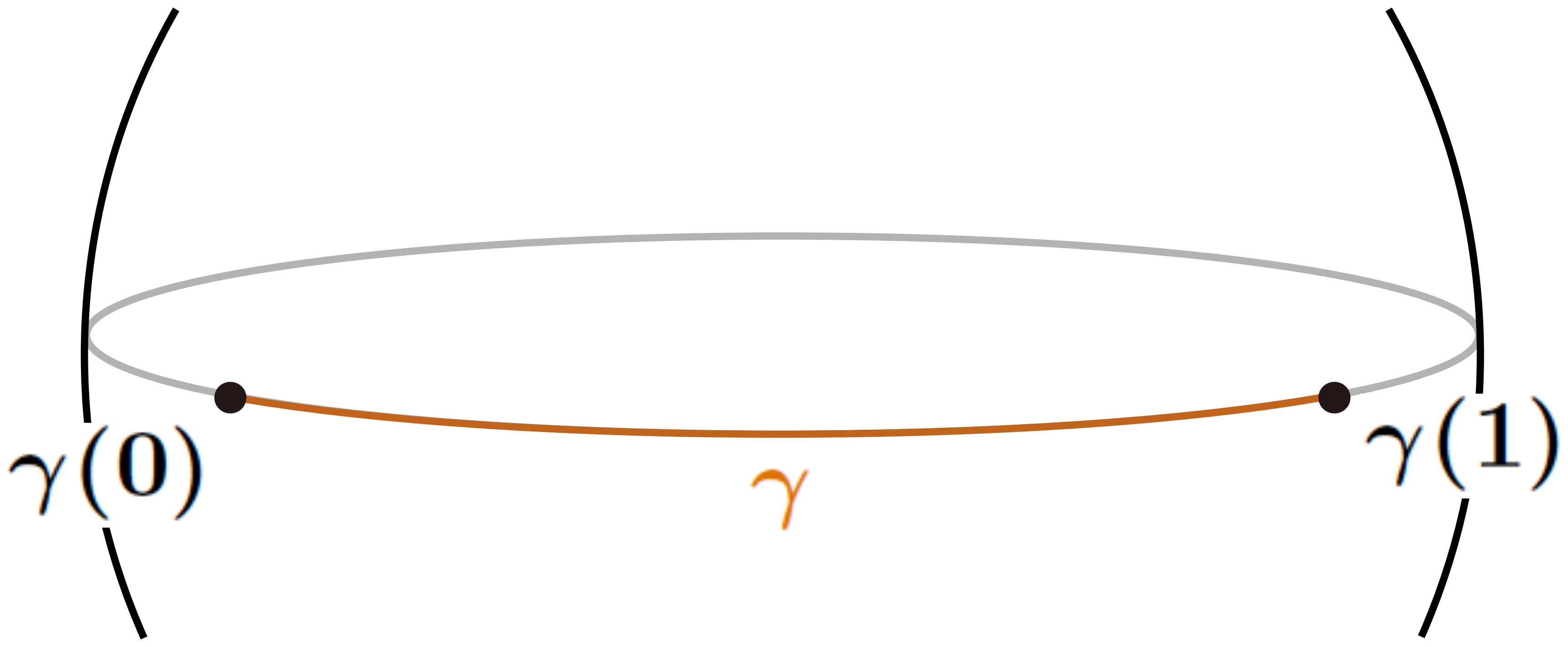
Bakry-Émery's L^2-gradient estimate

$$\mathrm{Ric} \geq K \Leftrightarrow |\nabla P_t f| \leq e^{-Kt} P_t(|\nabla f|^2)^{1/2}$$

(Averaged) variation of arclength

$\gamma : [0,1] \to M$ geodesic



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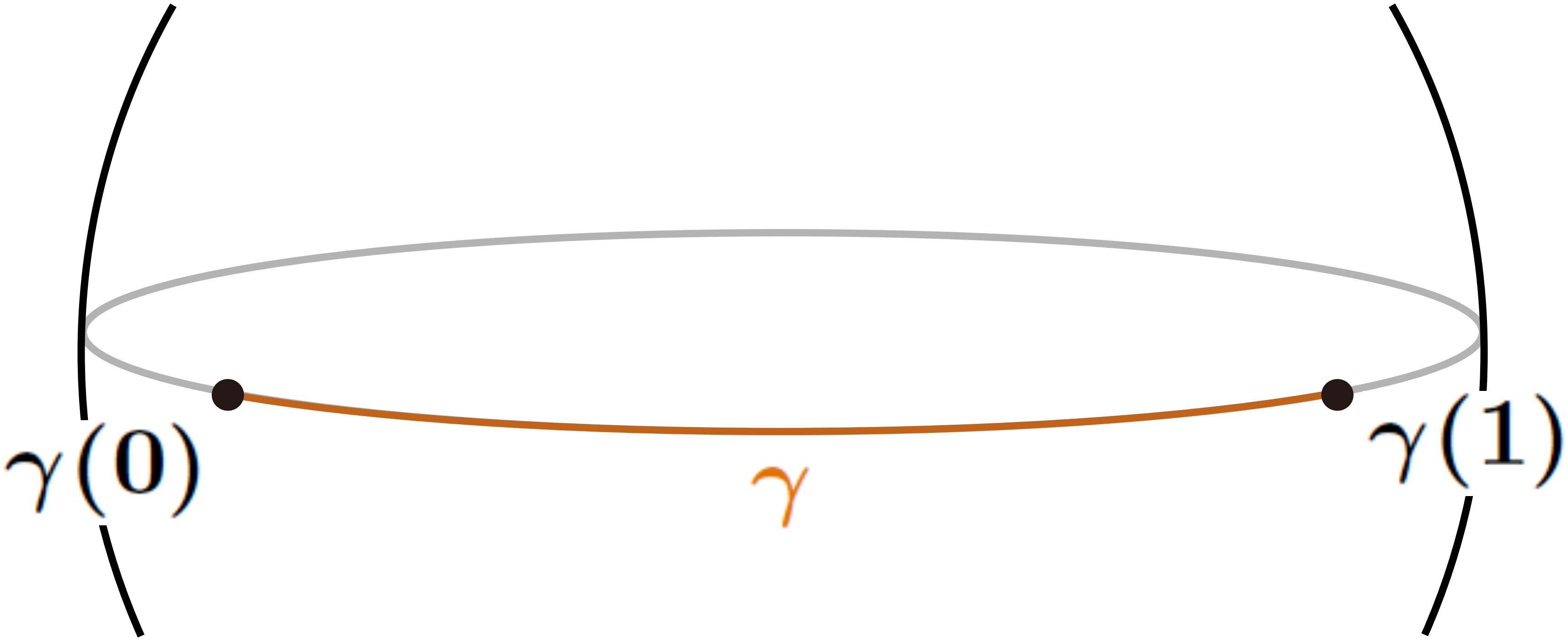

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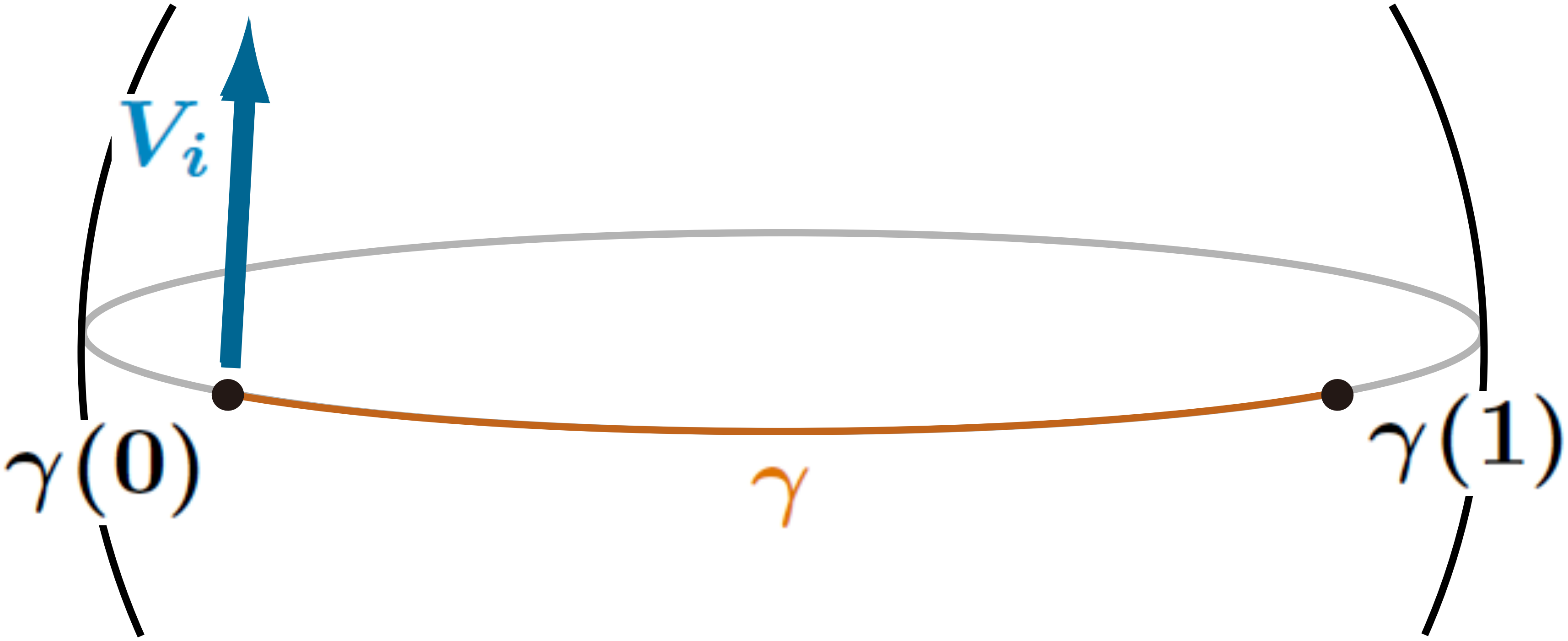

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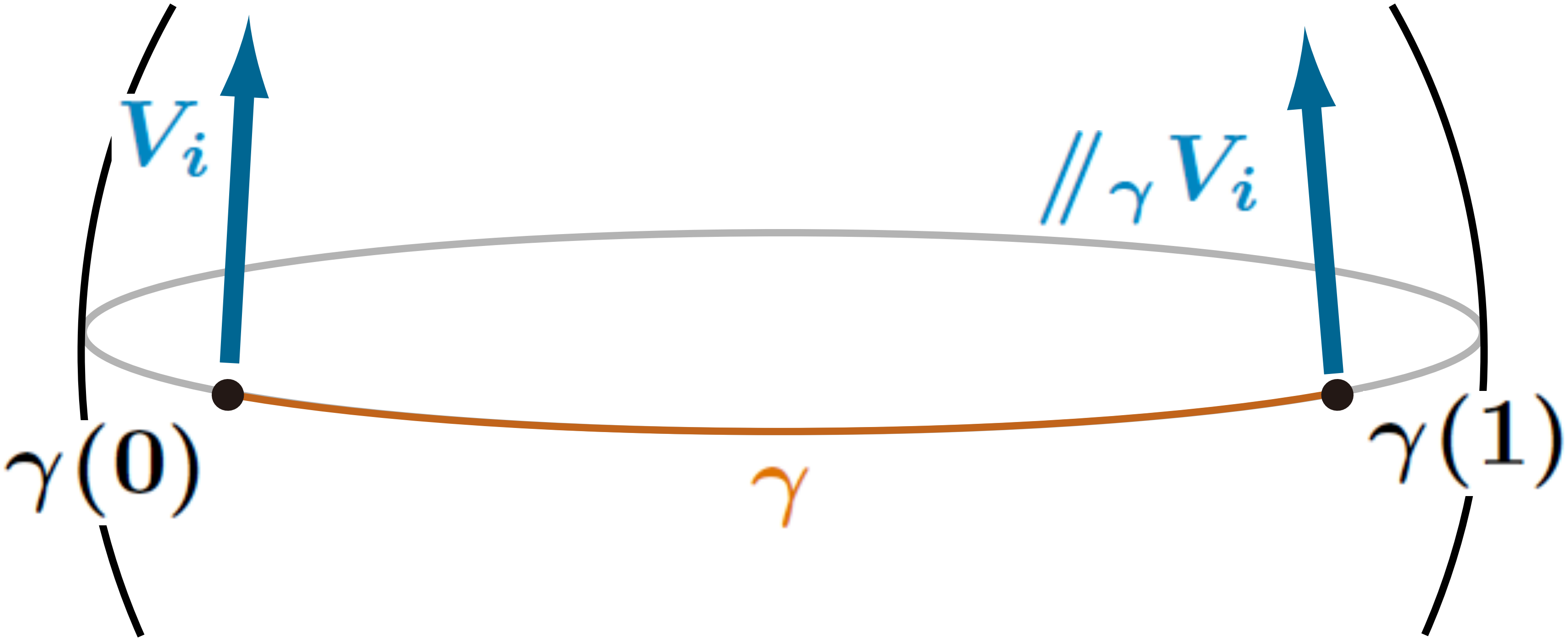

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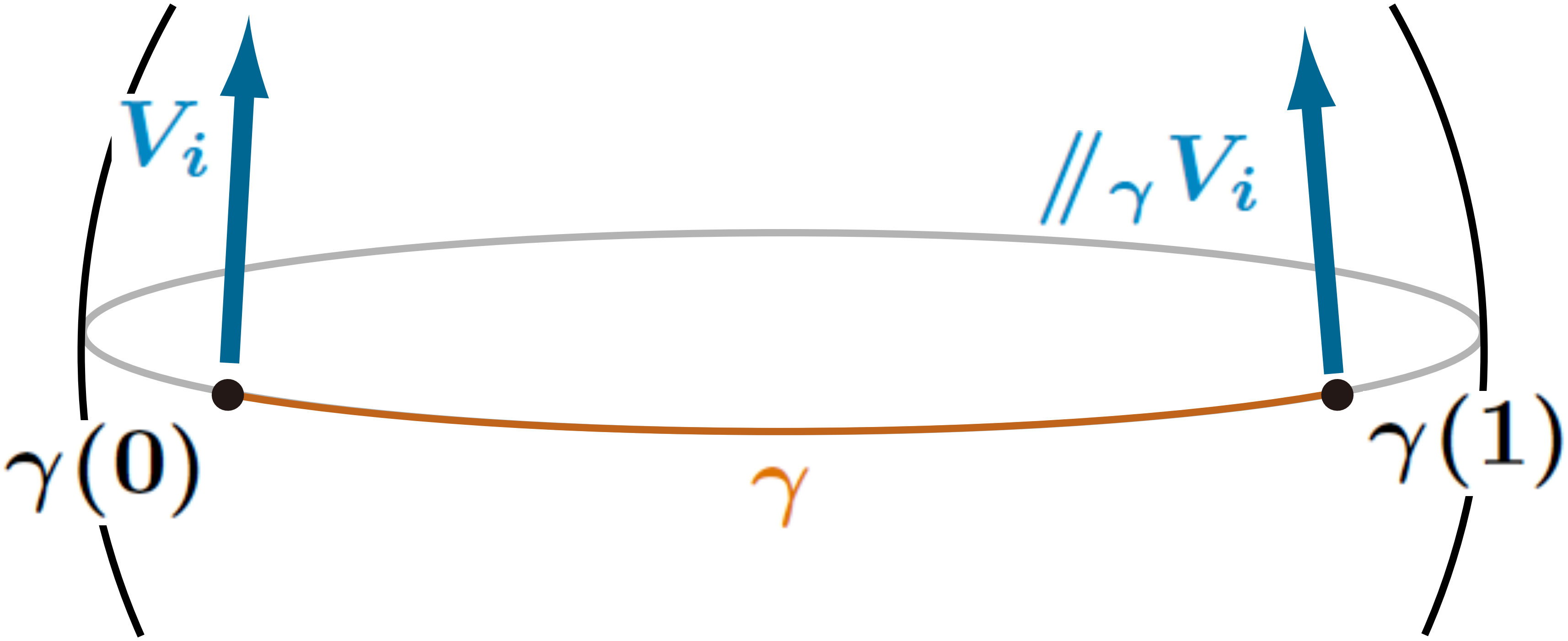

