Monotonicity of time-dependent transportation costs and coupling by reflection

Kazumasa Kuwada

(Ochanomizu University)

[joint work with K.-Th. Sturm (Bonn)]

Optimal transportation and differential geometry

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1. Introduction

M: complete Riem. mfd, $\dim M = n \geq 2$

$X^x(t)$: Brownian motion on M with $X(0) = x$

$$\mathrm{Ric} \geq K$$

$$\Downarrow$$

nice estimates for

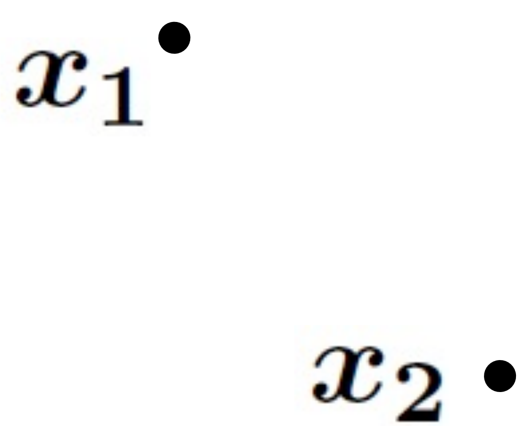
couplings by parallel transport / reflection of BMs

$(X_1(t), X_2(t))$: a coupling of $X^{x_1}(t)$ & $X^{x_2}(t)$

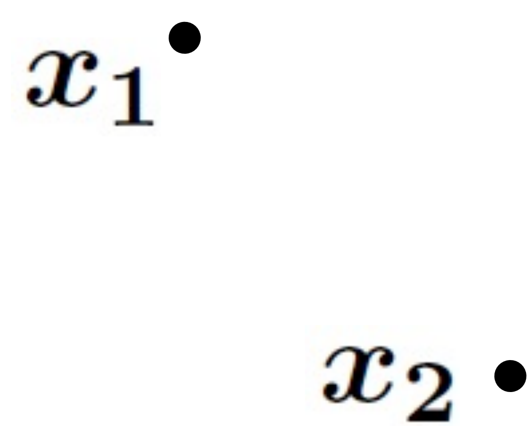
$\overset{\text{def}}{\Leftrightarrow} (X_i(t))_{t \geq 0} \overset{d}{=} (X^{x_i}(t))_{t \geq 0}$ $(i = 1, 2)$

Example ($M = \mathbb{R}^n$)

parallel transport



reflection



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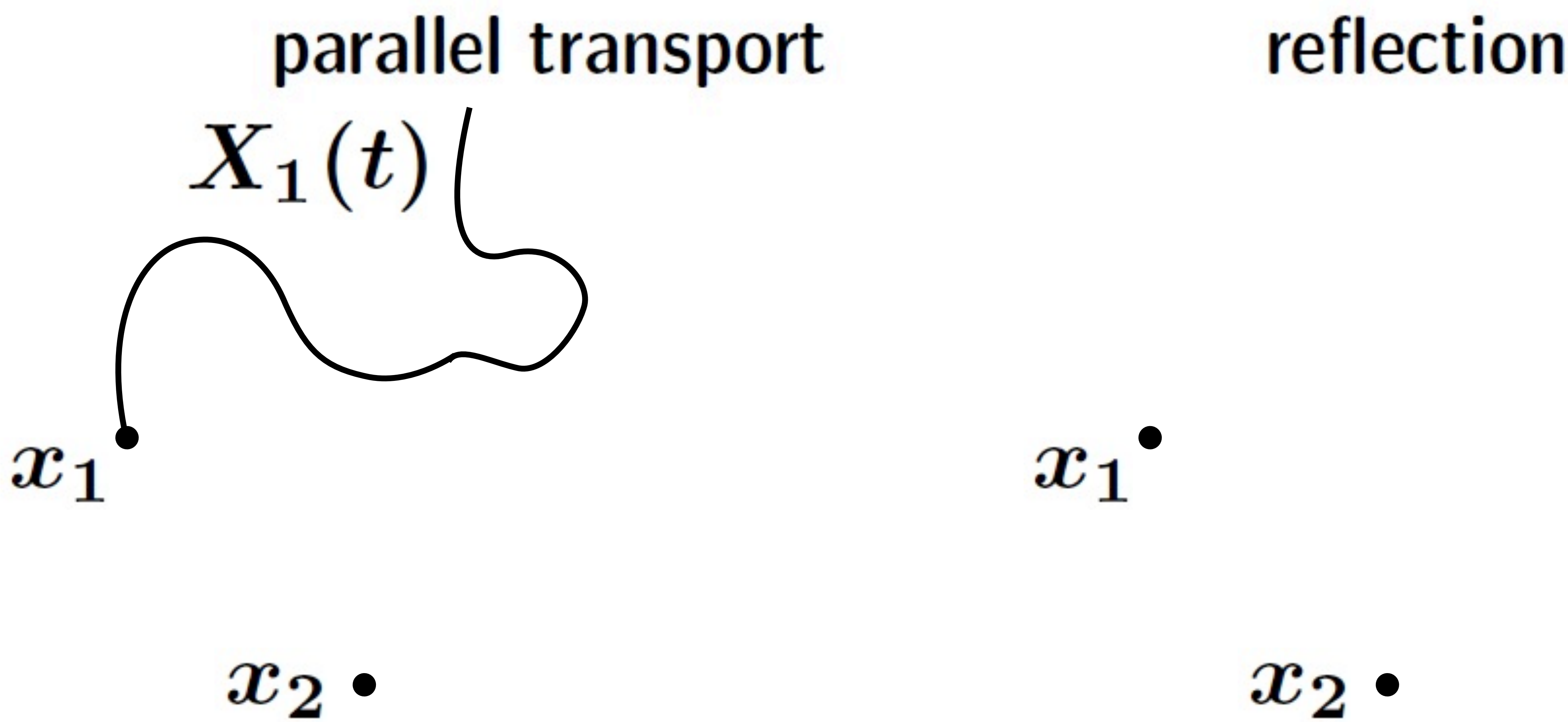

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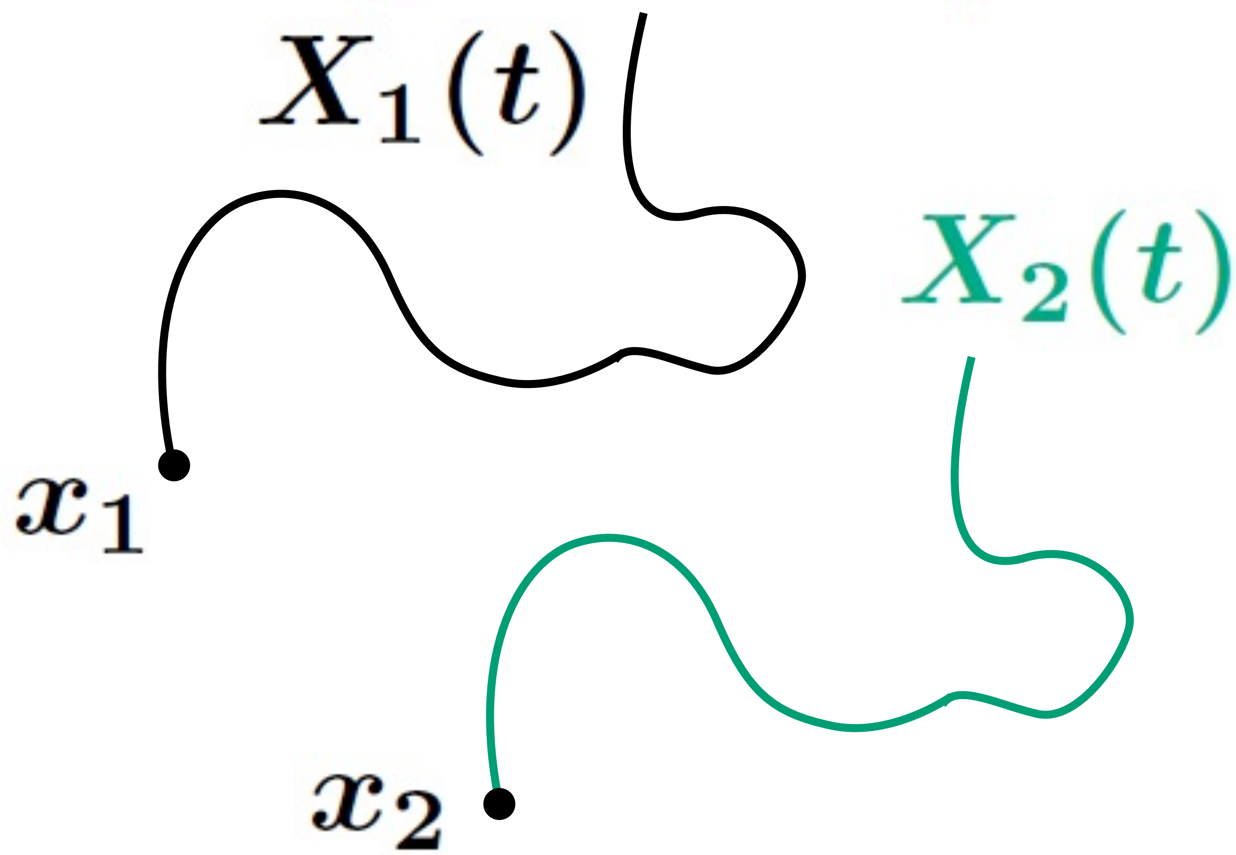

