Applications of Hopf-Lax formulae to analysis of heat distributions

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Let (X, d) be a metric space. Let $p \in (1, \infty)$. For $f \in C_b(X)$, we define $Q_t f \in C_b(X)$ by

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + \frac{t}{p} \left(\frac{d(x,y)}{t} \right)^p \right].$$

We call it Hopf-Lax semigroup (also called Hamilton-Jacobi semigroup). When (X, d) is an Euclidean space, $Q_t f$ is nothing but the Hopf-Lax formula, which gives a solution to the Hamilton-Jacobi equation

$$\partial_t Q_t f(x) = -\frac{1}{q} |\nabla Q_t f|(x)^q$$

in an appropriate sense, where q is the Hölder conjugate of p. This property is still valid even on more abstract metric spaces. It has been revealed that the notion of Hopf-Lax semigroup is strongly related with many functional inequalities including logarithmic Sobolev inequalities and transport-entropy inequalities. The purpose of this talk is to explain recent developments in this direction in connection with the heat semigroup.

For probability measures $\mu_0, \mu_1 \in \mathcal{P}(X)$, we denote the L^p-Wasserstein distance between μ_0 and μ_1 by $W_p(\mu_0, \mu_1)$. That is,

$$W_p(\mu_0, \mu_1) := \inf \left\{ \|d\|_{L^p(\pi)} \;\middle|\; \pi \in \mathcal{P}(X \times X)\text{: coupling of } \mu_0 \text{ and } \mu_1 \right\},$$

where we call π a coupling of μ_0 and μ_1 when the marginal distribution of π is μ_0 and μ_1 respectively. The dual representation of W_p is called the Kantorovich duality. By using $Q_t f$, it can be stated as follows:

$$W_p(\mu_0, \mu_1) = \sup_{f \in C_b(X)} \left[\int_X Q_1 f \, d\mu_1 - \int_X f \, d\mu_0 \right].$$

The Hopf-Lax semigroup appears here and this fact connects the study of Hopf-Lax formula with the theory of optimal transportation.

The first application of Hopf-Lax formula in this talk is a relation between a Lipschitz estimate of Wasserstein distance and a Bakry-Émery type gradient estimate for Markov kernels which in particular we can apply to the (Feller) heat semigroup. For $f : X \to \mathbb{R}$, we define the local Lipschitz constant $|\nabla_d f|(x)$ with respect to d by

$$|\nabla_d f|(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Theorem 1 (cf. [4])

Let (X, d) be a Polish length space and $\tilde{d}$ be another length metric on X. We denote the L^p-Wasserstein distance defined by using $\tilde{d}$ instead of d by $\tilde{W}_p$. Let $P(x, \cdot) \in \mathcal{P}(X)$ be a Markov kernel on X which depends continuously in $x \in X$. Then, for $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$, the following are equivalent:

(i) For $\mu_0, \mu_1 \in \mathcal{P}(X)$, $W_p(P^* \mu_0, P^* \mu_1) \le \tilde{W}_p(\mu_0, \mu_1)$.

(ii) For $f \in C_b^{\mathrm{Lip}}(X)$, $|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q}$ (When $q = \infty$, $\| |\nabla_{\tilde{d}} Pf| \|_\infty \le \| |\nabla_d f| \|_\infty$).

The second application is on the estimate of the speed of heat distributions with respect to W_2. For simplicity, we state it when X is a Riemannian manifold.

Theorem 2

Let X is a complete and stochastically complete Riemannian manifold and P_t the heat semigroup on X. Take $f : X \to [0, \infty)$ with $\|f\|_{L^1} = 1$ and set $\mu_t := P_t f \mathrm{vol}$. Then

$$|\dot{\mu}_t|_{W_2}^2 := \limsup_{s \downarrow 0} \frac{W_2(\mu_{t+s}, \mu_t)^2}{s^2} = \int_X \frac{|\nabla P_t f|^2}{P_t f} d\mathrm{vol}.$$

This estimate is first studied in [3] on Alexandrov spaces in the context of identification problem of heat flows. On Riemannian manifolds, there are two different ways to formulate a "heat flow". The one is a gradient flow of the Dirichlet energy in L^2-space of functions and the other is a gradient flow of the relative entropy on $\mathcal{P}(X)$ endowed with a metric structure by W_2. Thus Theorem 2 is an estimate related with the second formulation in the sense that it is a bound of the speed of curves in $\mathcal{P}(X)$ with respect to W_2 while the object μ_t is given by the first formulation. It plays a fundamental role for identifying those two formulation on non-smooth metric measure spaces as Alexandrov spaces (see [1, 3]). As a result of the identification, we can obtain the Bakry-Émery gradient estimate for the heat semigroup under a generalized notion of lower Ricci curvature bound (see [2, 3]).

The third application is a sort of extension of Theorem 1. Inequalities of the form (i) or (ii) are first introduced in connection with the notion of lower Ricci curvature bound. Recently, F.-Y. Wang introduced an extension of the Bakry-Émery gradient estimate involving an upper bound of $\dim X$ (property (v) below; see [5]). We obtain the condition corresponding to (i):

Theorem 3

Let X be a complete and stochastically complete Riemannian manifold with $\dim X \geq 2$. Then, for $N \in [2, \infty]$ and $K \in \mathbb{R}$, the following are equivalent:

(iii) $\dim X \leq N$ and $\mathrm{Ric} \geq K$.

(iv) $W_2(P_{t_0}\mu_0, P_{t_1}\mu_1)^2 \leq \dfrac{\mathrm{e}^{-2Kt_1} - \mathrm{e}^{-2Kt_0}}{2K(t_0 - t_1)} W_2(\mu_0, \mu_1)^2 + (t_1 - t_0) \displaystyle\int_{t_0}^{t_1} \frac{NK}{\mathrm{e}^{2Ku} - 1} du$ for $t_1 > t_0 > 0$ and $\mu_0, \mu_1 \in \mathcal{P}(X)$.

(v) $|\nabla P_t f|(x)^2 \leq \mathrm{e}^{-2Kt} P_t(|\nabla f|^2) - \dfrac{1 - \mathrm{e}^{-2Kt}}{NK} (\Delta P_t f)^2$ for $t > 0$ and $f \in C_b^{\mathrm{Lip}}(X)$.

References

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