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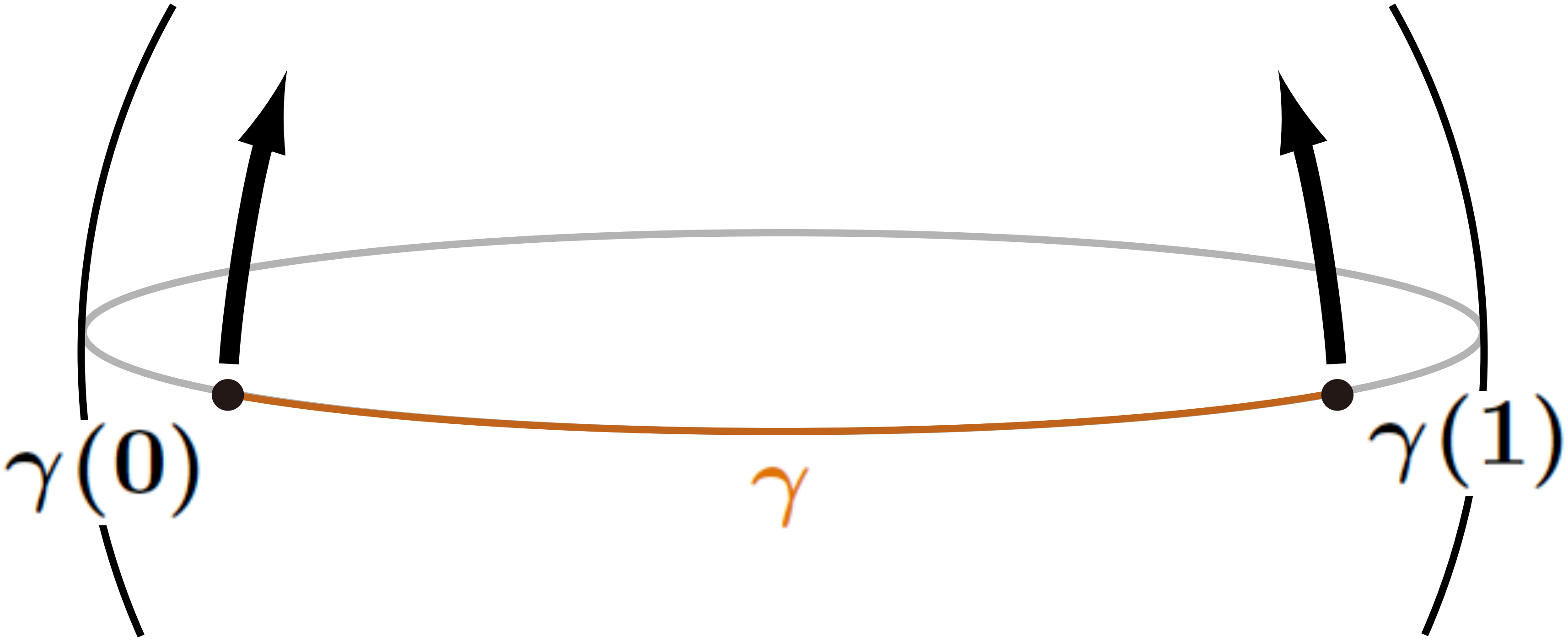

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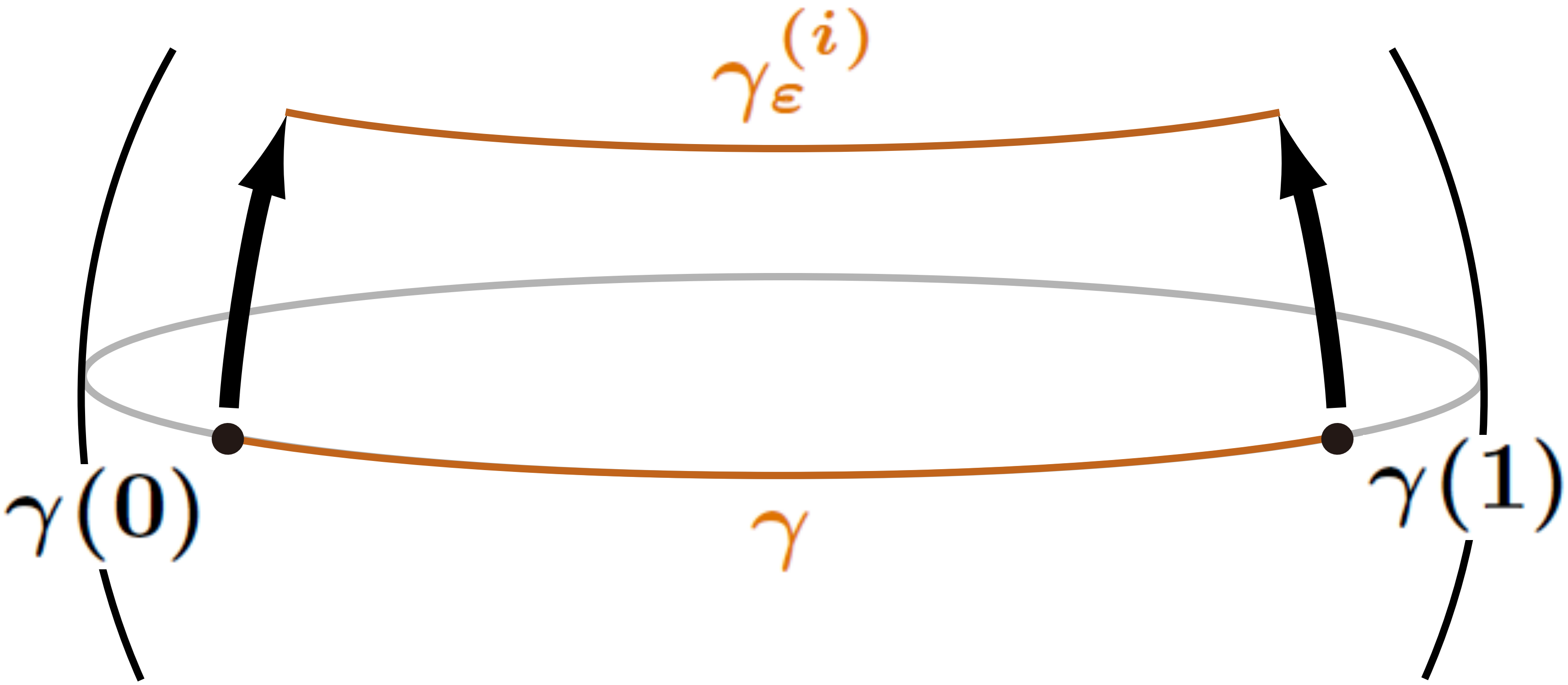

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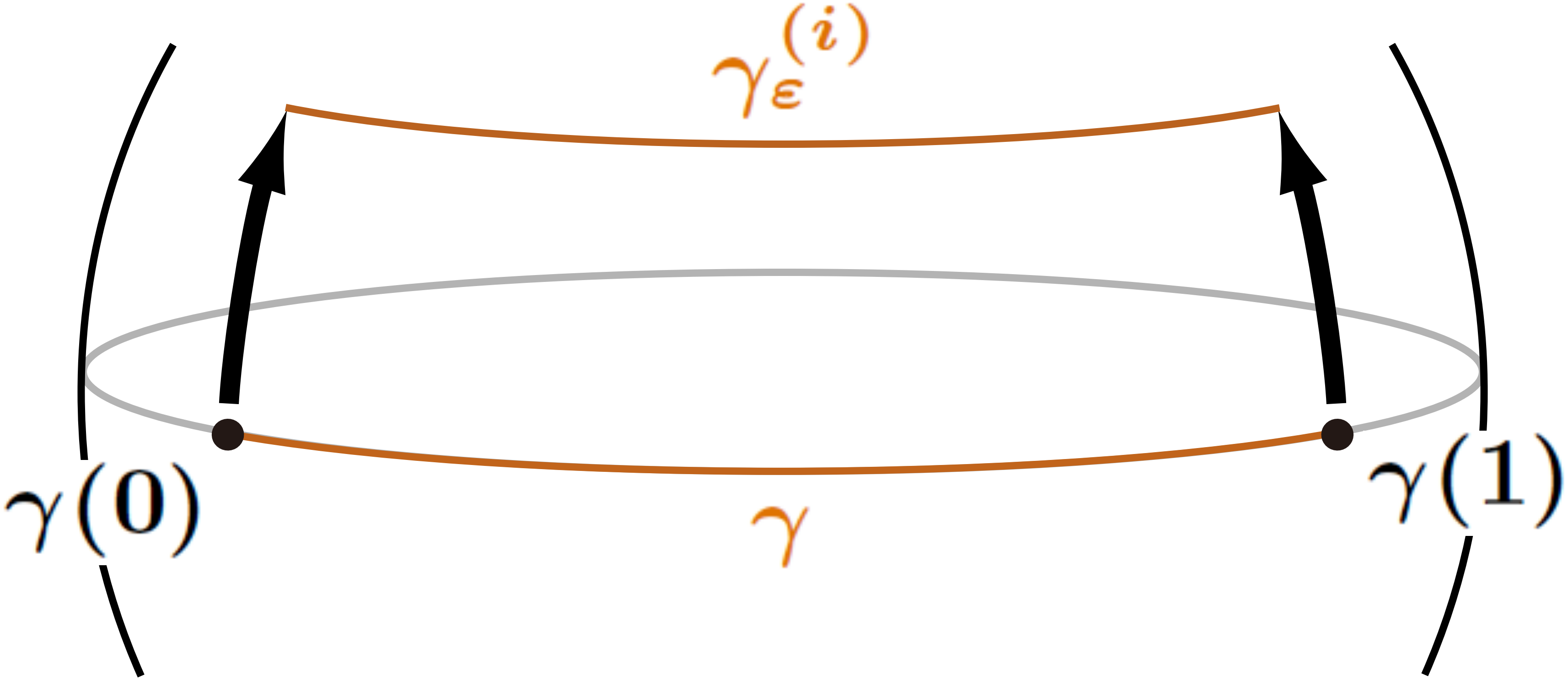

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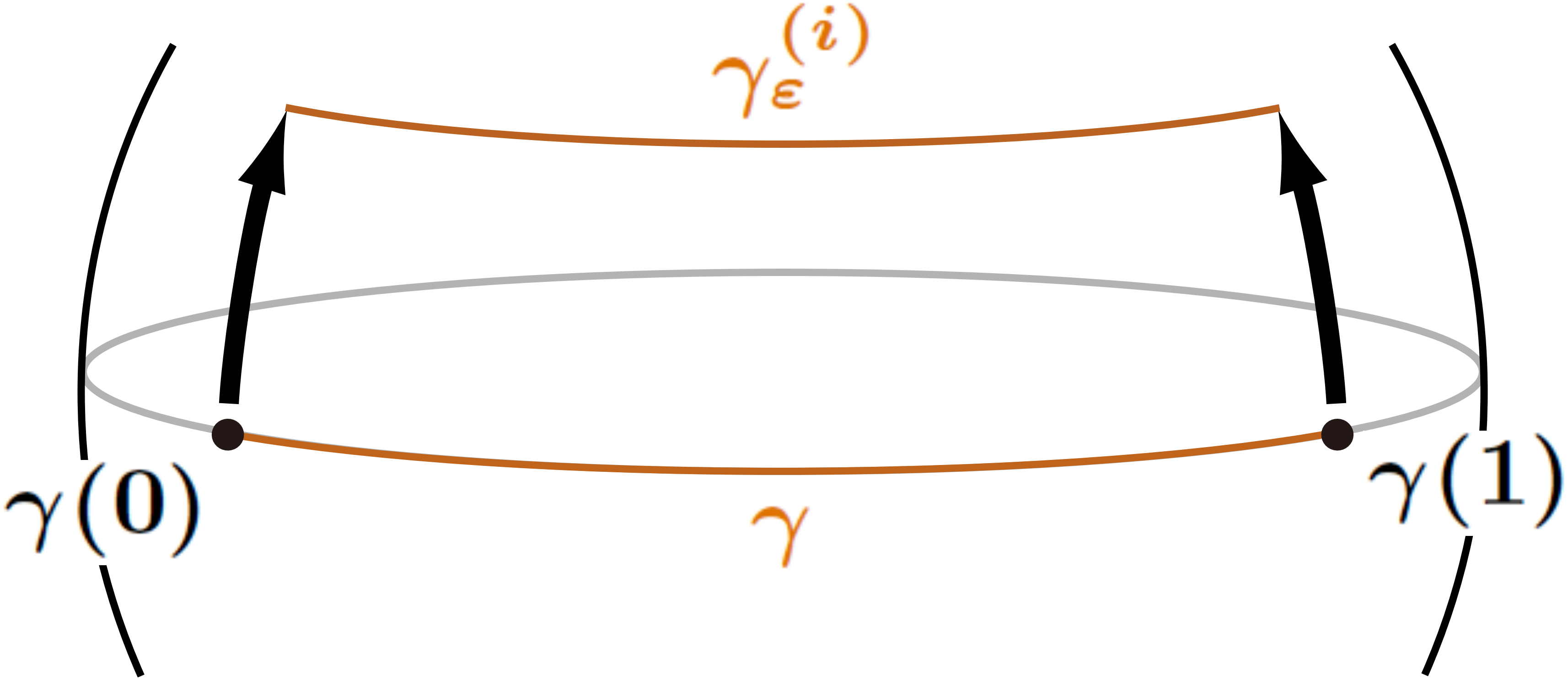

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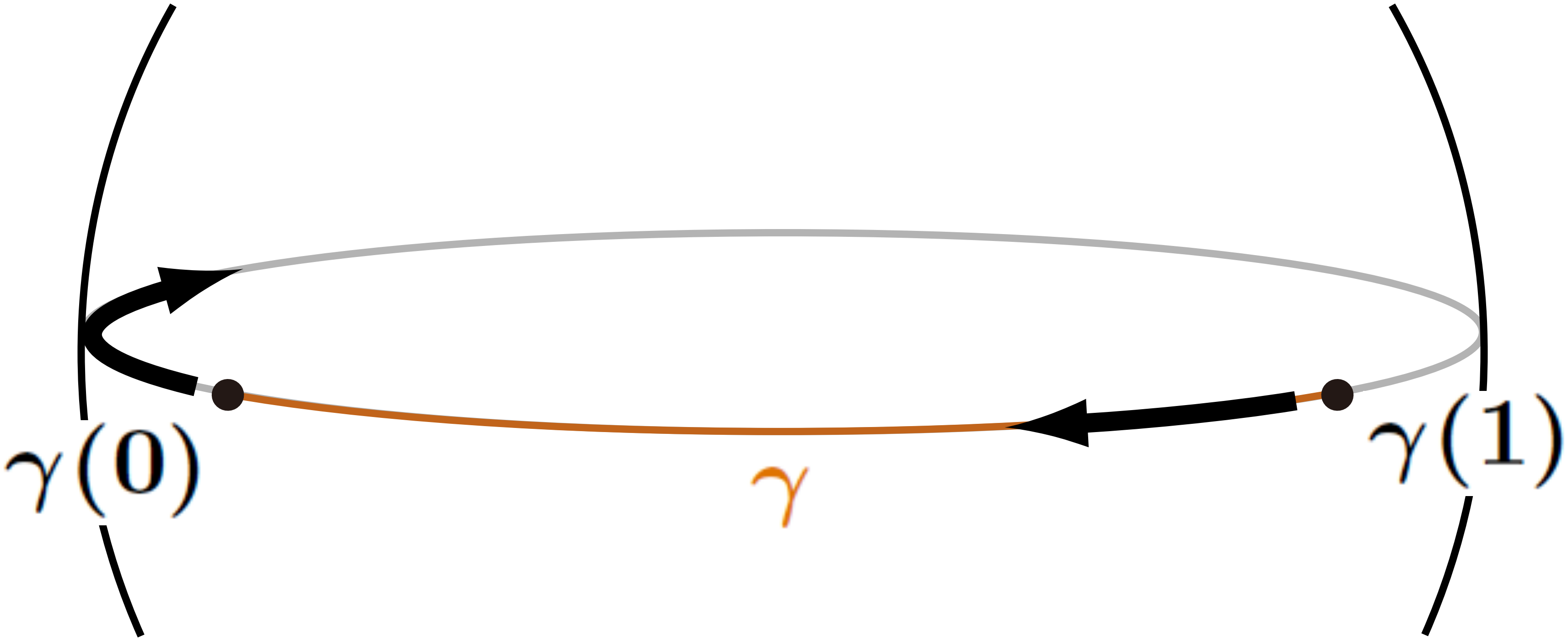