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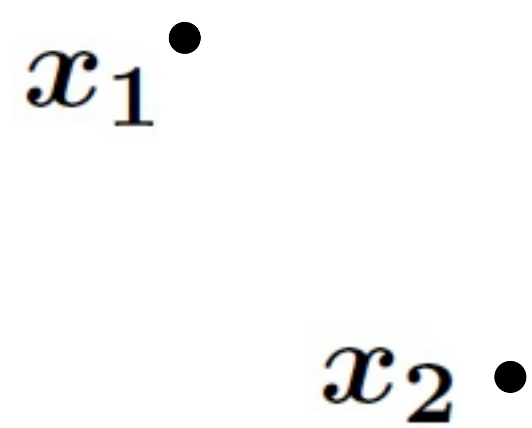

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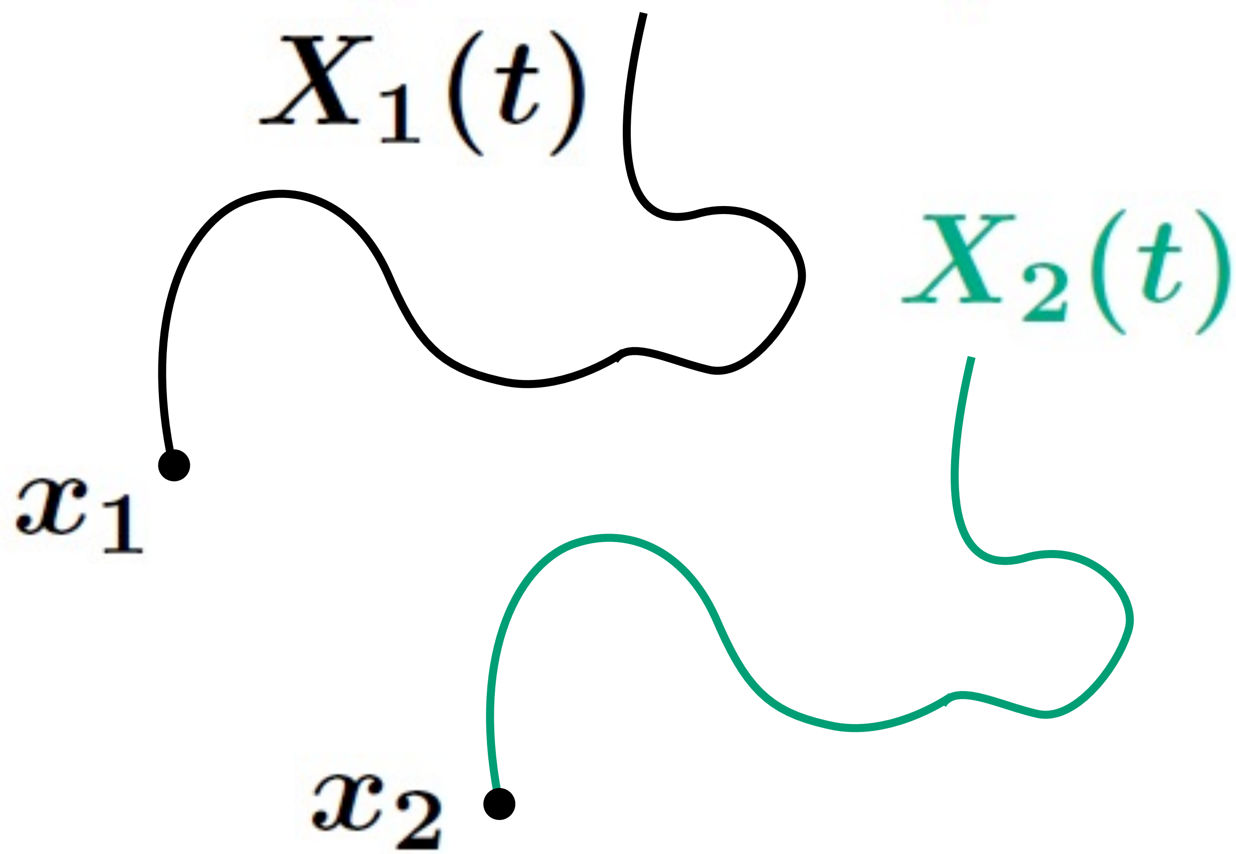

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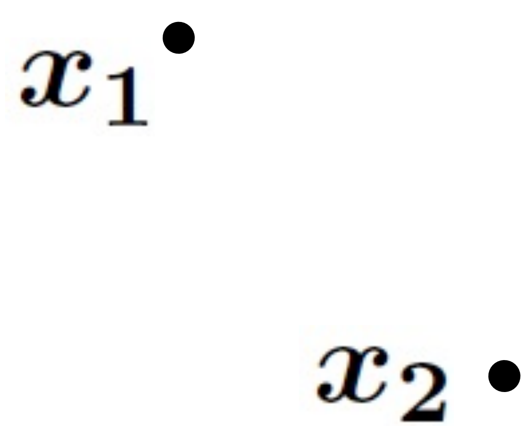

