# A probabilistic approach to the maximal diameter theorem

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# §1. Main theorem and related results

$(M,g)$: $n$-dim. complete. Riem. mfd, $\partial M = \emptyset$

## Bonnet-Myers' theorem

$$\mathbf{Ric} \geq \boldsymbol{K} > \mathbf{0} \;\Rightarrow\; \mathbf{diam}(\boldsymbol{M}) \leq \sqrt{\frac{\boldsymbol{n}-\mathbf{1}}{\boldsymbol{K}}}\boldsymbol{\pi}.$$

in particular, $M$: compact

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## Bonnet-Myers' theorem

$$\mathrm{Ric} \geq K > 0 \;\Rightarrow\; \mathrm{diam}(M) \leq \sqrt{\frac{n-1}{K}}\pi.$$

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## Cheng's maximal diameter theorem

$$\mathrm{Ric} \geq K > 0, \; \mathrm{diam}(M) = \sqrt{\frac{n-1}{K}}\pi$$

$$\Rightarrow\; (M, g) \simeq \mathbb{S}^n\left(\sqrt{\frac{n-1}{K}}\right) \text{ (isometry)}$$

$Z$: a vector field on $M$

$N \in [n, \infty]$

## Bakry-Émery Ricci tensor

$$\mathrm{Ric}_Z^N := \mathrm{Ric} - (\nabla Z)^{\mathrm{sym}} - \frac{1}{N-n} Z \otimes Z$$

**Correspondence**

$$\Delta \quad \leftrightarrow \quad \mathrm{Ric} = \mathrm{Ric}_0^n$$

$$\mathcal{L} = \Delta + Z \quad \leftrightarrow \quad \mathrm{Ric}_Z^N$$

$$(n < N)$$

**Theorem** [K.]

Let $K > 0$ & $N < \infty$

(i) $\mathrm{Ric}_Z^N \geq K \Rightarrow \mathrm{diam}(M) \leq \sqrt{\frac{N-1}{K}}\pi$

(ii) $\mathrm{Ric}_Z^N \geq K$ & $\mathrm{diam}(M) = \sqrt{\frac{N-1}{K}}\pi$

$\Rightarrow Z \equiv 0,\ N = n,$

$$(M, g) \simeq \mathbb{S}^n\left(\sqrt{\frac{n-1}{K}}\right)$$

Known results (When $Z = \nabla f$)

- Cptness under weaker assumption
  [X.-M. Li '95, X.-M. Li & F.-Y. Wang '03]

- $\mathcal{L}$: symm. $\rightsquigarrow$ Analytic approach
  [Bakry & Ledoux '96]

- $(M, d, \mathrm{e}^{f}\mathrm{vol})$: weighted Riem. mfd

  Riem. mfd: [Ruan '09]

  Metric meas. sp.: [Sturm '06, Ohta '07]

  Alexandrov sp.: [Zhang & Zhu '09]

(Green ref. studies a variant of the max. diam. thm)

# §2 Proof of the Bonnet-Myers theorem

$\boldsymbol{X_t}$: $\boldsymbol{\mathcal{L}}$-diffusion

$\boldsymbol{p \in M}$, $\boldsymbol{d_p(x) := d(p, x)}$

**Goal**

$$\mathbb{P}\left[\boldsymbol{d_p(X_t) > \sqrt{\frac{N-1}{K}}\pi}\right] = \mathbf{0}$$

($\Rightarrow$ Bonnet-Myers)

## Itô formula for the radial process

$$d_p(X_t) \le d_p(X_0) + \sqrt{2}\beta_t + \int_0^t \mathcal{L} d_p(X_s)\,ds,$$

($\beta_t$: 1-dim. standard Brownian motion)

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## (Extended) Laplacian comparison theorem

$$\mathcal{L} d_p(x) \leq \varphi_{N,K}(d_p(x)),$$

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$$\mathcal{L} d_p(x) \le \varphi_{N,K}(d_p(x)),$$

$$\varphi_{N,K}(r) \sim (N-1)\left( r - \sqrt{\frac{N-1}{K}}\pi \right)^{-1}$$

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$$\Rightarrow \quad \forall t,\ d_p(X_t) \le \sqrt{\frac{N-1}{K}}\pi \qquad \square$$

# §3 Proof of the maximal diameter theorem

Suppose $d(p,q) = \sqrt{\frac{N-1}{K}}\pi$

**Goal**

$$\mathcal{L} d_p(x) = \varphi_{N,K}(d_p(x)) \text{ a.e.} x$$

($\Rightarrow$ conclusion)

Take $\boldsymbol{x} \in \boldsymbol{M}$ with $\boldsymbol{d_p(x) + d_q(x) = \sqrt{\frac{N-1}{K}}\pi}$

Suppose $\boldsymbol{X_0 \equiv x}$

**Claim**

$$\boldsymbol{d_p(X_t) + d_q(X_t) \equiv \sqrt{\frac{N-1}{K}}\pi}$$

(NOTE: "$\geq$" is obvious)

By the Itô formula,

$$\mathbb{E}[d_p(\boldsymbol{X}_t) + d_q(\boldsymbol{X}_t)]$$

$$\leq \sqrt{\frac{N-1}{K}}\pi + \mathbb{E}\left[\int_0^t (\mathcal{L}d_p + \mathcal{L}d_q)(\boldsymbol{X}_s)ds\right]$$

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By the (extended) Laplacian comparison,

$$(\mathcal{L}d_p + \mathcal{L}d_q)(X_s)$$

$$\leq \varphi_{N,K}(d_p(X_s)) + \varphi_{N,K}(d_q(X_s))$$

$$\leq 0$$

By the Itô formula,

$$\mathbb{E}[d_p(X_t) + d_q(X_t)]$$

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$\Rightarrow$ Claim

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$$\leq 0$$

$\Rightarrow$ Claim $\Rightarrow$ all "$\leq$" must be "$=$" $\square$