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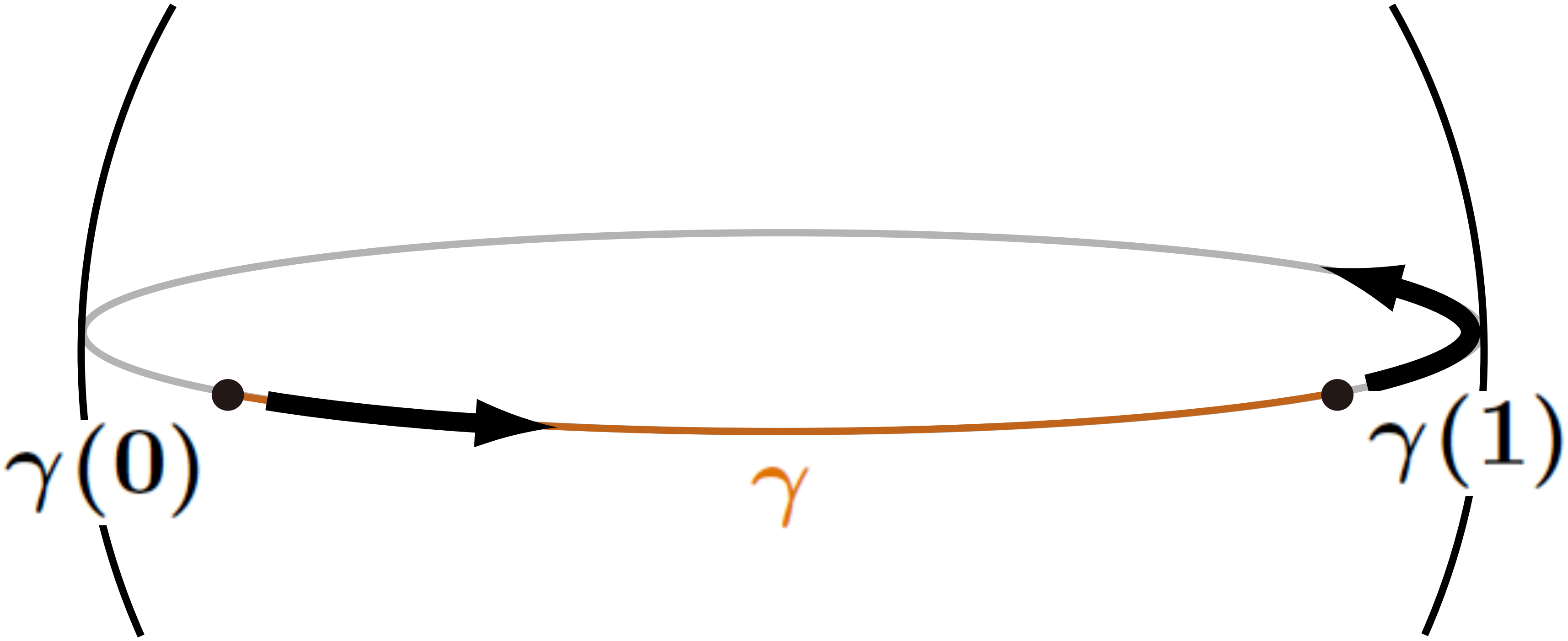

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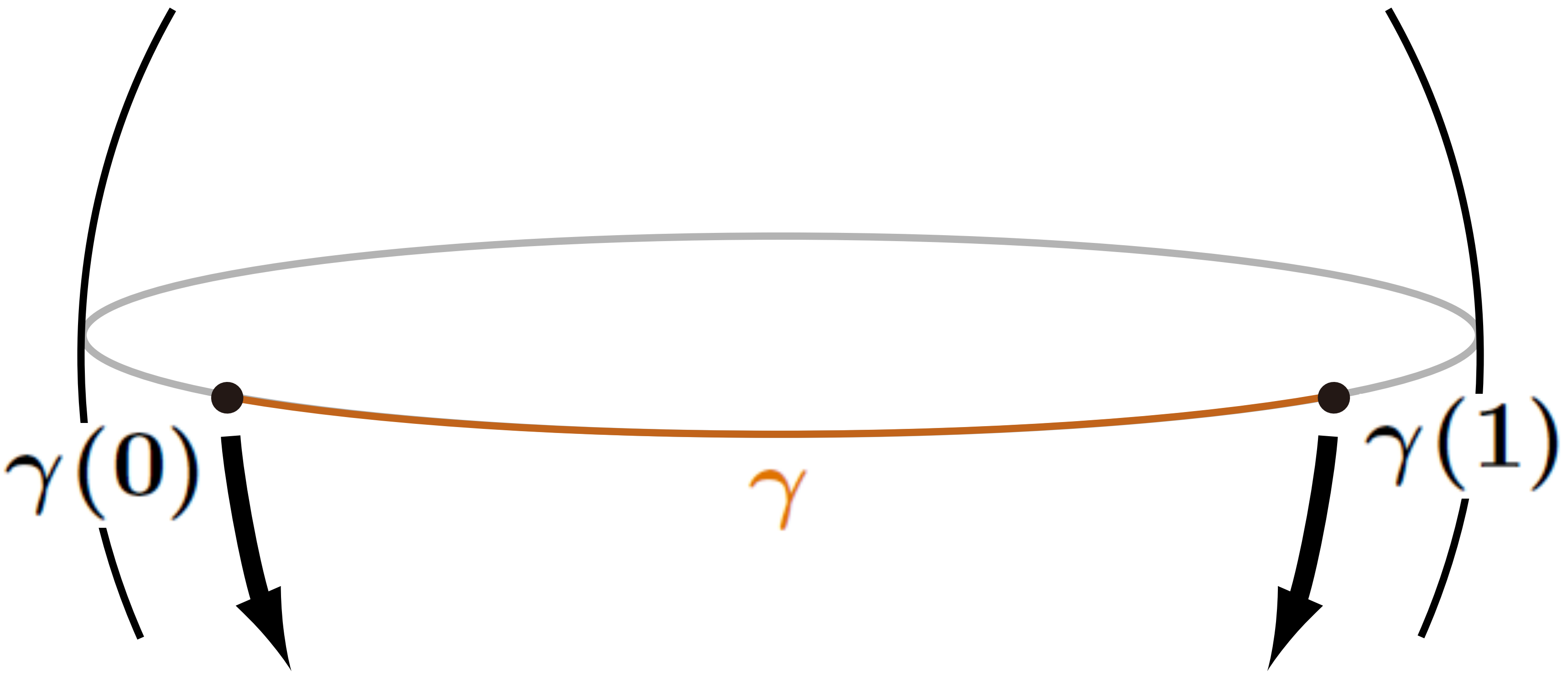

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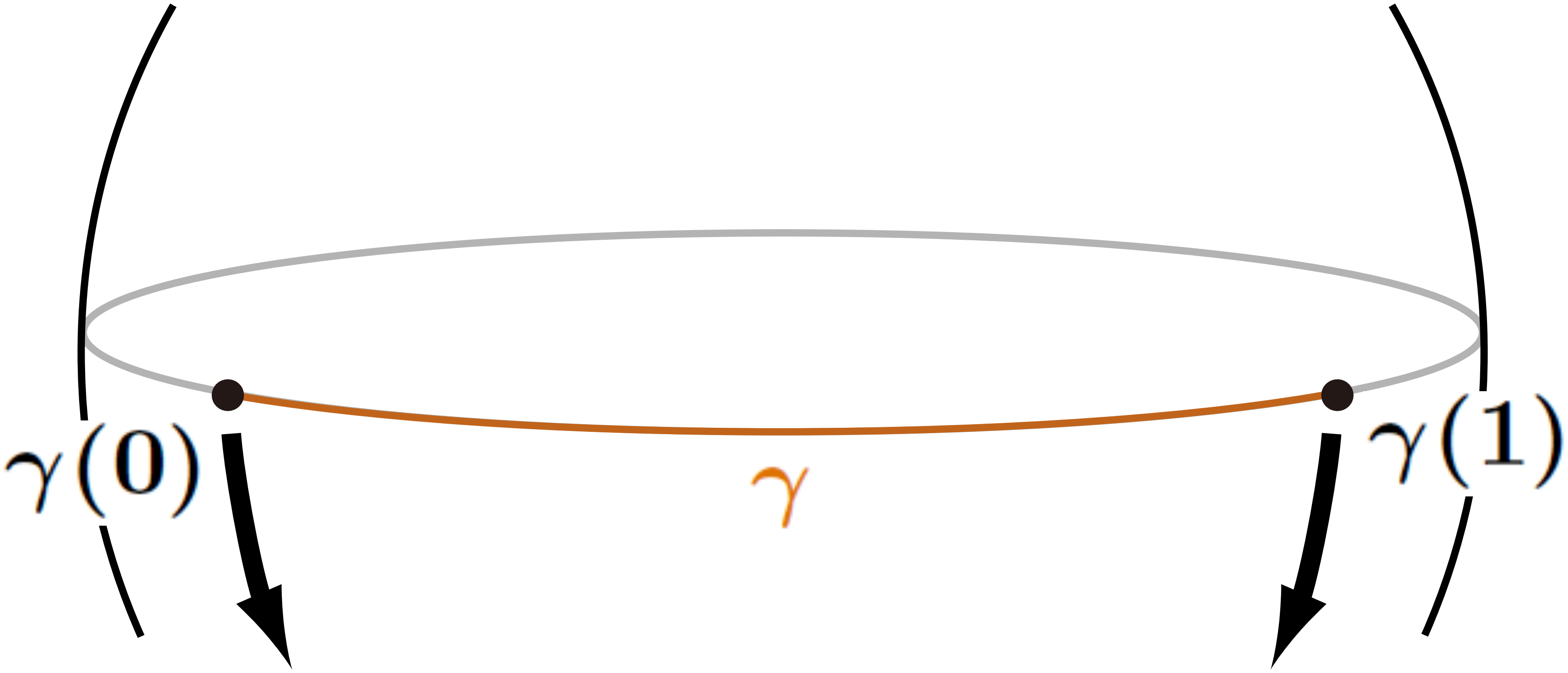

$$\sum_{i=1}^{m} \frac{\partial^2}{\partial \varepsilon^2} |\gamma_\varepsilon^{(i)}|$$

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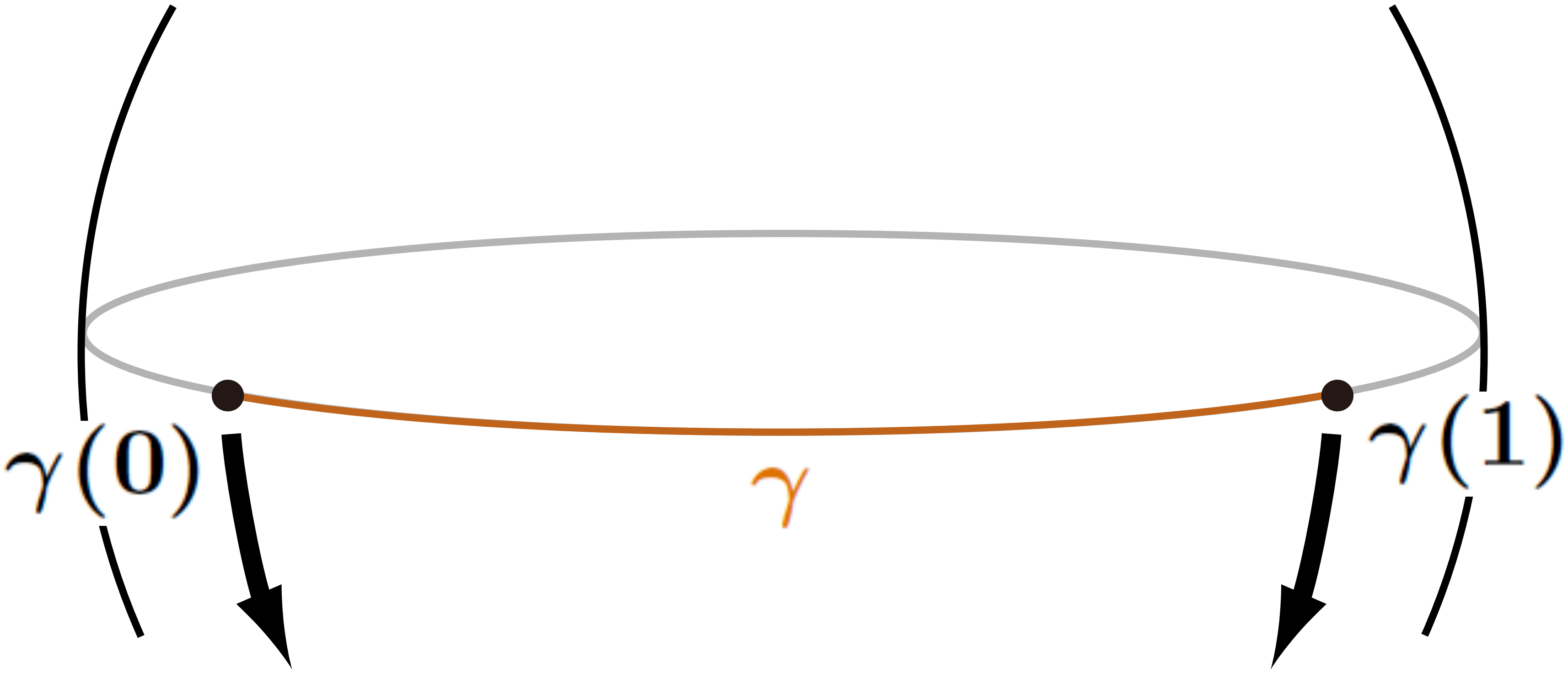

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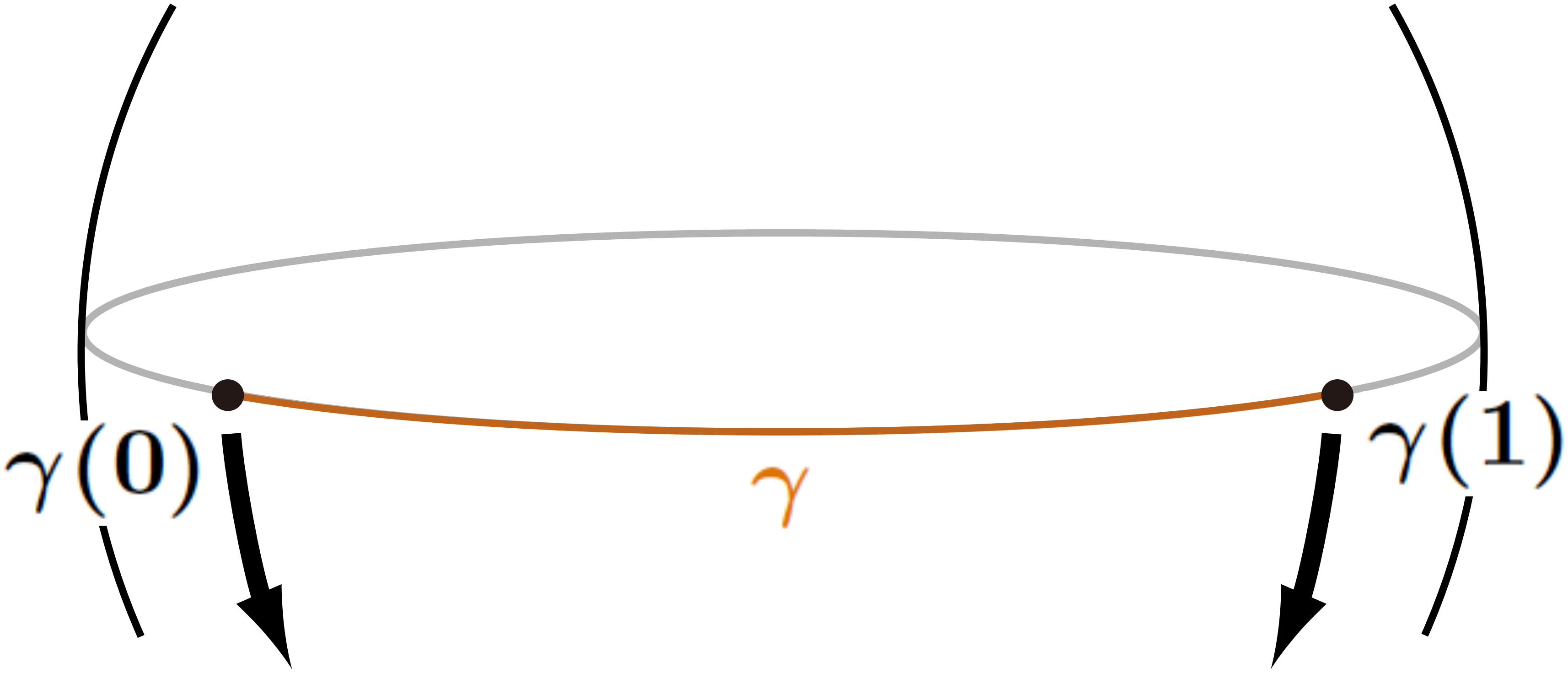
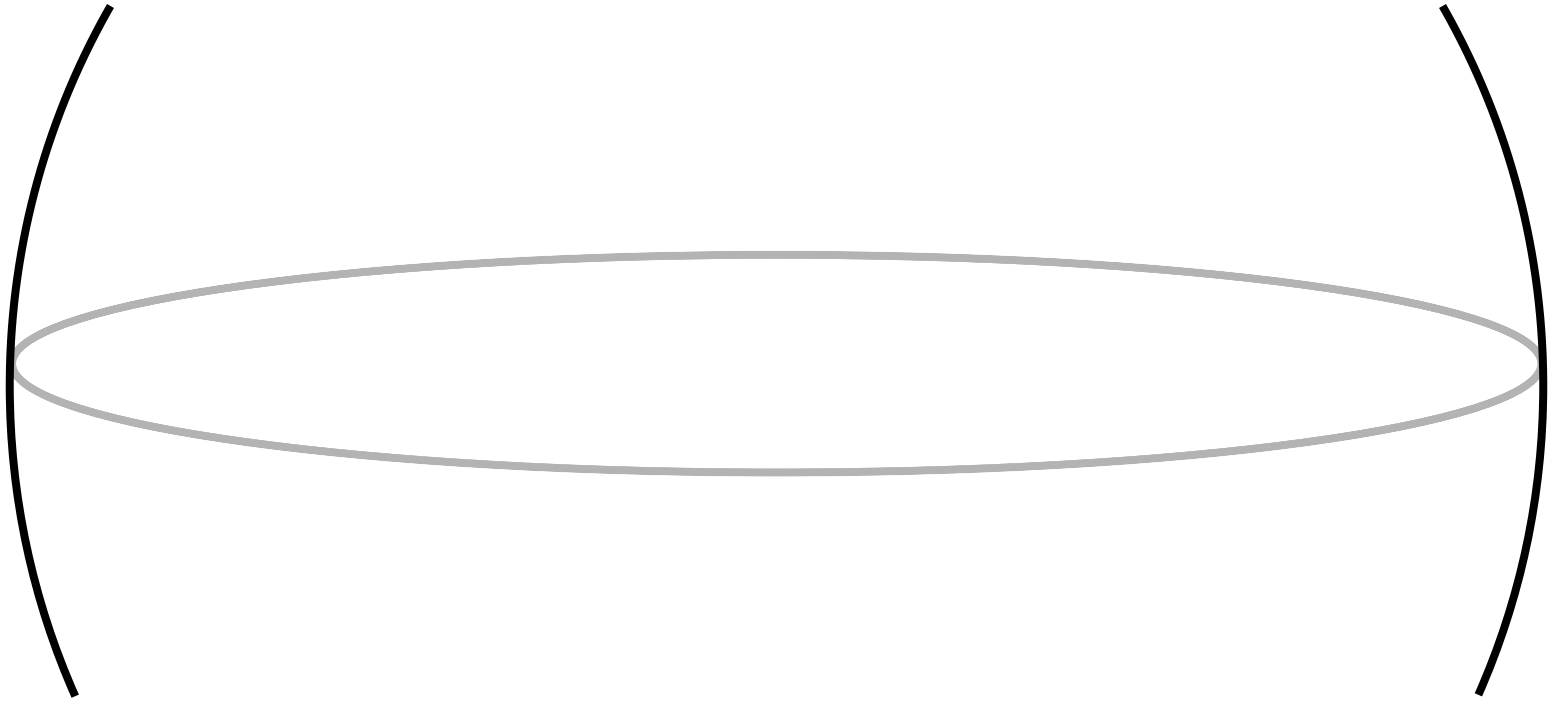