$$\Downarrow$$

$$d(X_1(t), X_2(t)) = d(X_1(0), X_2(0))$$

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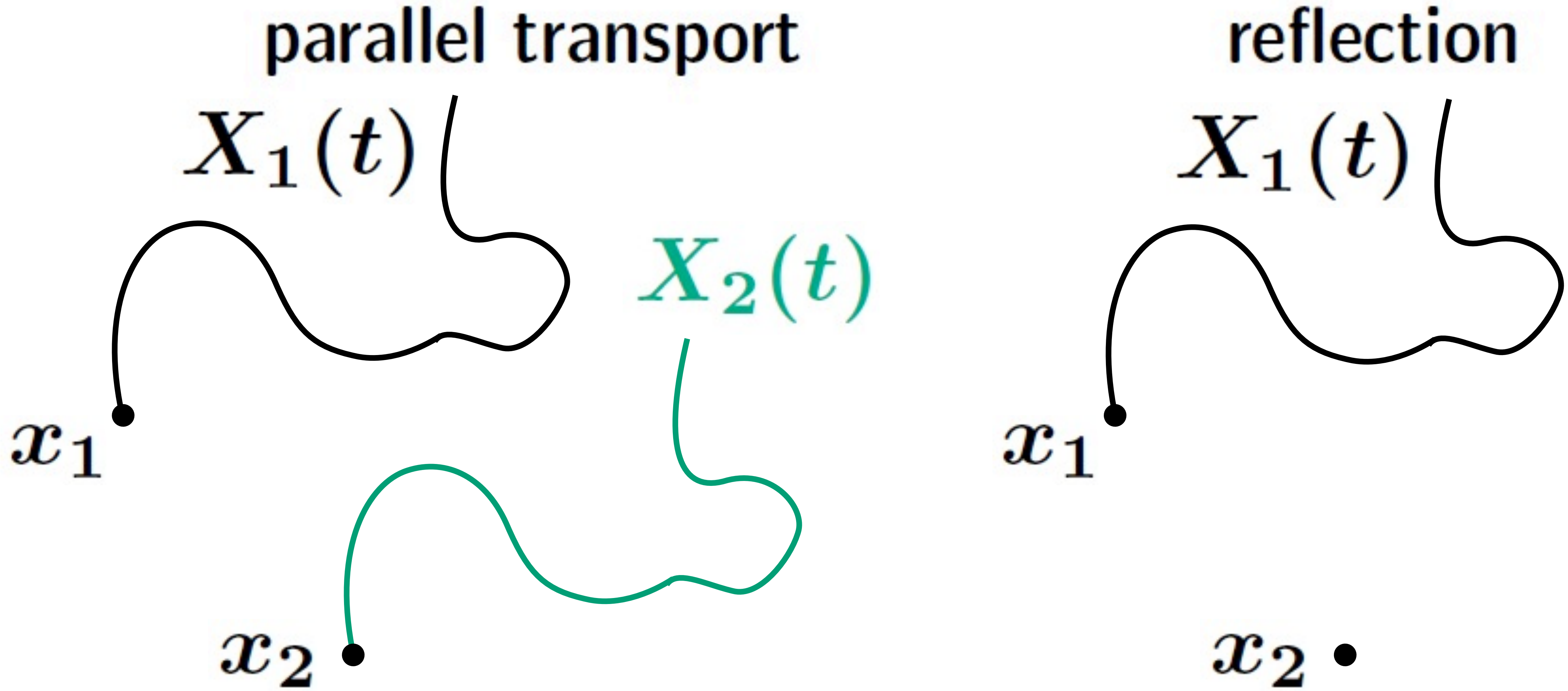

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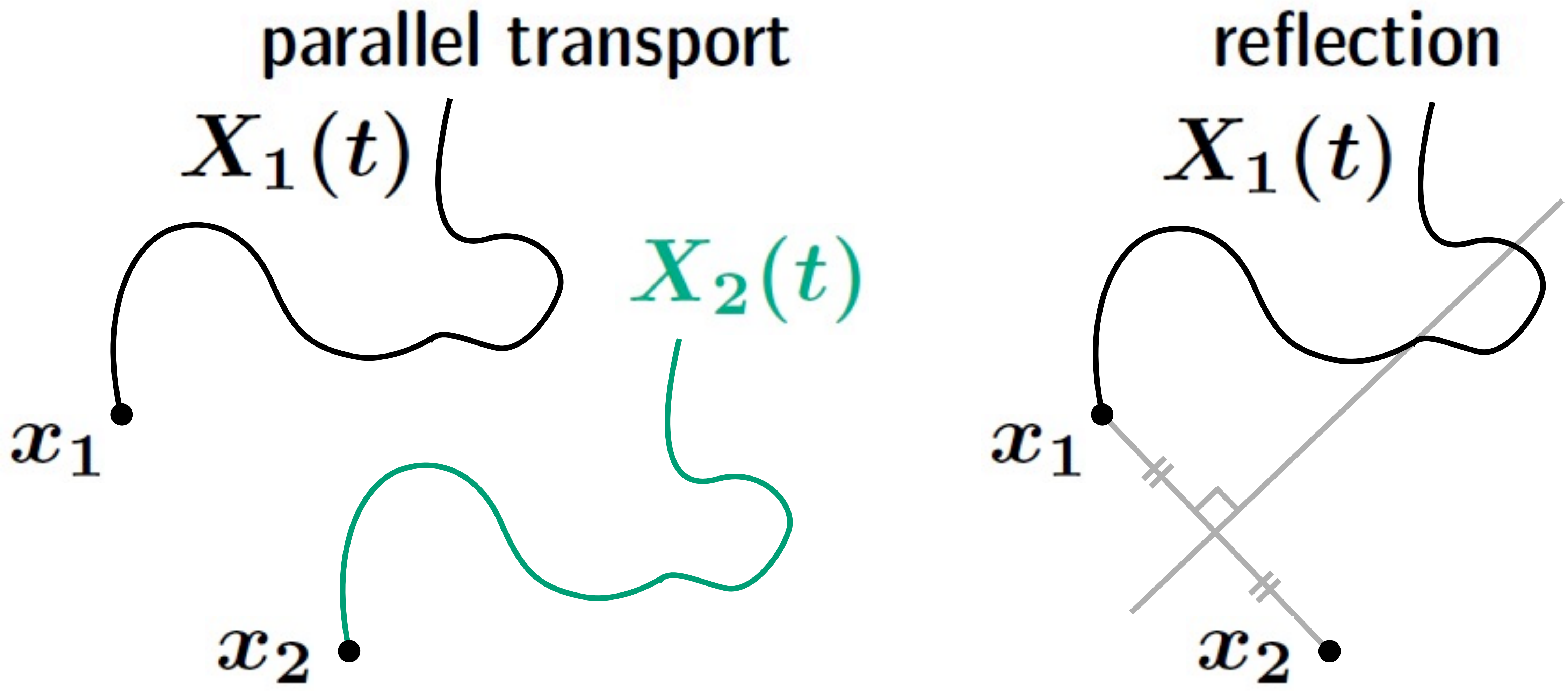

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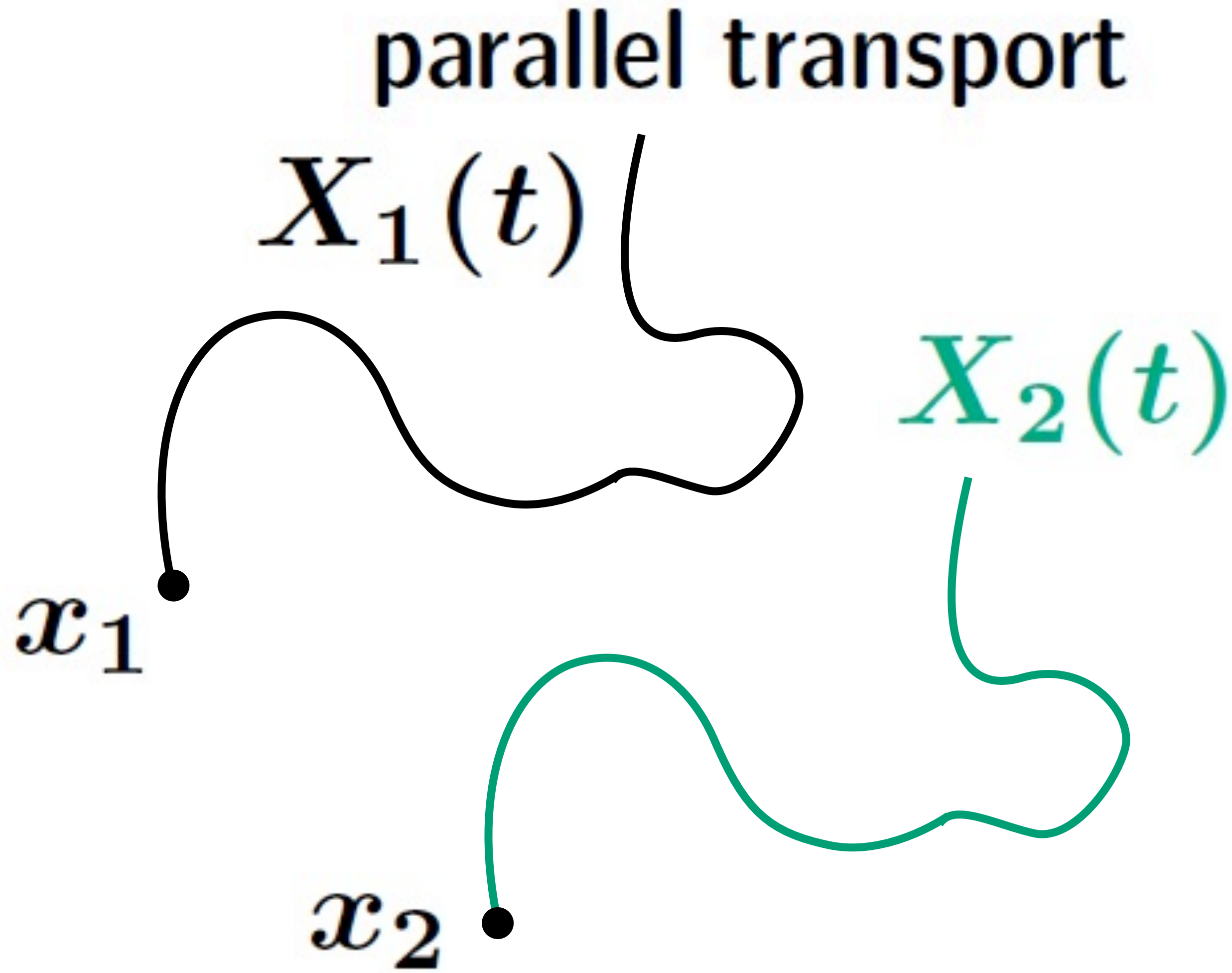

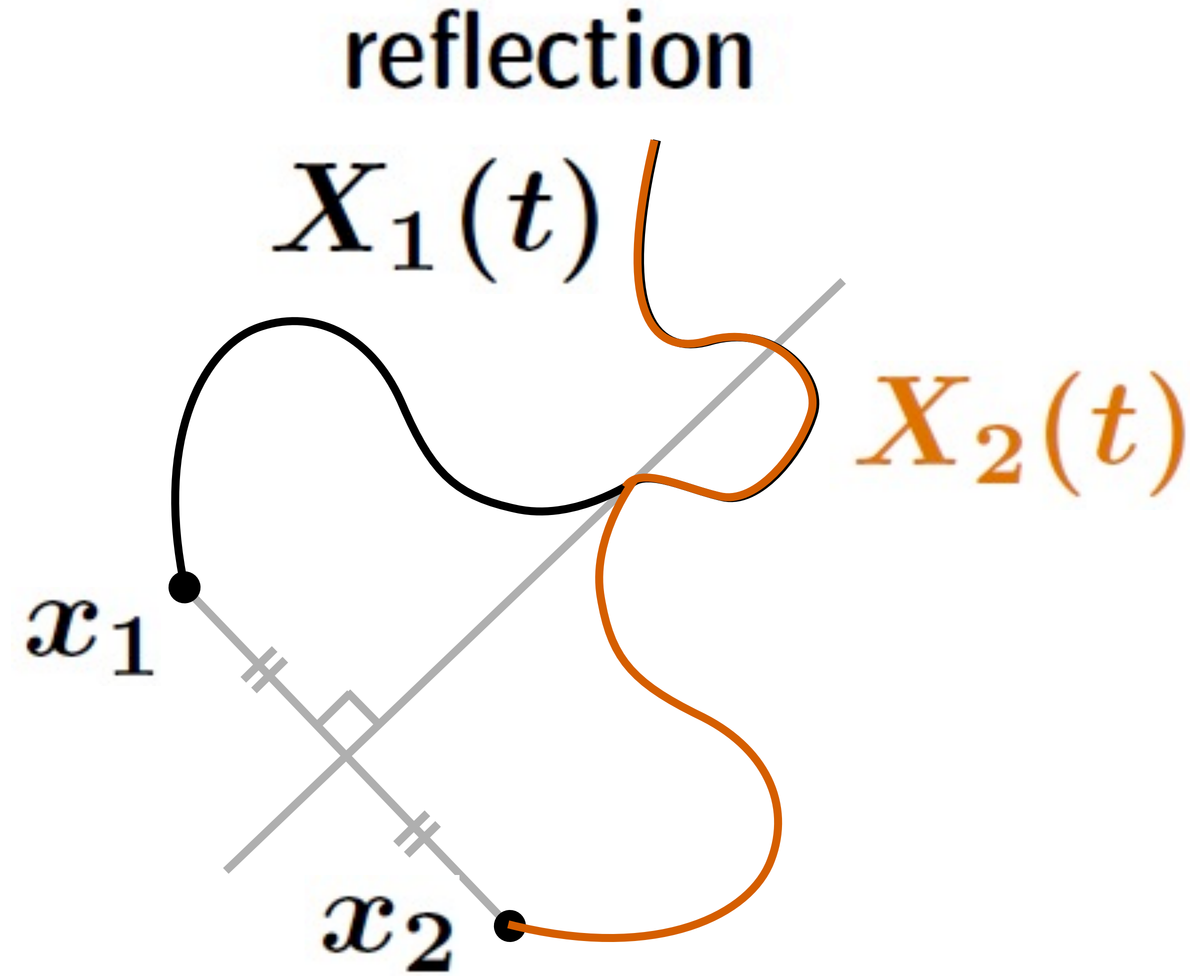

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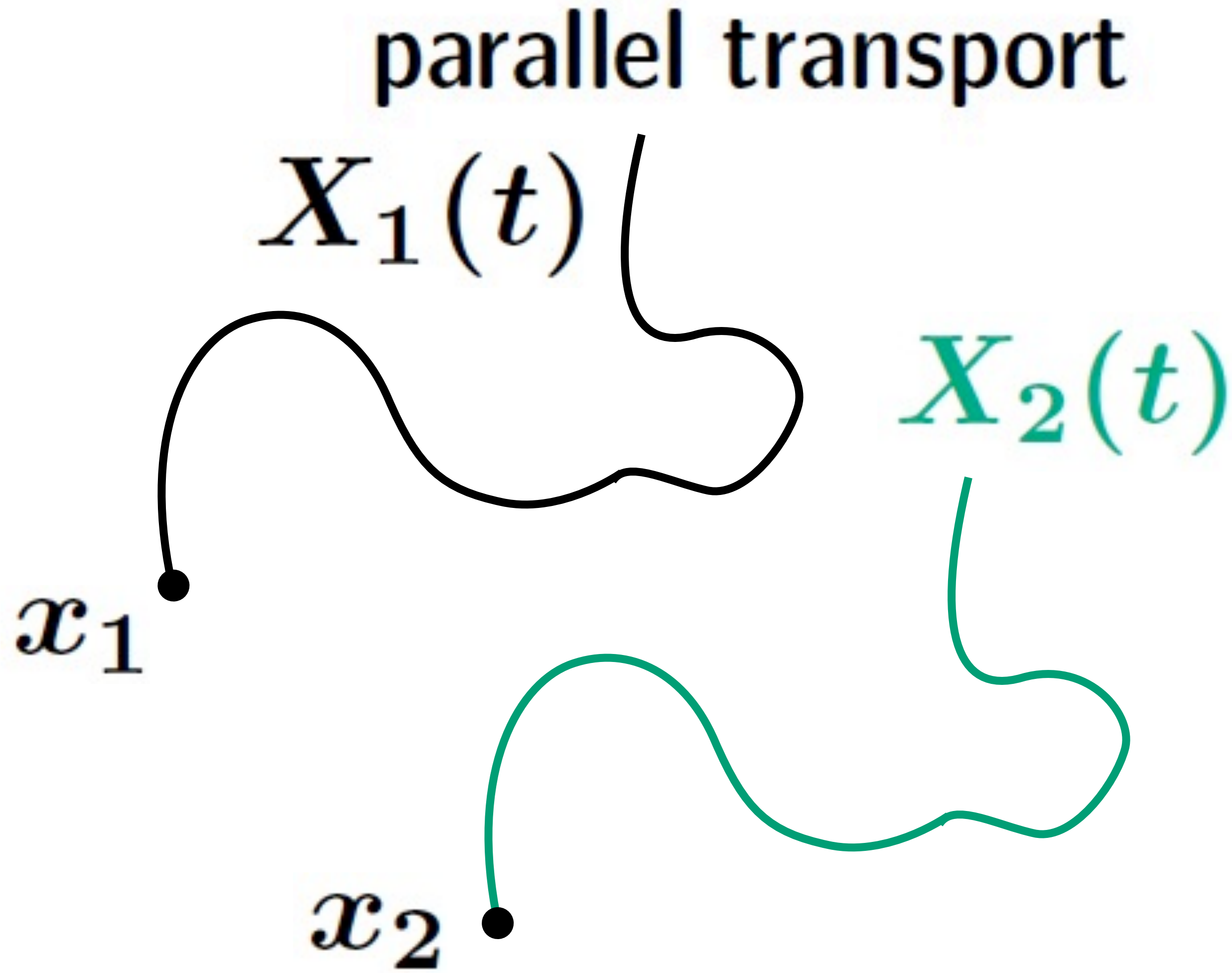