$$\mathrm{Ric} \geq K \Rightarrow \sum_{i=1}^{m} \frac{\partial^2}{\partial \varepsilon^2} |\gamma_\varepsilon^{(i)}| \leq -K|\gamma|$$

Coupling by parallel transport

$(B_0(t), B_1(t))$: coupling of BMs moving parallely



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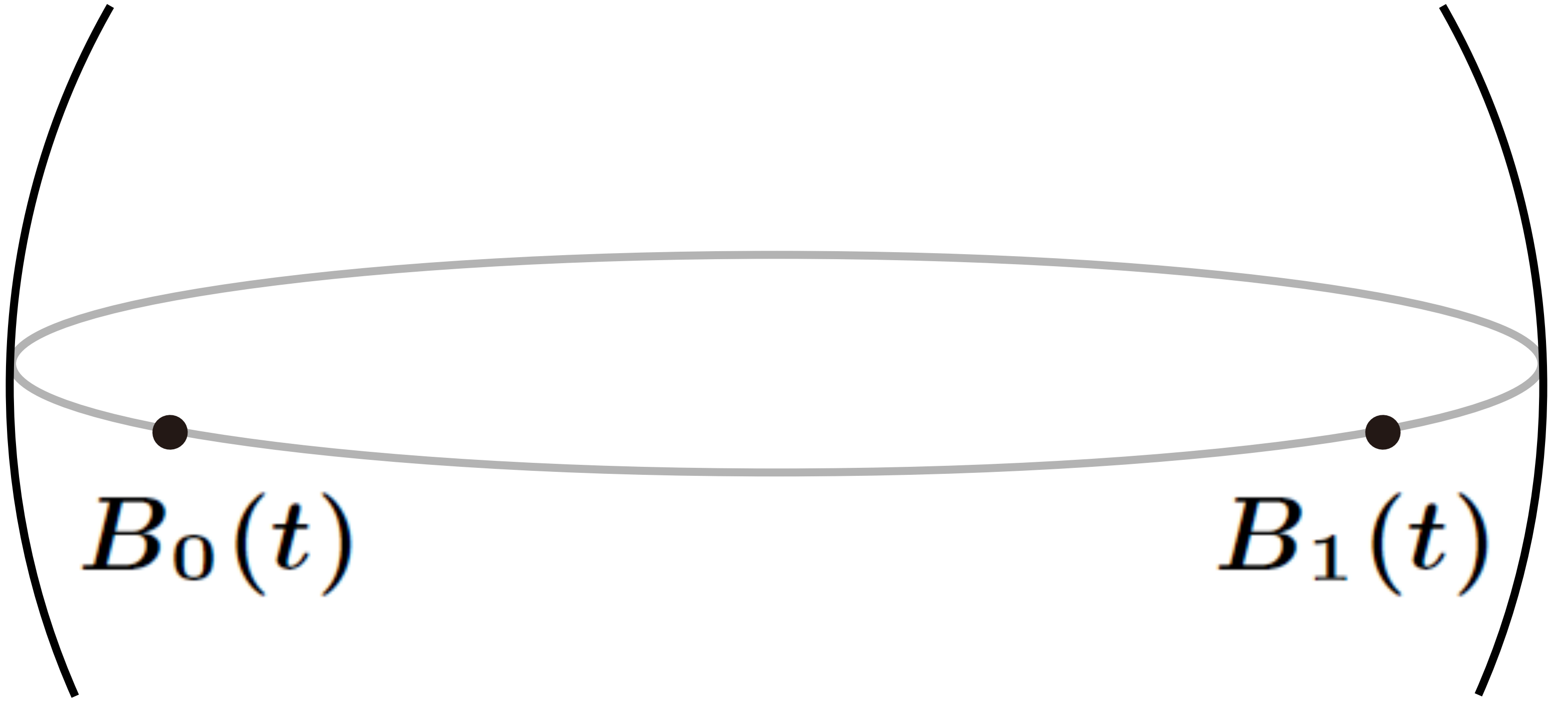

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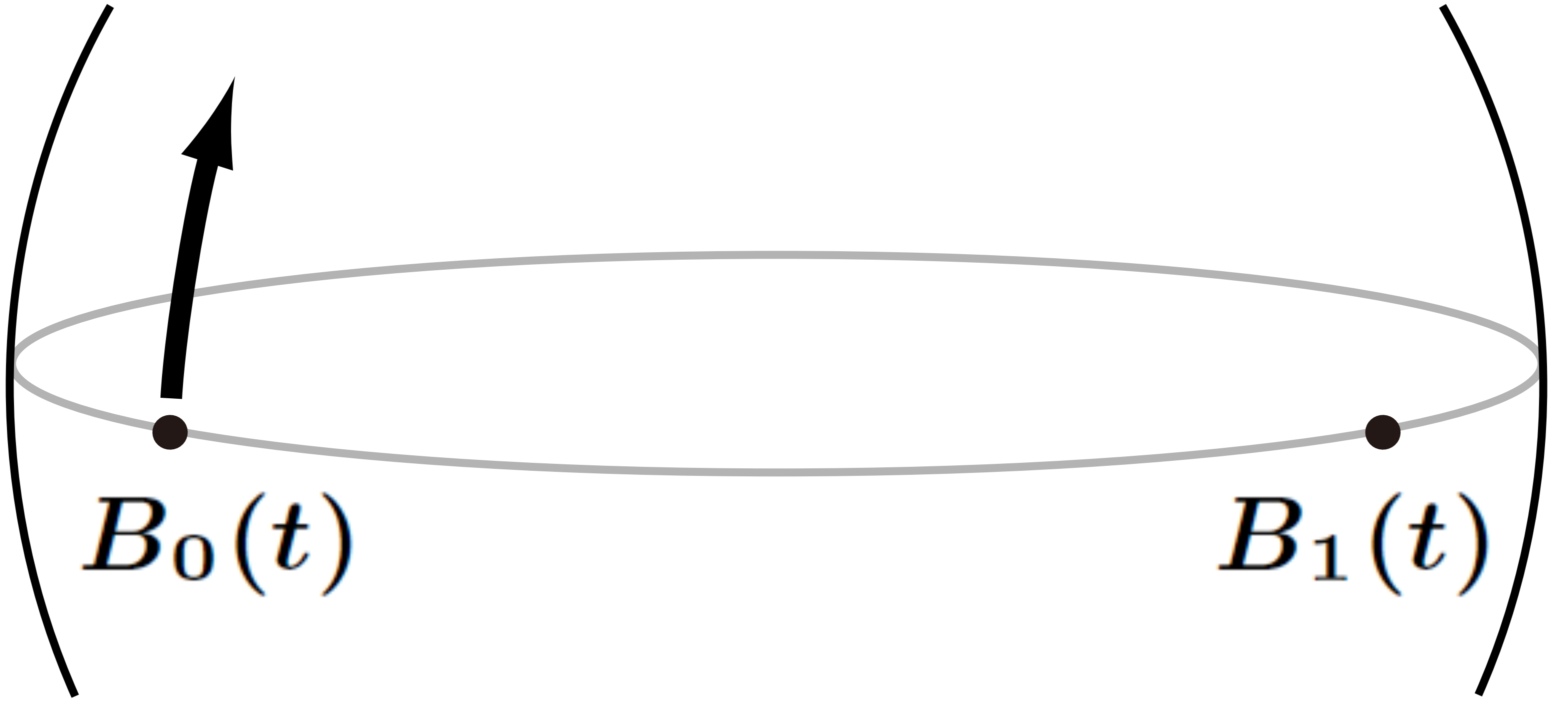

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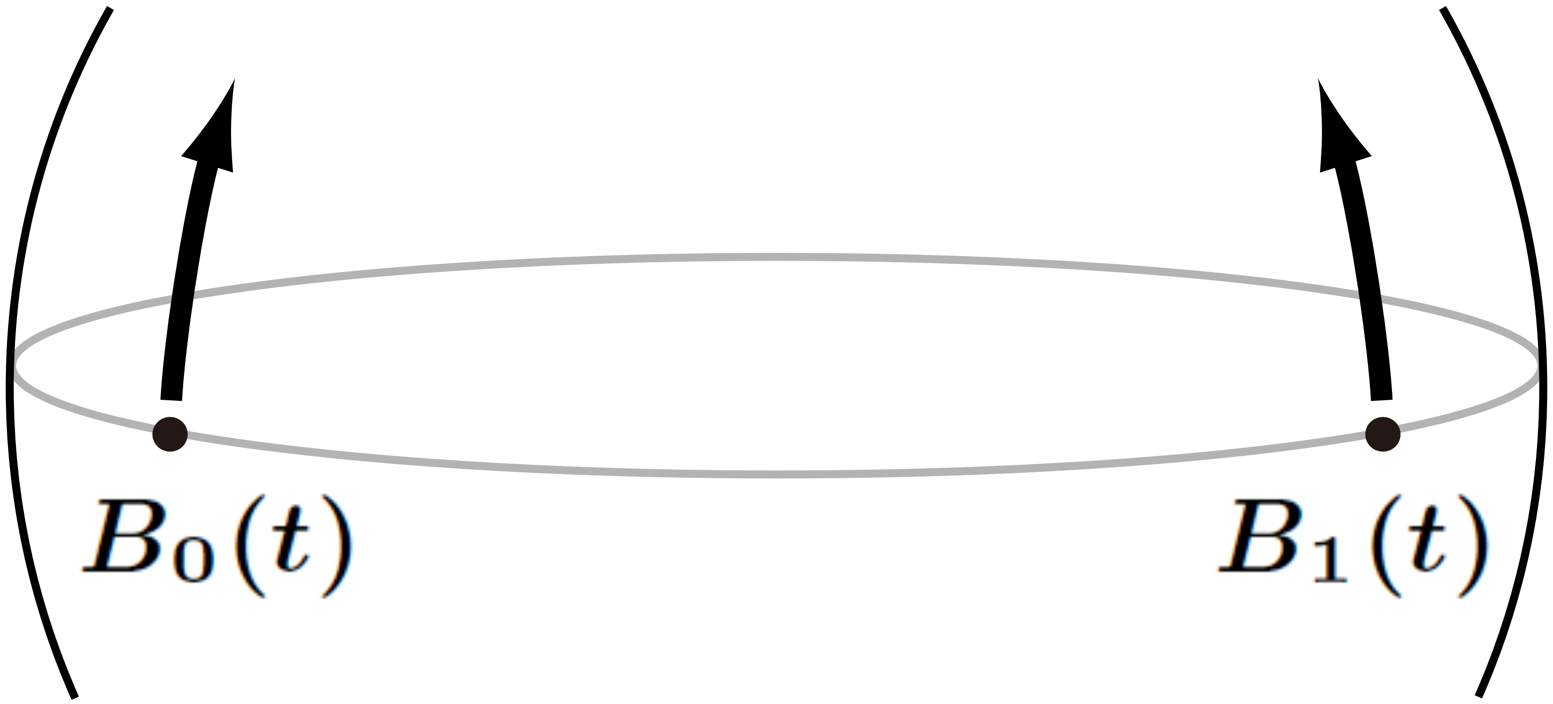

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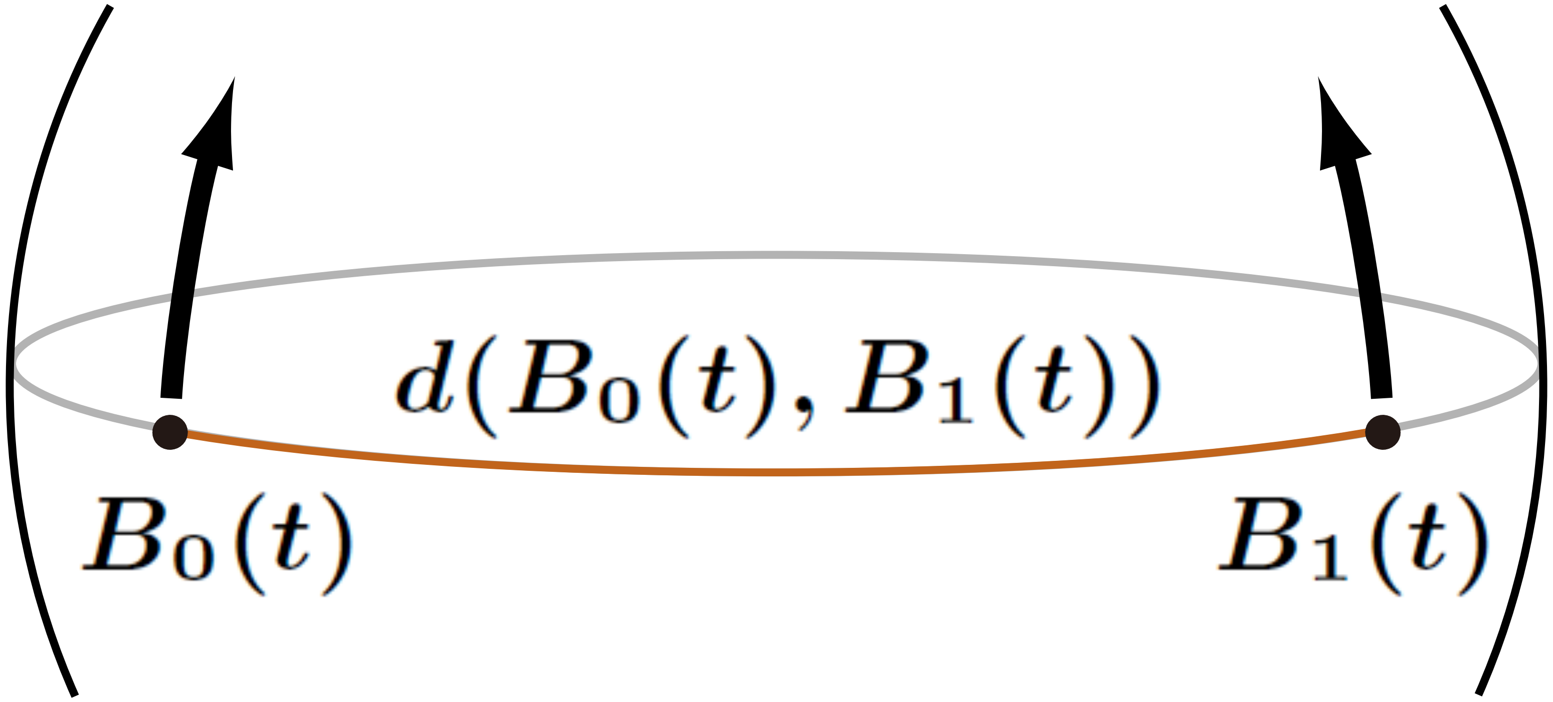

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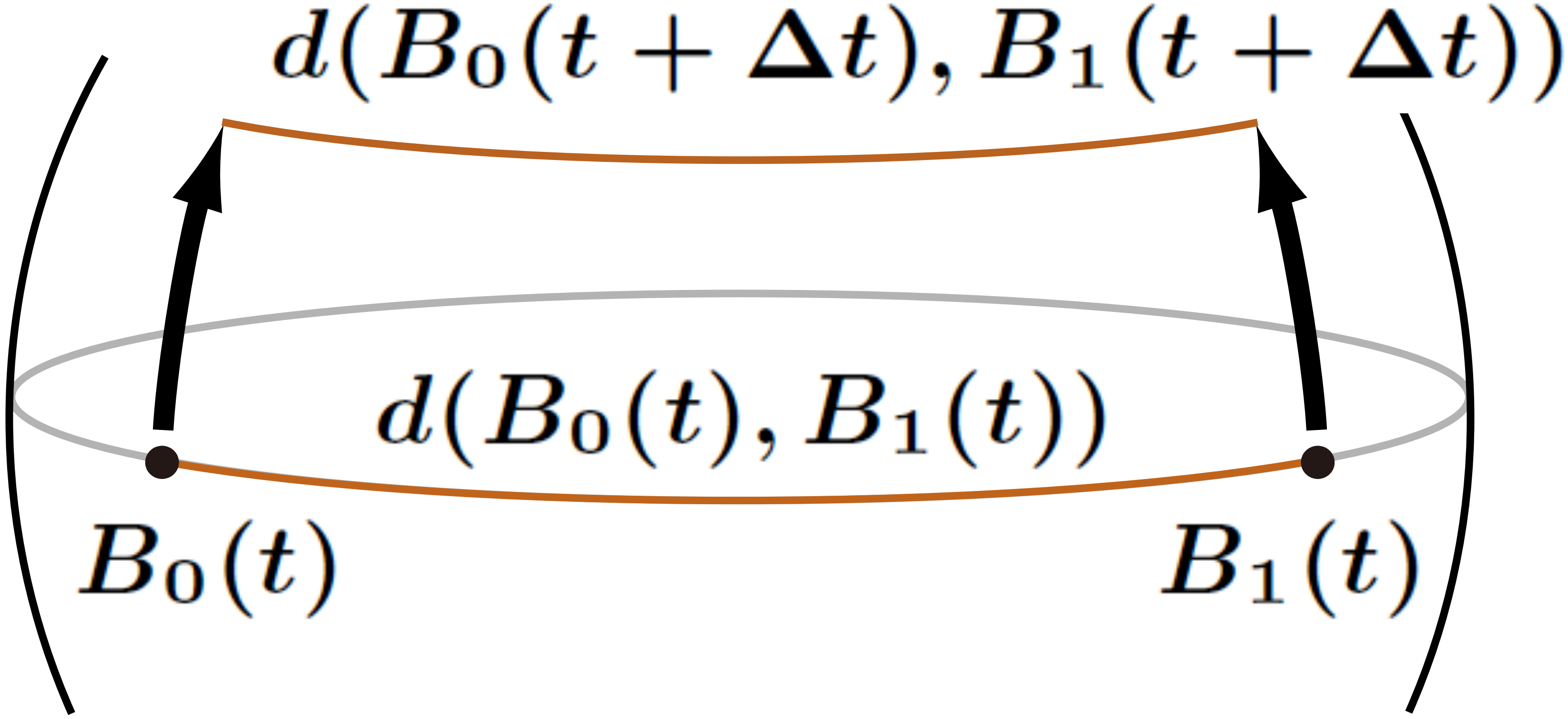

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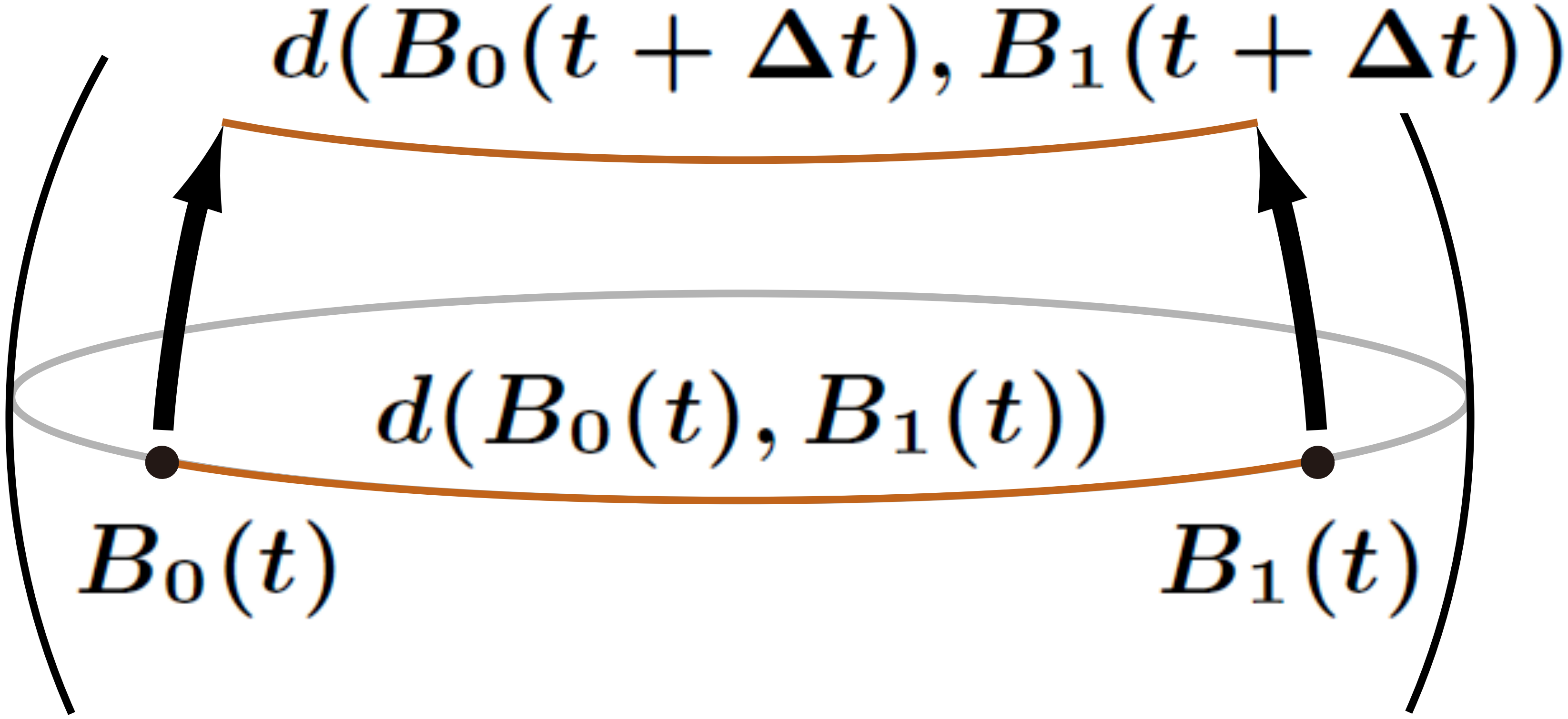

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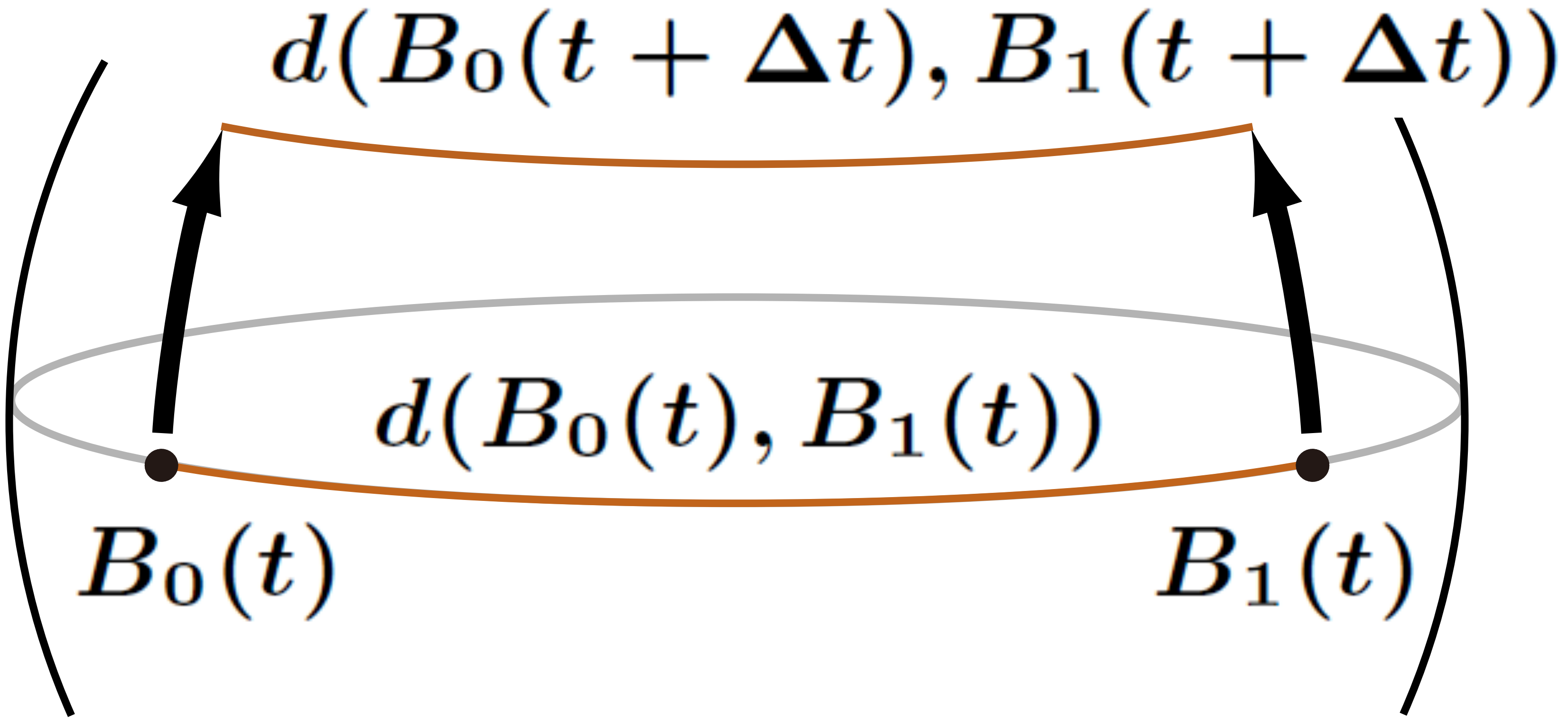

- [1st variation of $d(B_0(t), B_1(t))$] $= 0$
- "$\dfrac{\partial}{\partial t} d(B_0(t), B_1(t)) \leq -K d(B_0(t), B_1(t))$"

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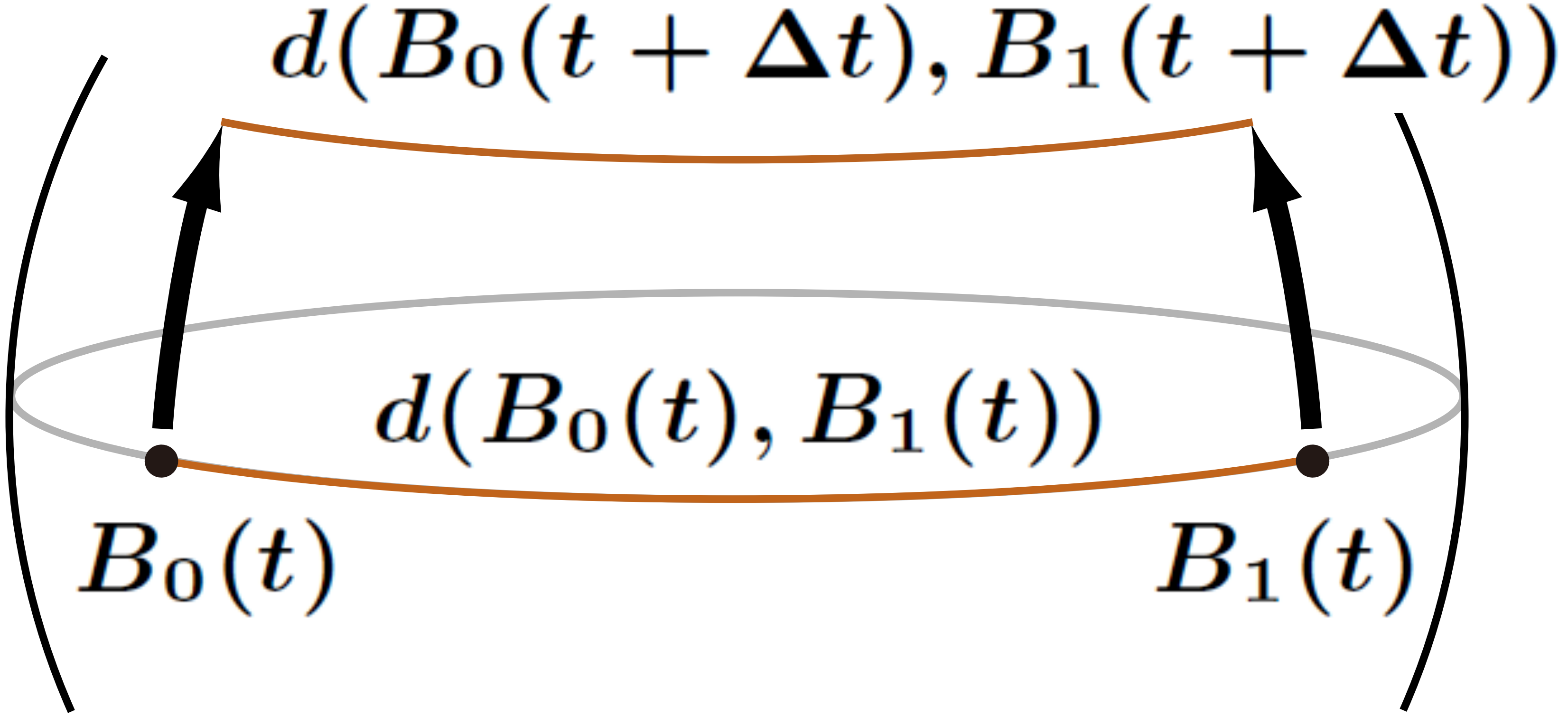