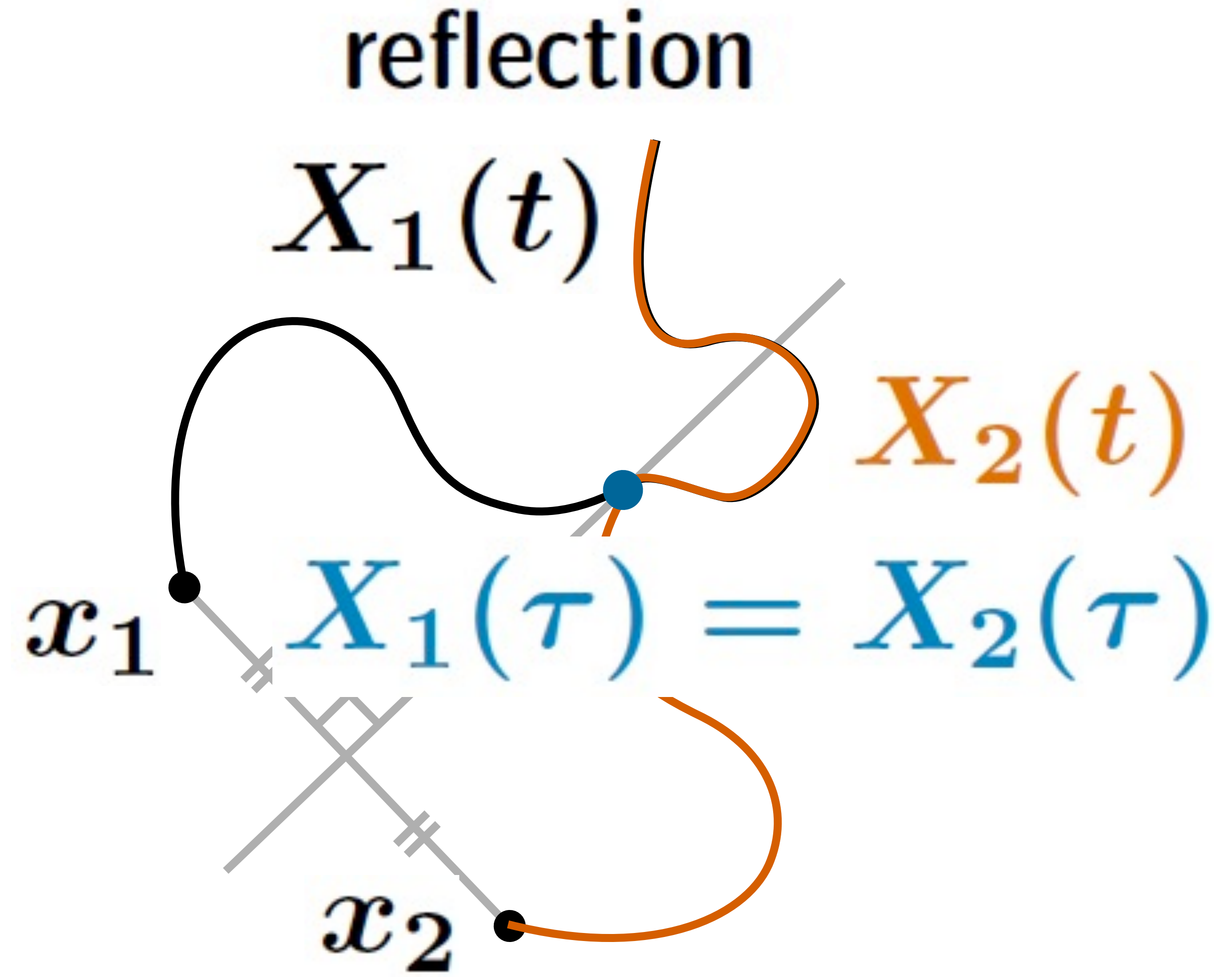

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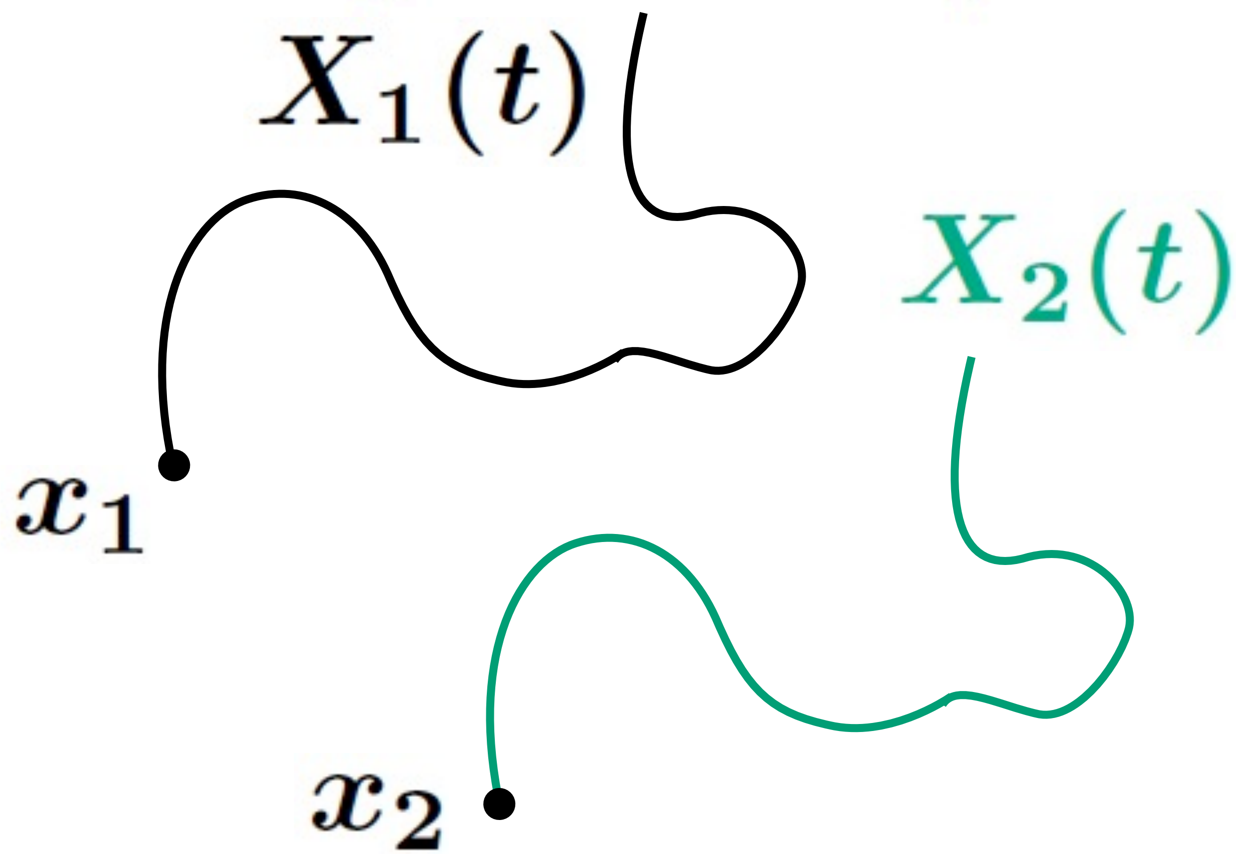

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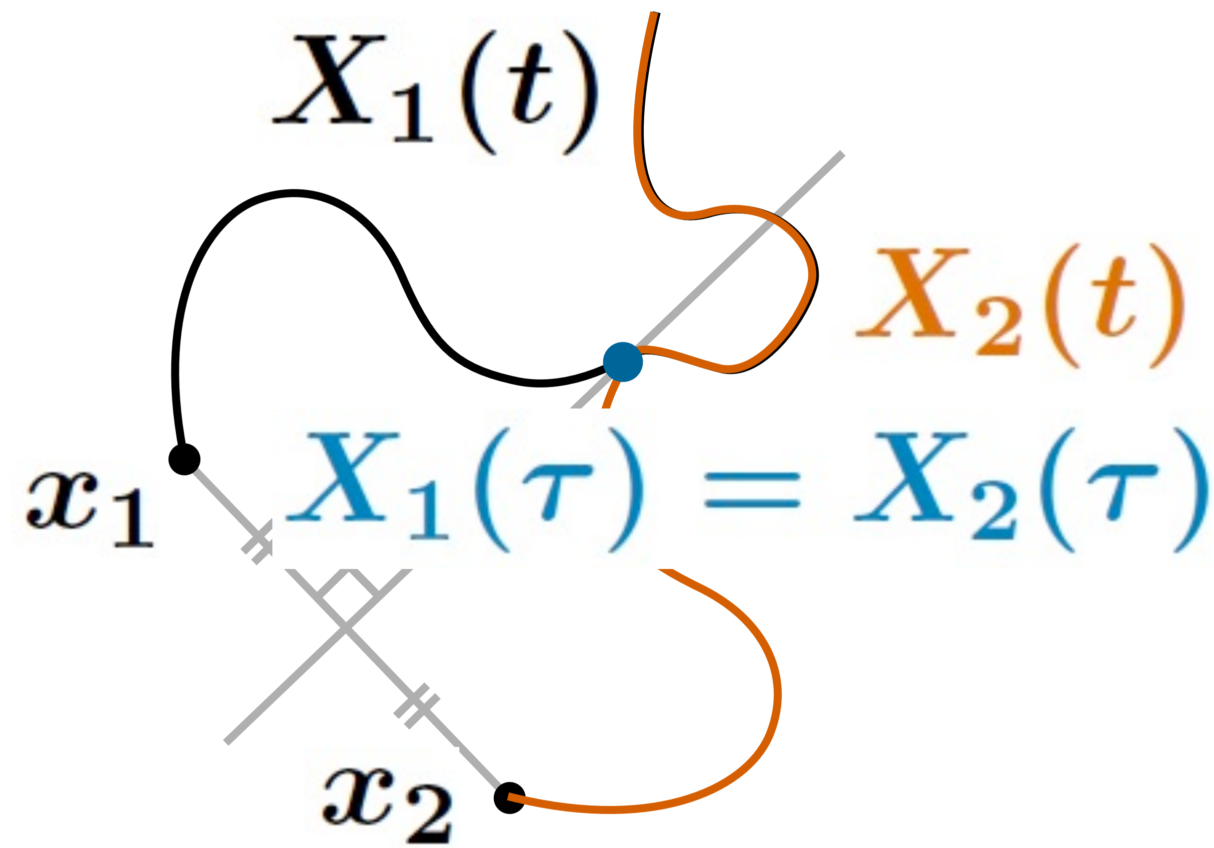

$\Downarrow$

Estimate of $\mathbb{P}[\tau \geq t]$

(τ: first time to meet)

On a Riem. mfd M:

parallel transport/reflection of infinitesimal motions along a geodesic joining particles

- Coupling by parallel transport
  [F.-Y.Wang '97, von Renesse '04, Arnaudon & Coulibaly & Thalmaier '09, K., . . .]

- Coupling by reflection
  [Kendall '86, Cranston '91, F.-Y.Wang '97, '05, von Renesse '04, K., . . .]

What can be obtained from couplings when $\mathrm{Ric} \geq K$?

Optimal transportation cost

For $c : M \times M \to \mathbb{R}$, $\mu_1, \mu_2 \in \mathcal{P}(M)$,

$$\mathcal{T}_c(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_2)} \int_{M \times M} c \, d\pi$$

$\Pi(\mu_1, \mu_2)$: set of couplings of μ_1 and μ_2

⋆ $(X_1(t), X_2(t))$: coupling of $X^{x_1}(t)$ & $X^{x_2}(t)$

$\Downarrow$

$$\mathbb{P} \circ (X_1(t), X_2(t))^{-1} \in \Pi(P_t^* \delta_{x_1}, P_t^* \delta_{x_2})$$

(P_t: heat semigroup)

- Coupling by parallel transport

- Coupling by reflection

- Coupling by parallel transport

  $\Rightarrow$ Pathwise contraction:

  $e^{Kt} d(X_1(t), X_2(t)) \searrow$ $\mathbb{P}$-a.s.

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- Coupling by reflection

$\Rightarrow$ Estimate of $\mathbb{P}[\tau > t]$ in terms of K and n
($\Rightarrow$ Estimate of total variations)

- Coupling by parallel transport

  $\Rightarrow$ Pathwise contraction:
  $\mathrm{e}^{Kt} d(X_1(t), X_2(t)) \searrow$ $\mathbb{P}$-a.s.

  $\Rightarrow$ Monotonicity of (scaled) transportation costs:
  $\mathcal{T}_{(\mathrm{e}^{Kt} d)^p}(P_t^* \mu_1, P_t^* \mu_2) \searrow$ in t

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- Coupling by reflection

  $\Rightarrow$ Estimate of $\mathbb{P}[\tau > t]$ in terms of K and n
  ($\Rightarrow$ Estimate of total variations)

  $\Rightarrow$ (monotonicity of a transportation cost)

Motivation: Stochastic analysis on singular spaces

- Construction of coupling by reflection

  $\Leftarrow$ differentiable structure on M

  (How do we formulate it on singular sp.'s?)

- "Monotonicity of transportation cost" is robust

  $\Rightarrow$ Stable under Gromov-Hausdorff conv.

- Potential connection with gradient flow theory

$$\left(\begin{array}{l} \text{e.g. "}\mathrm{Hess}\,\mathrm{Ent} \ge K\text{"} \\ \qquad\qquad \Rightarrow \mathcal{T}_{(e^{Kt} d)^2}(P_t^* \mu_1, P_t^* \mu_2) \searrow \end{array} \right)$$

2. Framework and main results

General framework

Z: C^1-vector field

$X^x(t)$: diffusion process associated with $\Delta + Z$

($X(t)$: BM $\Leftrightarrow Z = 0$)

Bakry-Émery Ricci tensor

For $N \in [n, \infty]$,

$$\mathrm{Ric}^{Z,N} := \mathrm{Ric} - (\nabla Z)^{\mathrm{sym}} - \frac{1}{N-n} Z \otimes Z$$

Assumption

For $K \in \mathbb{R}$, $\mathrm{Ric}^{Z,N} \geq K$

Remarks

- Ass. $\Leftrightarrow$ Bakry-Émery's curv.-dim. cond.
- When $Z = 0$,

$$\text{Ass.} \Leftrightarrow n \le N \text{ and } \mathrm{Ric} \ge K$$

- The Riem. metric g and Z can depend on t:

$$\mathrm{Ric}_{g(t)}^{Z(t),\infty} \ge \frac{1}{2}\partial_t g(t) + K$$

$K = 0$, $Z = 0$ & "$=$" $\Leftrightarrow$ backward Ricci flow

- $K > 0$ & $N < \infty \Rightarrow$ max. diam. thm.[K. '11]

Theorem 1 [K. & Sturm]

$(X_1(t), X_2(t))$: a coupling by refl. of two BMs.

$\Rightarrow$ For $\varphi_t = \varphi_t^{N,K} \; : \; [0,\infty) \to [0,1]$ given below,

$$\mathbb{E}[\varphi_{t-s}(d(X_1(s), X_2(s)))] \searrow$$

in $s \in [0,t]$

Theorem 2 [ibid.]