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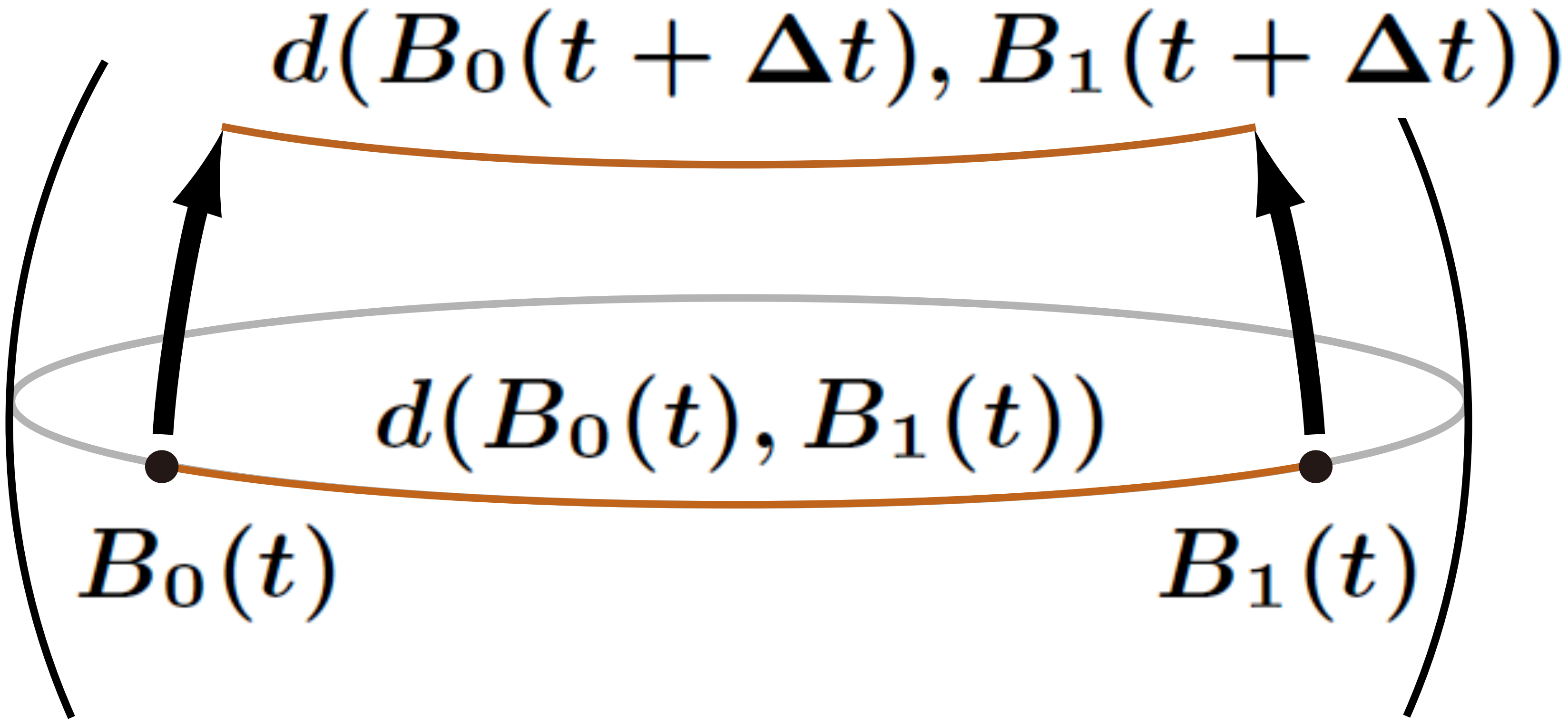

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  $$\Rightarrow d(B_0(t), B_1(t)) \leq e^{-Kt} d(B_0(0), B_1(0))$$

Wasserstein distance W_p

$$d(B_0(t), B_1(t)) \leq e^{-Kt} d(B_0(0), B_1(0))$$

$\Downarrow \mathbb{E}[\cdot]$, minimizing over all couplings

$$\boxed{W_p(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_p(\mu_0, \mu_1)}$$

$(1 \leq p \leq \infty)$

$$W_p(\mu, \nu) := \inf_{\pi} \|d\|_{L^p(\pi)}$$: L^p-Wasserstein distance

$\Lsh$ coupling of μ and ν $\quad (p \in [1, \infty])$

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- W_p: (pseudo-)distance on $\mathcal{P}(M)$
- Conv w.r.t. $W_p \Rightarrow$ weak conv. on $\mathcal{P}(M)$
- For $W_p(\mu_0,\mu_1)$, ${}^{\exists}(\mu_t)_{t\in[0,1]}$: geod. interpolation

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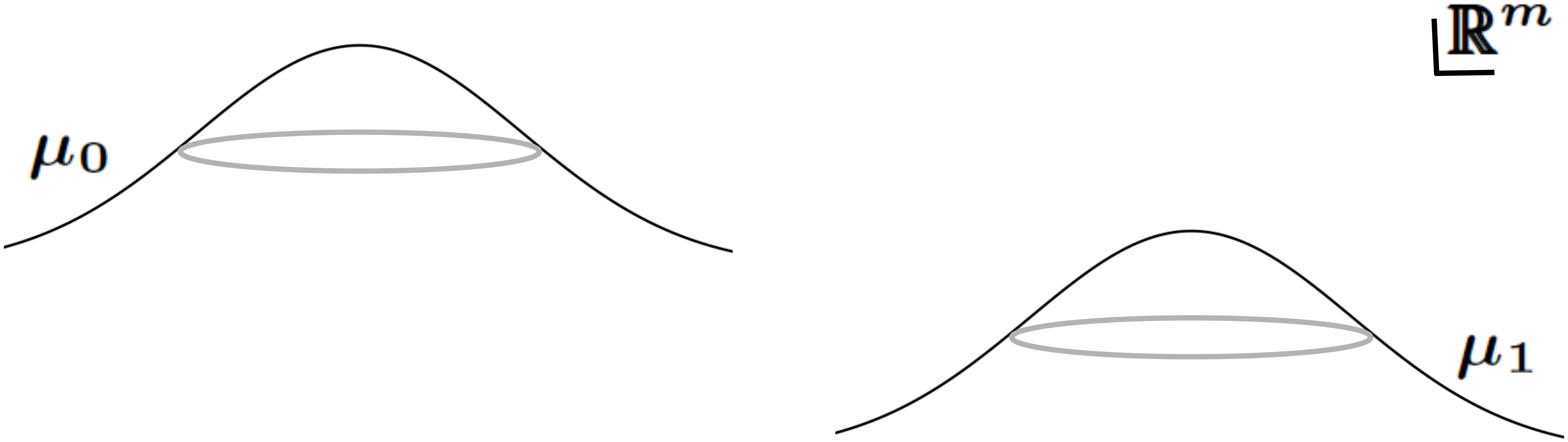

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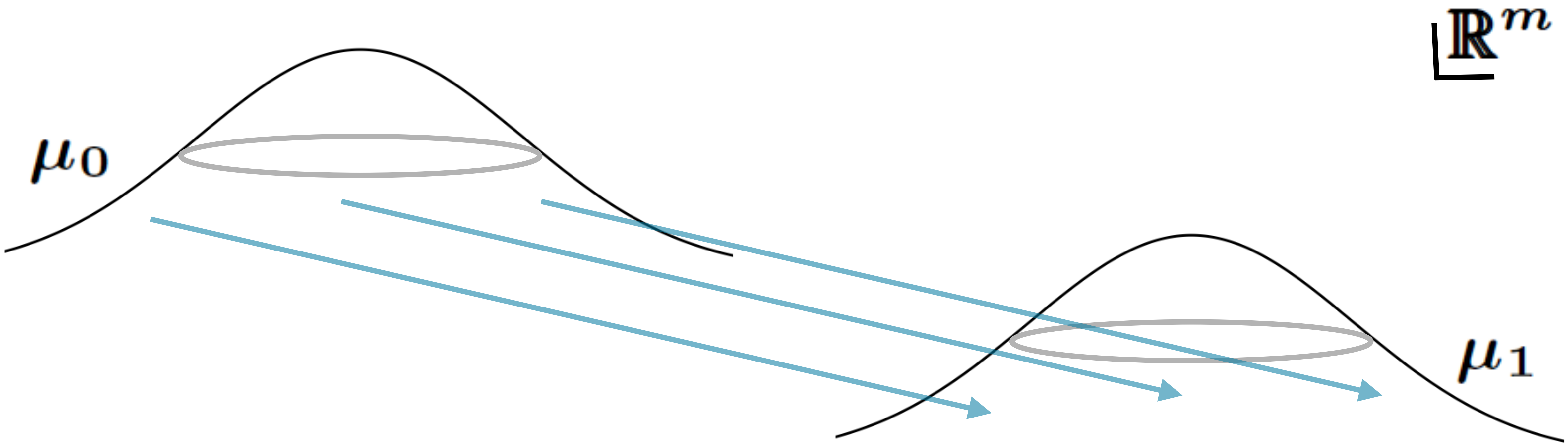

Wasserstein distance W_p

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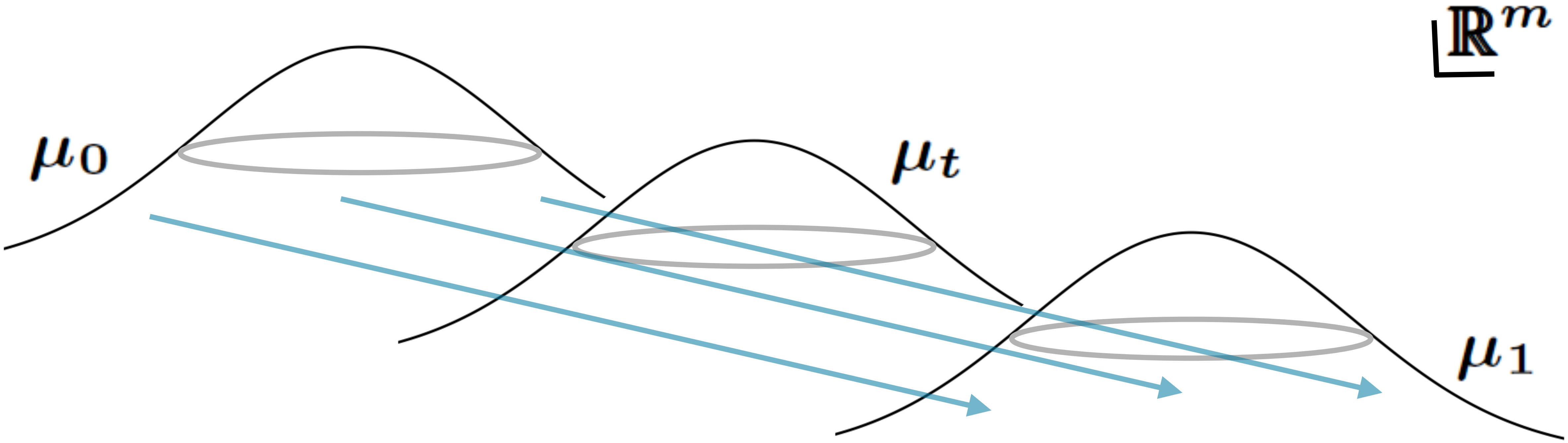

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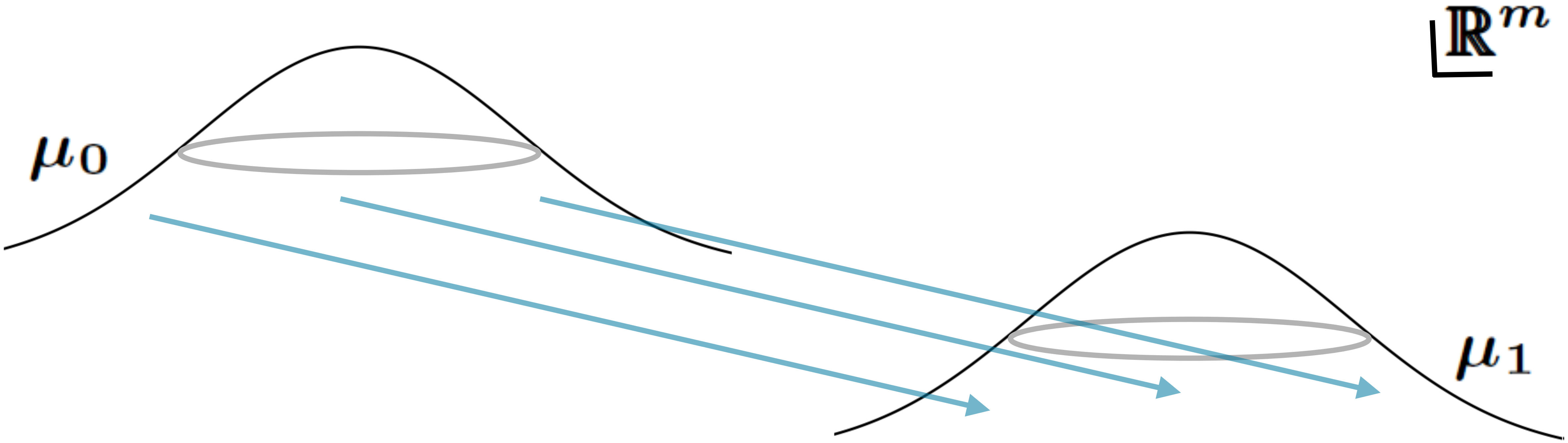

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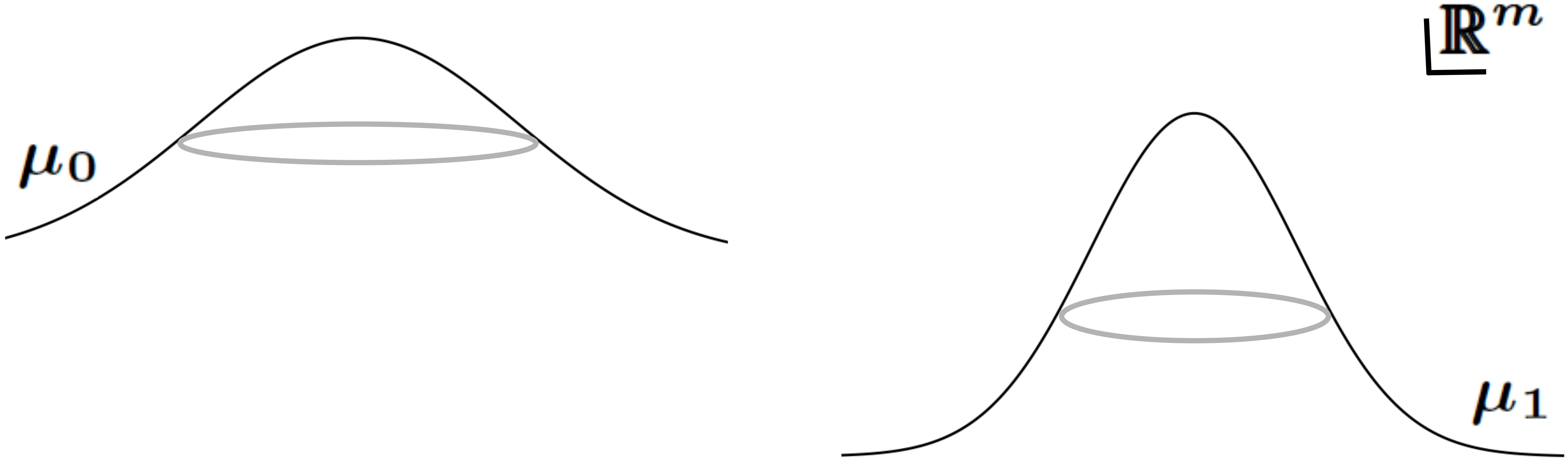

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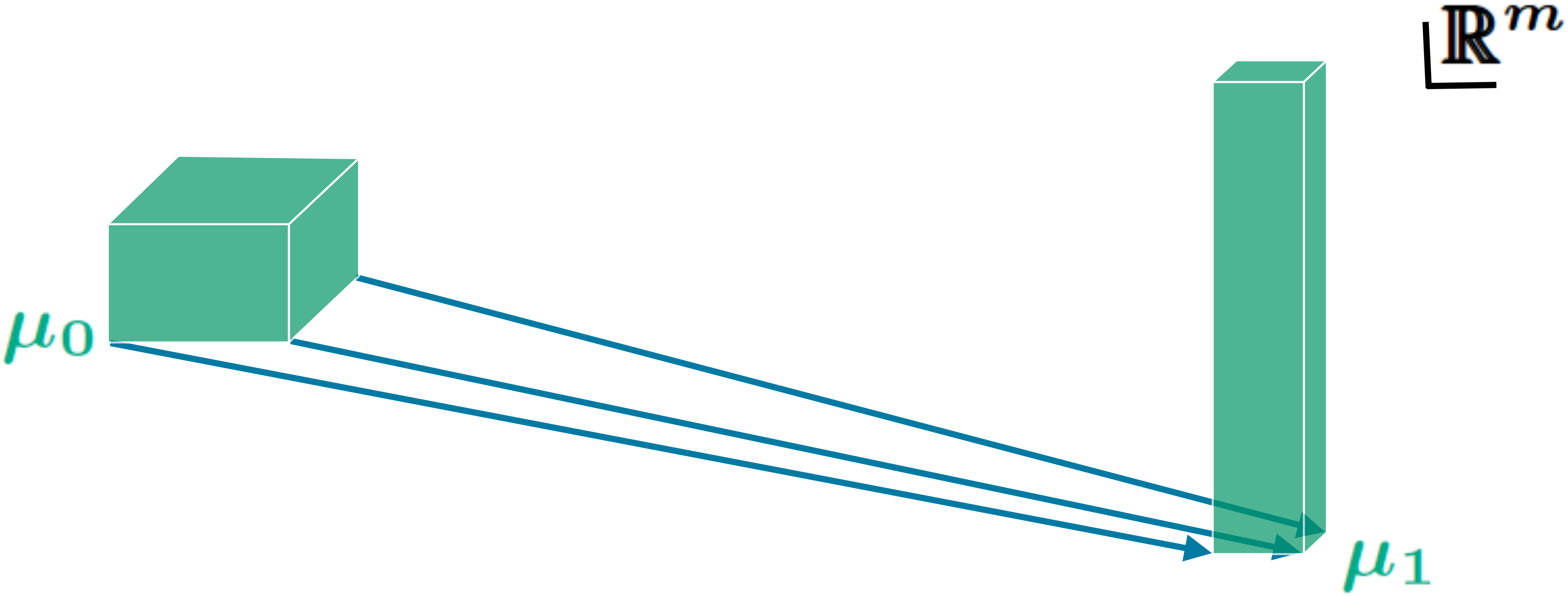