For $t > 0$, $\mu_1, \mu_2 \in \mathcal{P}(M)$,

$$\mathcal{T}_{\varphi_{t-s}(d)}(P_s^* \mu_1, P_s^* \mu_2) \searrow \text{ in } s \in [0,t]$$

Definition of $\varphi_t^{K,N}(a)$ (for $N \in \mathbb{N}$)

$$\varphi_t^{K,N}(a) := \frac{1}{2} \left\| \tilde{P}_t^* \delta_{\tilde{x}} - \tilde{P}_t^* \delta_{\tilde{y}} \right\|_{\mathrm{TV}}$$

- $\tilde{P}_t$: heat semigr. on the spaceform $\mathbb{M}_{K,N}$ ($\mathbb{M}_{N,K}$: sphere, Euclidean sp. or hyperbolic sp.)
- $d(\tilde{x}, \tilde{y}) = a$

Comparison functions / processes

$$s_K(u) := \frac{1}{\sqrt{K}} \sin(\sqrt{K} u),$$

$$c_K(u) := \cos(\sqrt{K} u)$$

$$\Psi_{K,N}(u) := -K \frac{s_{K/(N-1)}(u)}{c_{K/(N-1)}(u)}$$

- $\rho^a(t)$ solves the following SDE on $\mathbb{R}$:

$$d\rho^a(t) = \sqrt{2} d\beta(t) + \Psi(\rho^a(t)) dt,$$

$$\rho^a(0) = a$$

($\beta(t)$: BM on $\mathbb{R}$)

Definition of $\varphi_t^{K,N}(a)$ (general case)

P_t^ρ: transition semigroup of $\rho^a(t)$

$$\varphi_t^{K,N}(a) := \frac{1}{2}\left\|(P_t^\rho)^*\delta_{a/2} - (P_t^\rho)^*\delta_{-a/2}\right\|_{\mathrm{TV}}$$

Remark (Why does these definitions coincide?)

- $\boldsymbol{\rho}(t) = \begin{cases} (\rho^{a/2}(t), -\rho^{a/2}(t)) & \text{(before meet)} \\ (\rho^{a/2}(t), \rho^{a/2}(t)) & \text{(after meet)} \end{cases}$
  is a coupling by refl. of $\rho^{a/2}(t)$ & $\rho^{-a/2}(t)$

- When $M = \mathbb{M}_{K,N}$, $a = d(x_1, x_2)$ &
  $\mathbb{X}(t) = (X_1(t), X_2(t))$: coupling by refl.,

$$(d(\mathbb{X}(t)))_{t\geq 0} \overset{\mathcal{L}}{=} (d_{\mathbb{R}}(\boldsymbol{\rho}_{K,N}(t)))_{t\geq 0}$$

(Cont'd) Coupling inequality and maximality

For $\ell : [0, \infty) \to \mathbb{R}$,

$$\tau(\ell) := \inf\{t > 0 \mid \ell(t) = 0\}$$

- For any (Markovian) coupling $\hat{\mathbb{X}}(t)$ of (Markov) processes $\hat{X}^{x_1}(t)$ and $\hat{X}^{x_2}(t)$,

  $\mathbb{P}[\tau(d(\hat{\mathbb{X}})) > t]$

  $$\geq \frac{1}{2} \|\mathbb{P} \circ \hat{X}^{x_1}(t)^{-1} - \mathbb{P} \circ \hat{X}^{x_2}(t)^{-1}\|_{\mathrm{TV}}$$

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- "$=$" for $\mathbb{X}$ on $\mathbb{M}_{K,N}$ or $\boldsymbol{\rho}$ on $\mathbb{R}$

  (e.g., [K. '07, J. Theoret. Probab.])

Example of $\rho^a(t)$

- $K = 0$
  $\Rightarrow d\rho^a(t) = \sqrt{2}d\beta(t)$
  (1-dim. BM, independent of N)

- $N = \infty$
  $\Rightarrow d\rho^a(t) = \sqrt{2}d\beta(t) - K\rho^a(t)dt$
  (Ornstein-Uhlenbeck processes)

Properties of φ_t

- $\varphi_t \nearrow$, concave, $\varphi_t(0) = 0$ ($\Rightarrow \varphi_t(d)$: dist.)
- $\varphi_\cdot(a) \searrow$

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  ($\Rightarrow \mathcal{T}_{\varphi_0(d)}(\mu_1, \mu_2) = \frac{1}{2}\|\mu_1 - \mu_2\|_{\mathrm{TV}}$)

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- $\left.\begin{matrix} N \le N' \\ K \ge K' \end{matrix}\right\} \Rightarrow \varphi_t^{K,N}(a) \le \varphi_t^{K',N'}(a)$

3. Idea of the proof of Thm 1

Proposition 1

For $a = d(x_1, x_2)$,

$$\text{“}d(X_1(s), X_2(s)) \leq d_{\mathbb{R}}(\boldsymbol{\rho}(s))\text{”}\ \mathbb{P}\text{-a.s.}$$

for a BM $\beta(t)$ defining $\rho^{a/2}(t)$

$$\Downarrow$$

$$\mathbb{E}[\varphi_{t-s}(d(X_1(s), X_2(s))] \leq \mathbb{E}[\varphi_{t-s}(d_{\mathbb{R}}(\boldsymbol{\rho}(s)))]$$

Strategy of the proof of Proposition 1

- Itô formula:

$$d d(X_1(s), X_2(s)) \leq 2\sqrt{2} d\beta(s) + \Lambda(s) ds$$

- Index lemma:

$$\Lambda(s) \leq \Psi(d(X_1(s), X_2(s)))$$

$\Rightarrow$ SDE comparison thm implies conclusion $\square$

($\exists$ further technical difficulties to make it rigorous)

Lemma 1 (cf. [K.'07, J. Theoret. Probab.])

$$\mathbb{E}[\varphi_{t-s}(d_{\mathbb{R}}(\boldsymbol{\rho}(s)))]\text{: const. in } s$$

Proof

Since

$$\varphi_{t-s}(a) = \mathbb{P}[\boldsymbol{\tau}(d_{\mathbb{R}}(\boldsymbol{\rho})) > t - s],$$

the Markov property of $\boldsymbol{\rho}$ implies

$$\mathbb{E}[\varphi_{t-s}(d_{\mathbb{R}}(\boldsymbol{\rho}(s)))] = \mathbb{P}[\boldsymbol{\tau}(d_{\mathbb{R}}(\boldsymbol{\rho})) > t]$$

□

(i) Prop. 1 + Lem. 1

$$\Downarrow$$

$$\begin{aligned}\mathbb{E}[\varphi_{t-s}(d(X_1(s), X_2(s))] \\ &\le \mathbb{E}[\varphi_{t-s}(d_{\mathbb{R}}(\boldsymbol{\rho}(s)))] \\ &= \mathbb{E}[\varphi_t(d_{\mathbb{R}}(\boldsymbol{\rho}(0)))] \\ &= \varphi_t(d(x_1, x_2))\end{aligned}$$

(ii) The Markov property of $(X_1(t), X_2(t))$ yields the conclusion $\square$

4. Applications

Theorem 2: $\mathcal{T}_{\varphi_{t-s}(d_s)}(P_s^* \mu_1, P_s^* \mu_2) \searrow$

$$\Downarrow$$

$$\mathcal{T}_{\varphi_0(d)}(P_t^* \delta_x, P_t^* \delta_y) \leq \mathcal{T}_{\varphi_t(d)}(\delta_x, \delta_y)$$

Theorem 2: $\mathcal{T}_{\varphi_{t-s}(d_s)}(P_s^* \mu_1, P_s^* \mu_2) \searrow$

$$\Downarrow$$

$$\mathcal{T}_{\varphi_0(d)}(P_t^* \delta_x, P_t^* \delta_y) \le \mathcal{T}_{\varphi_t(d)}(\delta_x, \delta_y)$$

$$\Downarrow$$

Corollary 1 (Comparison thm for total variations)

$$\left\| P_t^* \delta_x - P_t^* \delta_y \right\|_{\mathrm{TV}} \le 2 \varphi_t(d(x,y))$$

When $N \in \mathbb{N}$, for $d_{\mathbb{M}_{K,N}}(\tilde{x}, \tilde{y}) = d(x,y)$,

$$(\text{RHS}) = \left\| \tilde{P}_t^* \delta_{\tilde{x}} - \tilde{P}_t^* \delta_{\tilde{y}} \right\|_{\mathrm{TV}}$$

When $K < 0$,

$$\exists \lim_{t \to \infty} \varphi_t^{K,N}(a) =: \Phi^{K,N}(a) \quad (> 0 \text{ iff } a > 0)$$

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$\Downarrow$

Corollary 2 (Monotonicity when $K < 0$)

$\mathcal{T}_{\Phi^{K,N}(d)}(P_t^* \mu_1, P_t^* \mu_2) \searrow$ in $t \geq 0$

★ Φ will be given explicitly below

Stability under GH-convergence

(M_m, g_m): n-dim. cpt. Riem. mfds, $\mathrm{Ric}_{g_m} \geq K$

Suppose

$$(M_m, d_m, \mathrm{vol}_{g_m}) \overset{\text{mGH}}{\longrightarrow} (M_\infty, d_\infty, v_\infty)$$

$\Downarrow$

For $\mu^{(m)} \in \mathcal{P}(M_m)$
with $\mu^{(m)} \to \mu^{(\infty)} \in \mathcal{P}(M_\infty)$,
$P_t \mu^{(m)} \longrightarrow$ a "heat distribution" μ_t^∞ on M_∞

[Gigli '10]

Theorem 3 [K. & S., op.sit.]

$(M_\infty, d_\infty, v_\infty)$: as above, $N \geq n$

$\mu_1(t), \mu_2(t)$: heat distributions on M_∞

$\Rightarrow$ For $t > 0$,

$$\mathcal{T}_{\varphi_{t-s}^{K,N}(d)}(\mu_1(t), \mu_2(t)) \searrow$$

in $s \in [0, t]$

5. The function φ_t

Properties of φ_t

- $\varphi_t(0) = 0$, $\varphi_0 = 1_{(0,\infty)}$

(A) $\varphi_t \nearrow$, $\varphi_\cdot(a) \searrow$

(B) $$\left.\begin{array}{l} N \le N' \\ K \ge K' \end{array}\right\} \Rightarrow \varphi_t^{K,N}(a) \le \varphi_t^{K',N'}(a)$$

(C) φ_t: concave

(A) $\varphi_t(\cdot) \nearrow$ & $\varphi_\cdot(a) \searrow$

Recall: $$\boxed{\mathbb{P}[\tau(\boldsymbol{\rho}) > t] = \varphi_t(a)}$$

($\boldsymbol{\rho}(t)$: coupling by refl. of $\rho^{a/2}(t)$ and $\rho^{-a/2}(t)$)

$\Rightarrow \varphi_\cdot(a) \searrow$

- $\tau(\boldsymbol{\rho}) =$ the first time that $\rho^{a/2}$ hits 0

$\Rightarrow \tau(\boldsymbol{\rho}) \nearrow$ as $a \nearrow \Rightarrow \varphi_t(\cdot) \nearrow$

(B) $\varphi_t^{K,N}(a) \le \varphi_t^{K',N'}(a)$

SDE comparison:

$$d_{\mathbb{R}}(\boldsymbol{\rho}_{K,N}(t)) \le d_{\mathbb{R}}(\boldsymbol{\rho}_{K',N'}(t))$$

(for a suitable $\beta(t)$)

$$\Downarrow$$

$$\boxed{\boldsymbol{\tau}(\boldsymbol{\rho}_{K,N}) \le \boldsymbol{\tau}(\boldsymbol{\rho}_{K',N'})}$$

$$\Downarrow$$

$$\boxed{\boldsymbol{\varphi}_t^{K,N}(a) \le \boldsymbol{\varphi}_t^{K',N'}(a)}$$

(C) φ_t: concave

Proposition 2

$\exists \xi_t^{K,N} \in \mathcal{P}([0,\infty))$ s.t.

$$\varphi_t(a) = \int_{[0,\infty)} \chi\left(\frac{a}{2\sqrt{2u}}\right) \xi_t^{K,N}(du),$$

$$\chi(r) := \frac{1}{\sqrt{2\pi}} \int_{-r}^{r} \mathrm{e}^{-x^2/2} dx$$

- χ: concave. $\therefore$ φ_t: concave

Expression of $\xi_t^{K,N}$

(i) $\xi_t^{K,\infty} = \xi_t^{0,N} = \delta_{\gamma(t)}, \quad \gamma(t) := \dfrac{e^{2Kt} - 1}{2K}$

(ii) When $N < \infty$ & $K \neq 0$

$$\xi_t^{K,N}(A) = \mathbb{P}\left[\int_0^t \frac{ds}{c_{K/(N-1)}(\theta_s)^2} \in A\right],$$

$$d\theta_t = \sqrt{2}d\beta_t + \hat{\Psi}(\theta_t)dt,$$

$$\hat{\Psi}(a) := (N-2)\frac{c_{K/(N-1)}(a)}{s_{K/(N-1)}(a)} + \frac{\Psi_{K,N}(a)}{(N-1)}$$

Another expression of $\varphi_t(a)$

(Let $K^* := \dfrac{K}{N-1}$)

(i) When $N < \infty$ & $K > 0$,

$$\varphi_t(a) = \sum_{n=0}^{\infty} \mathrm{e}^{-(2n+1)(2n+N)K^* t} \times \frac{(-1)^n (4n+N+1)}{\pi(2n+N)} \times B\left(\frac{N-1}{2}, n+\frac{1}{2}\right) P_{2n+1}(\tilde{a})$$

- $B(\cdot,\cdot)$: Beta function,
- $\tilde{a} := \sin(\sqrt{K^*}a/2)$
- $P_n(x)$: Gegenbauer polynomial of param. $\dfrac{N-1}{2}$

(ii) When $N < \infty$ and $K < 0$,

$$\varphi_t(a) = \mathbb{E}\left[\chi\left(\frac{s_{K^*}(a/2)}{2\sqrt{2}}\left(\int_0^t \theta'(s)^2 ds\right)^{-1/2}\right)\right],$$

where $\theta'(t) := \exp\left(\sqrt{-2K^*}\beta(t) + Kt\right)$

More explicitly,

$$\varphi_t(a) = \int_{-\infty}^{\infty} \frac{du}{u} \int_0^{\infty} dx$$

$$\chi\left(\frac{1}{2}\sqrt{\frac{-K^*}{u}}\, s_{K^*}\left(\frac{a}{2}\right)\right) \vartheta\left(\frac{e^x}{u}, -2K^* t\right)$$

$$\times \exp\left(\frac{(N-1)}{2}(Kt - x) - \frac{1+e^{2x}}{2u}\right),$$

where

$$\vartheta(r,t) := \frac{r}{2\pi^3 t} e^{\pi^2/(2t)} \int_0^{\infty} e^{-\xi^2/(2t)}$$

$$e^{-r\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi$$

The limit of $\varphi_t(a)$ as $t \to \infty$ (when $K < 0$)

- $\Phi^{K,\infty}(a) = \lim_{t\to\infty} \varphi_t^{K,\infty}(a) = \chi\left(\frac{a\sqrt{-K}}{2}\right)$
- $\Phi^{K,N}(a) = \lim_{t\to\infty} \varphi_t^{K,N}(a)$

$$= \int_0^\infty \chi\left(\sqrt{\frac{-K^* u}{2}} s_{K^*}\left(\frac{a}{2}\right)\right) \nu(du),$$

where ν: Gamma distribution of param. $\frac{N-1}{2}$,

i.e. $\nu(dx) = \Gamma\left(\frac{N-1}{2}\right)^{-1} x^{(N-3)/2} \mathrm{e}^{-x} dx$