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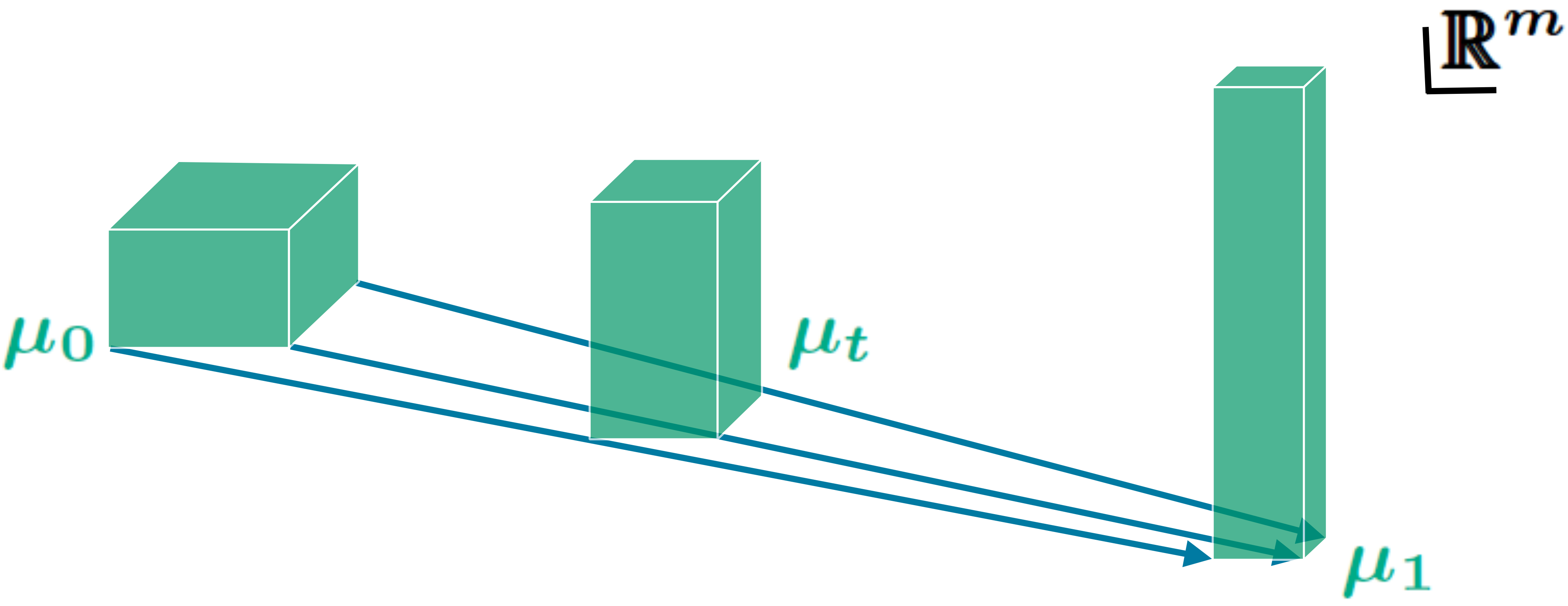

Volume distortion

Volume meas.: $\mathfrak{m} := \sqrt{\det g_{ij}}\,dx_1 \cdots dx_n$

$\rightsquigarrow$ $\mathbf{Ric}$ in volume distortion along W_2-geod:

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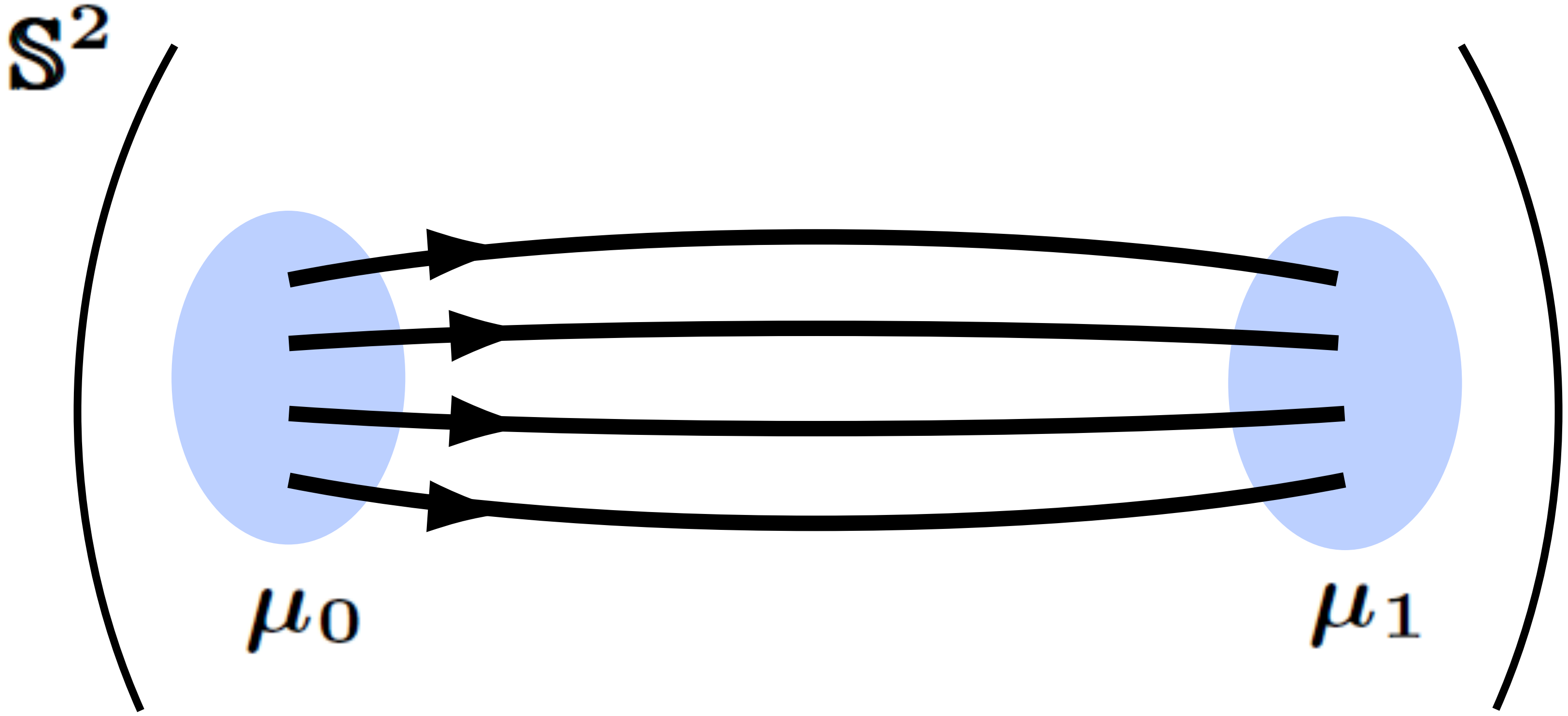

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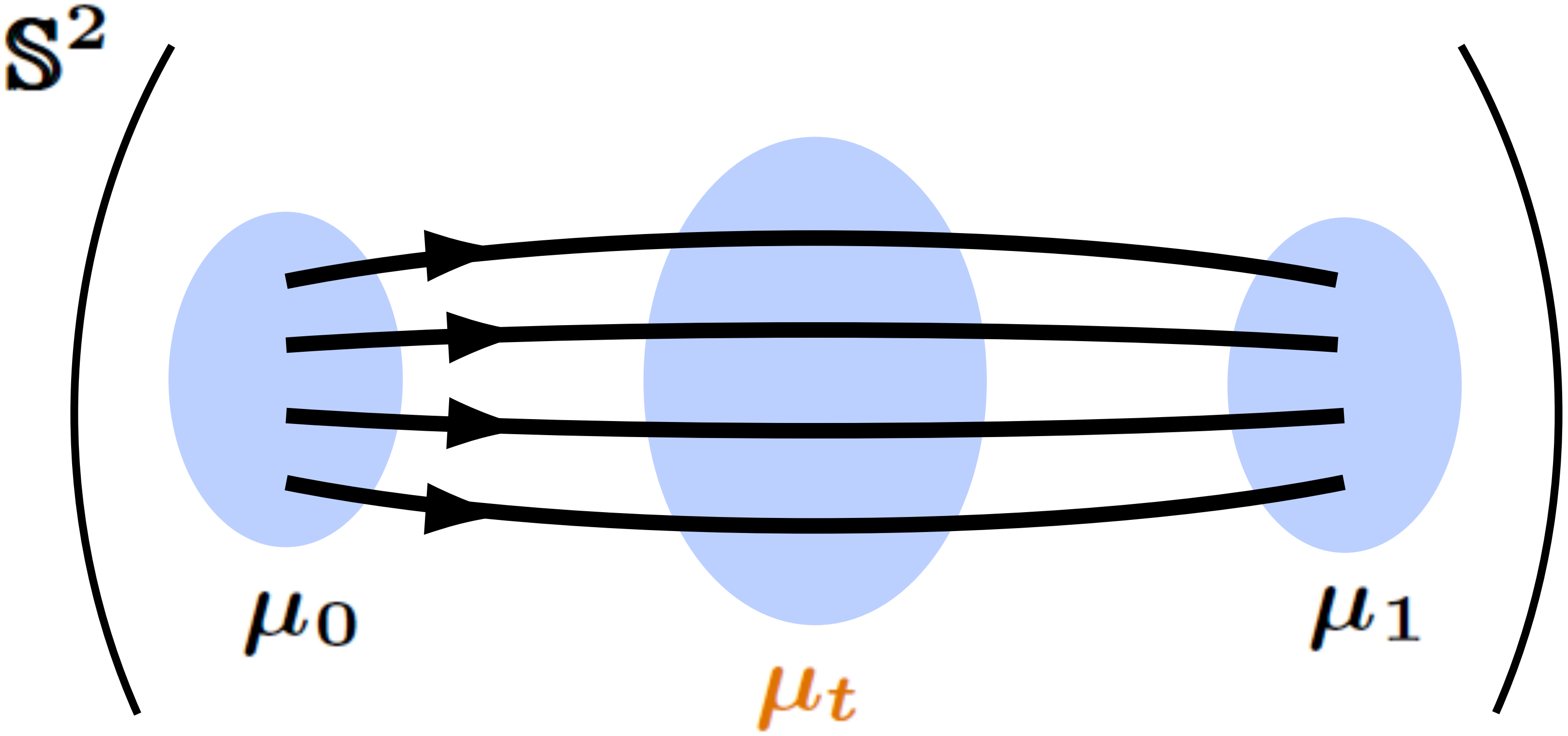

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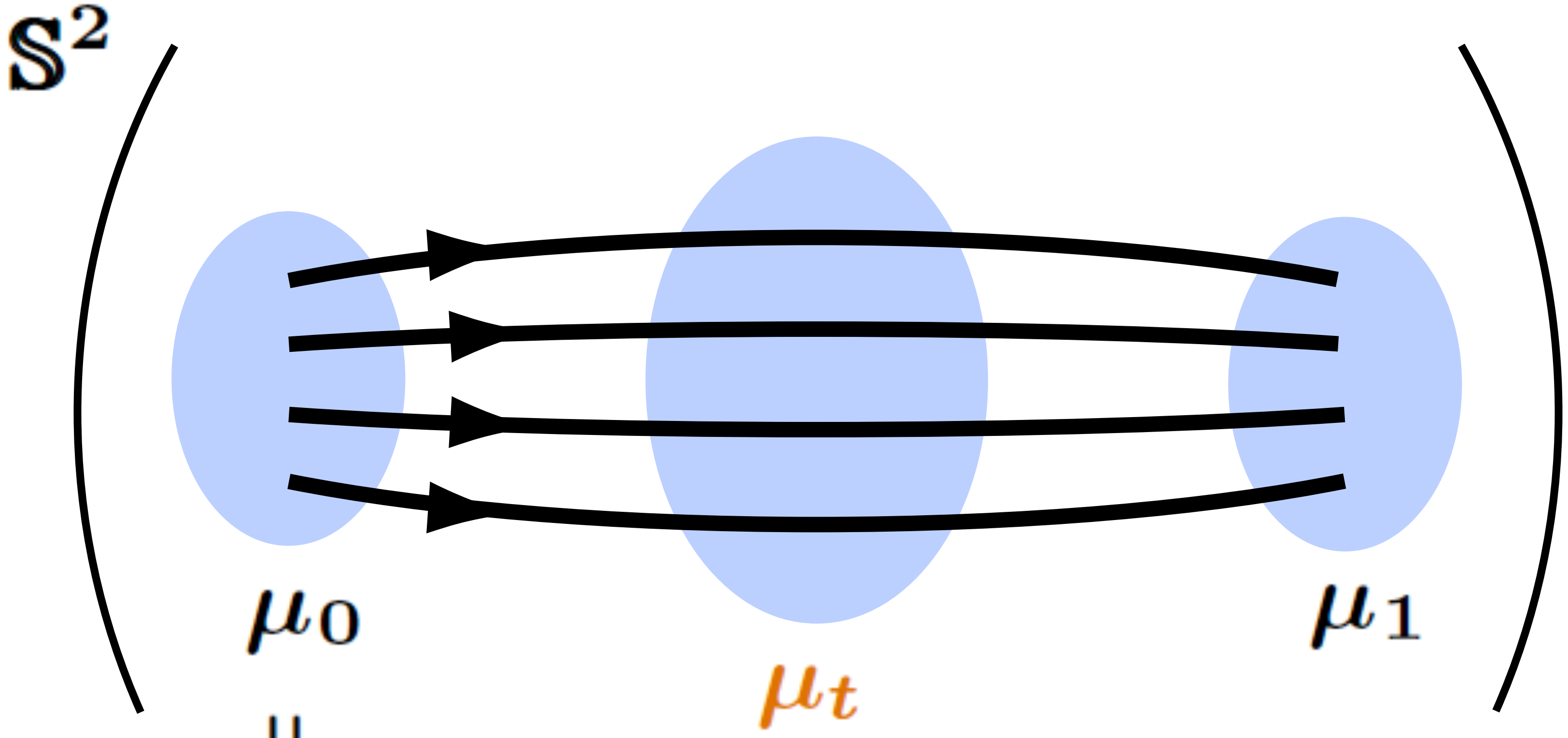


$\Downarrow$

$\mathbf{Ric} \geq K \Rightarrow$ relative entropy $\mathbf{Ent}$ is K-convex

[Cordero-Erausquin, McCann & Schmuckenschläger '01]

[von Renesse & Sturm '05]

Convexity of Ent

$$\mathrm{Ent}(\mu) := \int_M \rho \log \rho \, d\mathfrak{m} \quad (\mu = \rho \mathfrak{m})$$

Ent is K-convex:

$$\mathrm{Ent}(\mu_t) \le (1-t)\,\mathrm{Ent}(\mu_0) + t\,\mathrm{Ent}(\mu_1) - \frac{K}{2} t(1-t) W_2(\mu_0, \mu_1)^2$$

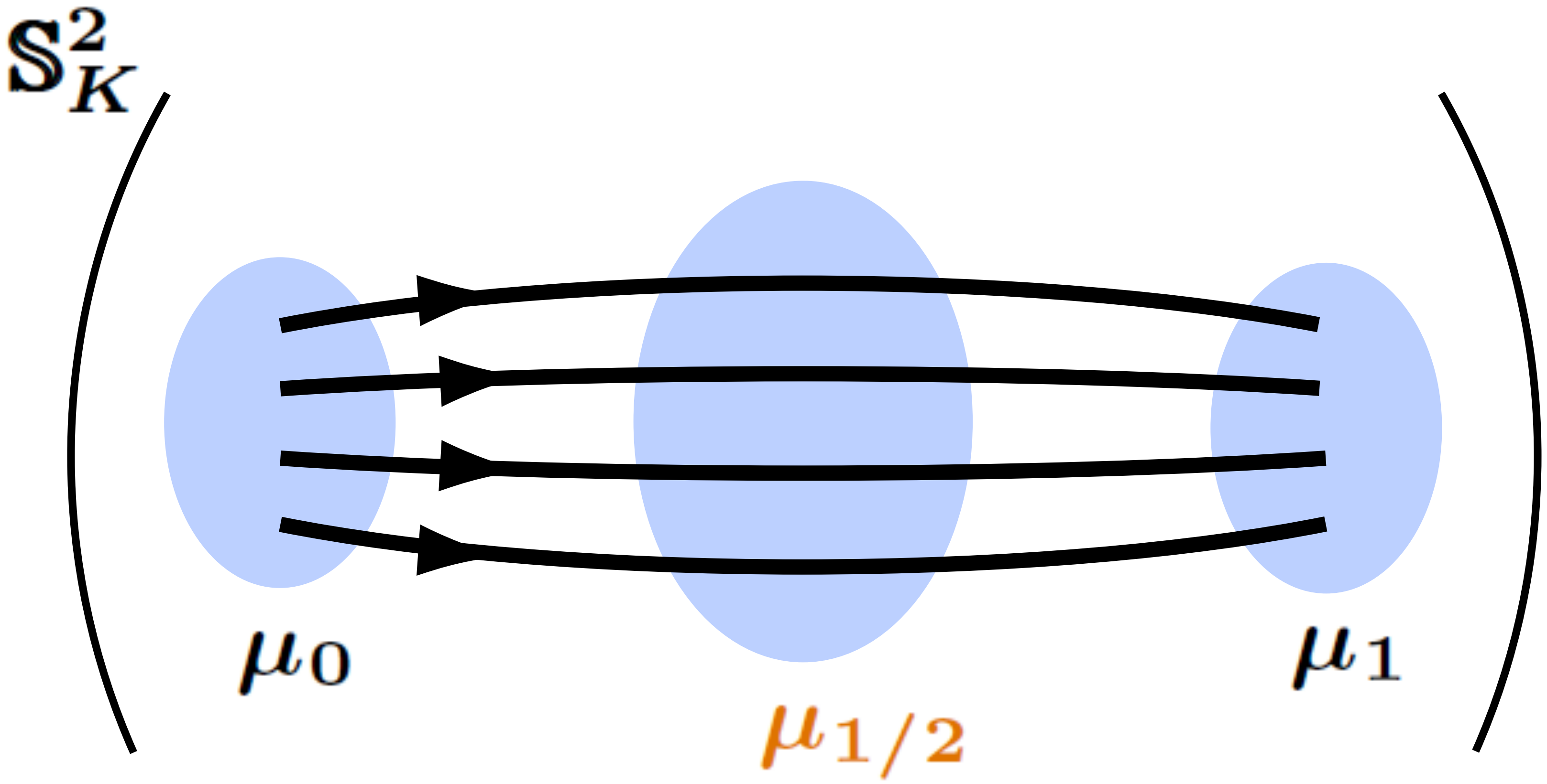
Ex.

$$M = \mathbb{S}^2_K,\ \mu_i = \frac{1}{\mathfrak{m}(B_\delta(x_i))} \mathfrak{m}|_{B_\delta(x_i)}$$

$$\Rightarrow \mathrm{Ent}(\mu_i) = -\log \mathfrak{m}(B_\delta)$$

$$\Rightarrow \mathrm{Ent}(\mu_{1/2}) \le -\log \mathfrak{m}(B_\delta) - \frac{K}{8} W_2(\mu_0, \mu_1)^2$$

Convexity of Ent



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Characterization of $\mathrm{Ric} \geq K$

TFAE for $K \in \mathbb{R}$ ([von Renesse & Sturm '05] etc.)

(i) $\mathrm{Ric} \geq K$

(ii) Ent is K-convex on $(\mathcal{P}(X), W_2)$

(iii) $W_p(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_p(\mu_0, \mu_1)$ (W_p)
for some $p \in [1, \infty]$

(iv) $|\nabla P_t f|(x) \leq e^{-Kt} P_t(|\nabla f|^q)(x)^{1/q}$ (G_q)
for some $q \in [1, \infty]$

(v) $\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$

Table of implications

$$\begin{tikzcd}
 & \textbf{(iii)} & \textbf{(iv)} & \\
\textbf{(i)} \arrow[r, Rightarrow, dashed] \arrow[d, Leftrightarrow, dashed] & (W_\infty) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & (G_1) \arrow[d, Rightarrow] & \\
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Q. Which's valid on singular spaces?

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[Bakry & Émery '84]

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[K '10 / K.]

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[Ambrosio, Gigli & Savaré / Koskela & Zhou]

General duality: (iii) $\Leftrightarrow$ (iv)

M: Polish geodesic sp., $(P(x, \cdot\,))_{x \in M} \subset \mathcal{P}(M)$

For $f : M \to \mathbb{R}$, $|\nabla f|(x) := \overline{\lim_{y \to x}} \dfrac{|f(x) - f(y)|}{d(x, y)}$

Theorem 1 ([K.'10]/[K.])

For $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$ and $C > 0$, TFAE:

(a) $W_p(P^*\mu_0, P^*\mu_1) \le C W_p(\mu_0, \mu_1)$

(b) $|\nabla P f|(x) \le C P(|\nabla f|^q)(x)^{1/q}$

(When $q = \infty$, (RHS of (G_q)) will be $\| |\nabla f| \|_\infty$)

General duality: (iii) $\Leftrightarrow$ (iv)

Ingredients of the proof

(a) $\Rightarrow$ **(b)**: For ${}^{\forall}\boldsymbol{\pi}$: coupling of $\boldsymbol{P}(\boldsymbol{x},\cdot)$ & $\boldsymbol{P}(\boldsymbol{y},\cdot)$

$$\boldsymbol{P}\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{P}\boldsymbol{f}(\boldsymbol{y}) = \int_{M\times M} (\boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{w}))\boldsymbol{\pi}(\boldsymbol{d}\boldsymbol{z}\boldsymbol{d}\boldsymbol{w})$$

(b) $\Rightarrow$ **(a)**: Kantorovich duality

$$W_2(P^*\mu_0, P^*\mu_1)^2 = \sup_f \left[\int_M \hat{f}\, dP^*\mu_1 - \int_M f\, dP^*\mu_0 \right]$$

$$\left(\hat{f}(x) := \inf_{y\in M} \left[f(y) + d(x,y)^2 \right] \right)$$

& interpolation: $[\cdots] = \int_0^1 \partial_r \left(\int_M P f_r \, d\mu_r \right) dr \; \square$

General duality: (iii) $\Leftrightarrow$ (iv)

Ingredients of the proof

(a) $\Rightarrow$ **(b)**: For $^\forall\pi$: coupling of $P(x,\cdot)$ & $P(y,\cdot)$

$$Pf(x) - Pf(y) = \int_{M \times M} (f(z) - f(w))\pi(dzdw)$$

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General duality: (iii) ⇔ (iv)

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Gradient flow of Ent: (ii) $\Rightarrow$ (W_2)

★ μ_t^1, μ_t^2: gradient curves of Ent in $(\mathcal{P}(M), W_2)$

Then, heuristically,

$$\boxed{\begin{array}{l}\mathrm{Ent}\text{: } K\text{-convex w.r.t. } W_2 \\ \quad \Rightarrow W_2(\mu_t^1, \mu_t^2) \le e^{-Kt} W_2(\mu_0^1, \mu_0^2)\end{array}} \quad (\spadesuit)$$

★ On $\mathbb{R}^m$, $\boxed{[\text{GF of } \mathrm{Ent}] = P_t^* \mu}$ $(\heartsuit)$

[Jordan, Kinderlehrer & Otto '98]

Q.: When are $(\spadesuit)(\heartsuit)$ rigorous?

Gradient flow of Ent: (ii) $\Rightarrow$ (W_2)

Ans. for $\boxed{[\text{GF of } \mathbf{Ent}] = \boldsymbol{P_t^* \mu}}$: Known results (I)

Smooth case

- $\mathbb{R}^m$ [Jordan, Kinderlehrer & Otto '98]
- cpl. Riem. mfd [Erbar '10]
- Finsler mfd [Ohta & Sturm '09] (nonlinear)
- Wiener sp. [Fang, Shao & Sturm '09] (∞-dim)
- Heisenberg gr. [Juillet] (sub-elliptic)

Gradient flow of Ent: (ii) $\Rightarrow$ (W_2)

Ans. for $\boxed{[\text{GF of } \mathbf{Ent}] = \boldsymbol{P}_t^* \boldsymbol{\mu}}$: Known results (II)

Non-smooth case

- cpt. Alexandrov sp. [Gigli, K. & Ohta] (singular)
- metric measure sp. [Ambrosio, Gigli & Savaré]
  [Koskela & Zhou]

Gradient flow of Ent: (ii) $\Rightarrow$ (W_2)

Ans. for $\boxed{\text{[GF of } \mathbf{Ent}] = \boldsymbol{P_t^* \mu}}$: Known results (III)

Other cases

- finite set [Maas '11] (discrete Markov chain)
- Lévy processes [Erbar] (nonlocal generator)

Gradient flow of Ent: (ii) $\Rightarrow$ (W_2)

Ent: $\boldsymbol{K}$-convex w.r.t. $\boldsymbol{W_2}$
$\Rightarrow \boldsymbol{W_2(\mu_t^1, \mu_t^2) \le e^{-Kt} W_2(\mu_0^1, \mu_0^2)}$ (♠)

Ans. for (♠)

- (♠) holds when GF is linear w.r.t. initial data [Ambrosio, Gigli & Savaré]
- When GF is nonlinear, $^{\exists}$GF s.t. (♠) fails [Ohta & Sturm '12]

Gradient flow of Ent: (ii) $\Rightarrow$ (W_2)

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Ans. for $(\spadesuit)$

- $(\spadesuit)$ holds when GF is linear w.r.t. initial data
  [Ambrosio, Gigli & Savaré]
- When GF is nonlinear, $^\exists$GF s.t. $(\spadesuit)$ fails
  [Ohta & Sturm '12]

Rem

GF of Ent satisfies $(\mathbf{EVI}_K)$ condition

$\Rightarrow$ Ent is $\boldsymbol{K}$-convex

[Daneri & Savaré '08 / Ambrosio, Gigli & Savaré]

Summary

$$\begin{tikzcd}
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\textbf{(i)} \arrow[r, Rightarrow, dashed] \arrow[d, Leftrightarrow, dashed] & (W_\infty) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & (G_1) \arrow[d, Rightarrow] & \\
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Summary

The following are valid even on metric measure spaces

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\end{tikzcd}$$

Q. When does $(W_\infty)/(G_1)$ hold?

Known results

(i) $\mathrm{Ric} \geq K$

$\Updownarrow$

(v) $\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$

$\Updownarrow$

(iv) $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2)$

Known results

(i)' $\mathrm{Ric} \geq K$ & $\dim M \leq N$

$$\Updownarrow$$

(v) $\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$

$$\Updownarrow$$

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Known results

(i)' $\mathrm{Ric} \geq K$ & $\dim M \leq N$

$\Updownarrow$ $\leftrightsquigarrow$ [Bakry & Émery '84]

(v)' $$\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2$$

$\Updownarrow$

(iv) $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2)$

Known results

(i)' $\mathrm{Ric} \geq K$ & $\dim M \leq N$

$\Updownarrow$ $\leftsquigarrow$ [Bakry & Émery '84]

(v)' $$\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2$$

$\Updownarrow$ $\leftsquigarrow$ [F.-Y. Wang '11]

(iv)' $$|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) - \frac{1 - e^{-2Kt}}{NK}(\Delta P_t f)^2$$

Known results

(i)' $\mathrm{Ric} \geq K$ & $\dim M \leq N$

$\Updownarrow$ $\leftsquigarrow$ [Bakry & Émery '84]

(v)' $$\frac{1}{2}\Delta(|\nabla f|^2) - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2$$

$\Updownarrow$ $\leftsquigarrow$ [F.-Y. Wang '11]

(iv)' $$|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) - \frac{1 - e^{-2Kt}}{NK}(\Delta P_t f)^2$$

(i)' $\Leftrightarrow$ **(ii)'**: CD(K, N) [Sturm '06 / Lott & Villani '09]

Known results

Table of implications

$$\begin{tikzcd}
\textbf{(i)'} \arrow[d, Leftrightarrow, dashed] & & & \\
\textbf{(ii)'} & & \textbf{(iv)'} \arrow[r, Leftrightarrow] & \textbf{(v)'} \arrow[d, Leftrightarrow, dashed] \\
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⋆ What's **(iii)'**?

Known results

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⋆ What's **(iii)'**?

Theorem 2 ([K.])

For $K \in \mathbb{R}$ *and* $N \in [2, \infty)$,
(iv)' *is equivalent to the following* **(iii)'**:

(iii)' $W_2(P_s^* \mu_0, P_t^* \mu_1)^2$

$$\leq \left(\fint_s^t e^{2Kr} \xi(dr) \right)^{-1} W_2(\mu_0, \mu_1)^2 + \frac{N}{2} \xi([s, t])^2$$

where $\xi(dr) = \left(\frac{2K}{1 - e^{-2Kr}} \right)^{-1/2} dr$

- ⋆ Sharper than the one in the abstract!

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- ⋆ **(i)'** ⇒ **(iii)'** via coupling method

The case $K = 0$

Corollary 3 ([K.])

For $N \in [2, \infty]$*, TFAE:*

(i)' $\mathrm{Ric} \geq 0 \ \&\ \dim M \leq N$

(iii)' $$W_2(P_s^* \mu_0, P_t^* \mu_1)^2 \leq W_2(\mu_0, \mu_1)^2 + \frac{N}{2}(\sqrt{t} - \sqrt{s})^2$$

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⋆ **(iii)'** ⇒ sharp est. for

$$\Delta(d(x, \cdot)^2)(y) = \partial_t W_2(\delta_x, P_t^* \delta_y)^2$$

Extended duality

Theorem 4 ([K.])

M: *Polish geod. met. sp.*, $P_t = e^{t\mathcal{L}}$: *Feller semigr.*

Then for $p, q \in (1, \infty)$ *with* $p^{-1} + q^{-1} = 1$

& $a, b : [0, \infty) \to (0, \infty)$, *TFAE:*

(A) $$W_p(P_s^* \mu_0, P_t^* \mu_1)^2 \le \left(\int_s^t \frac{\xi(dr)}{a(r)} \right)^{-1} W_p(\mu_0, \mu_1)^2 + \xi([s,t])^2$$

(B) $$|\nabla P_t f|^2 \le a(t) \Big[P_t(|\nabla f|^q)^{2/q} - b(t)(\mathcal{L} P_t f)^2 \Big]$$

where $\xi(dr) := b(r)^{-1/2} dr$

Summary

- Several ways to characterize $\mathrm{Ric} \geq K$

$$\begin{tikzcd}
\textbf{(i)} \arrow[r, Rightarrow, dashed] \arrow[d, Leftrightarrow, dashed] & (W_\infty) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & (G_1) \arrow[d, Rightarrow] & \textbf{(i)} \arrow[d, Leftrightarrow, dashed] \\
\textbf{(ii)} \arrow[r, Rightarrow] & (W_2) \arrow[r, Leftrightarrow] & (G_2) \arrow[r, Leftrightarrow] & \textbf{(v)} \arrow[ul, Rightarrow, dashed]
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Summary

- Several ways to characterize $\mathrm{Ric} \geq K$
- Coupling method $\Rightarrow$ stronger est. on smooth sp.'s

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Summary

- Several ways to characterize $\mathrm{Ric} \geq K$
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\textbf{(i)'} \arrow[dd, Leftrightarrow, dashed] \arrow[ddrr, Rightarrow, dashed] & & & & & \textbf{(i)'} \arrow[dd, Leftrightarrow, dashed] \\
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Summary

- Several ways to characterize $\mathrm{Ric} \geq K$
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- Characterization of $\mathrm{Ric} \geq K$ & $\dim \leq N$
- New condition (iii)'

$$\begin{tikzcd}
& \textbf{(i)} \arrow[r, Rightarrow, dashed] \arrow[dd, Leftrightarrow, dashed] \arrow[dddr, Rightarrow, dashed] & (W_\infty) \arrow[r, Leftrightarrow] \arrow[dd, Rightarrow] & (G_1) \arrow[dd, Rightarrow] & \textbf{(i)} \arrow[dd, Leftrightarrow, dashed] & \\
\textbf{(i)'} \arrow[dd, Leftrightarrow, dashed] & & & & & \textbf{(i)'} \arrow[dd, Leftrightarrow, dashed] \\
& \textbf{(ii)} \arrow[r, Rightarrow] & (W_2) \arrow[r, Leftrightarrow] & (G_2) \arrow[r, Leftrightarrow] & \textbf{(v)} \arrow[uul, Rightarrow, dashed] & \\
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\end{tikzcd}$$

Questions

- When does $(\boldsymbol{W_\infty})/(\boldsymbol{G_1})$ hold?
- When **(ii)'** $\Rightarrow$ **(iii)'**?
- How sharp **(iii)'** is?
  - ⇜ Weaker cond. than **(iii)'** can be equiv. to **(iv)'** ("self-improvement")
  - ★ Seems to be sharp when $\boldsymbol{K = 0}$
  - ⇝ Sharpen **(iii)'** by coupling method (in prog.)
- Counterpart of $(\boldsymbol{W_p})/(\boldsymbol{G_q})$ in curv.-dim.?
- Connection with the monotonicity of normalized $\mathcal{L}$-transp. cost under backward Ricci flow? [cf. Topping '09, K.-Philipowski '11]
- Weaker conditions and their relations?