Heat flow on Alexandrov spaces

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§1 Introduction

(X, d): compact Alexandrov space of curv. $\geq k$,

$\partial X = \emptyset$

(geodesic metric sp. of sect. curv. $\geq k$)

$n := \dim_H X \in \mathbb{N}$, $\mathcal{H}^n$: Hausdorff measure

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- It admits singularity of "curv. $= \infty$"
- Set of singular points can be dense
- It naturally appears in geometry
- Usual differential calculus is no longer available

$\Rightarrow$ We have to develop new techniques

which is also new in smooth cases

Two different ways to define a "heat distribution"

(1) Gradient flow of Dirichlet energy functional on L^2-sp. of functions (Dirichlet form)

(2) Gradient flow of relative entropy functional on a sp. of probability measures (Otto calculus)

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$\Downarrow$

We can combine properties of (1) and (2)

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Finer properties of the heat kernel etc.

§2 Framework and the main result

Dirichlet energy and its gradient flow

[Kuwae, Machigashira & Shioya '01]

$\exists(\mathcal{E}, W^{1,2}(X))$: (str. local, reg.) Dirichlet form

$$\mathcal{E}(u,u) := \int_X \langle \nabla u, \nabla u \rangle \, d\mathcal{H}^n$$

$$(\mathcal{E}, W^{1,2}(X)) \leftrightarrow (\Delta, \mathcal{D}(\Delta))\text{: generator}$$

$$\leftrightarrow T_t = \mathrm{e}^{t\Delta}\text{: semigroup,}$$

$\exists p_t(x,y)$: heat kernel

(Hölder conti.)

L^2-Wasserstein space $(\mathcal{P}(X), W_2)$ and entropy

For $\mu, \nu \in \mathcal{P}(X)$,

$$W_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \|d\|_{L^2(\pi)}$$

(L^2-Wasserstein distance)

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$$\mathrm{Ent}(\mu) := \begin{cases} \int_X \rho \log \rho \, d\mathcal{H}^n & \text{if } d\mu = \rho d\mathcal{H}^n \\ \infty & \text{other} \end{cases}$$

Grad. flow $(\mu_t)_{t\geq 0}$ of $\mathbf{Ent}$ on $(\mathcal{P}(X), d_2^W)$

$$\partial_t \mathbf{Ent}(\mu_t) + \frac{1}{2}|\dot{\mu}_t|^2 + \frac{1}{2}|\nabla_- \mathbf{Ent}(\mu_t)|^2 = 0$$

- $|\dot{\mu}_t|$: velocity of μ_t w.r.t. W_2
- $|\nabla_- \mathbf{Ent}(\mu)|$: subgradient w.r.t. W_2

Heuristically,

$$\partial_t \mathrm{Ent}(\mu_t) = \langle \dot{\mu}_t, \nabla \mathrm{Ent}(\mu_t) \rangle$$
$$\geq -\frac{1}{2}|\dot{\mu}_t|^2 - \frac{1}{2}|\nabla \mathrm{Ent}|^2(\mu_t)$$

$$\left(\because \langle u, v \rangle \geq -\frac{1}{2}(\langle u, u \rangle + \langle v, v \rangle) \right)$$

and "=" holds iff $\dot{\mu}_r = -\nabla \mathrm{Ent}(\mu_r)$

In what follows, suppose $\mathsf{CD}(K, \infty)$ for $K \in \mathbb{R}$.
(curvature-dimension cond. of Lott-Sturm-Villani;
generalized notion of "$\mathbf{Ric} \geq K$")

<u>Rem</u>

[Petrunin]: $(X, d, \mathcal{H}^n)$ enjoys $\mathsf{CD}((n-1)k, \infty)$.

$$\mathsf{CD}(K, \infty)$$

$$\Downarrow$$

$\exists!$ grad. flow of Ent for $\forall$ initial $\mu \in \mathcal{P}(X)$

[Ambrosio, Gigli & Savaré '05, Ohta '09, Gigli '10]

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$$\Downarrow$$

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<u>L^2-Wasserstein contraction</u>

For grad. flows μ_t and ν_t,

$$W_2(\mu_t, \nu_t) \le \mathrm{e}^{-Kt} W_2(\mu_0, \nu_0)$$

[Savaré '07, Ohta '09, Gigli & Ohta '10]

Theorem 1 [G.-K.-O.]

For any $\mu \in \mathcal{P}(X)$,

$T_t \mu$ is a gradient flow of Ent on $(\mathcal{P}(X), W_2)$

§3 Rough sketch of the proof

Let $\mu_t := T_t \mu$.

<u>Goal</u>

$$\partial_t \mathrm{Ent}(\mu_t) + \frac{1}{2}|\dot{\mu}_t|^2 + \frac{1}{2}|\nabla_{-}\mathrm{Ent}(\mu_t)|^2 = 0$$

by the uniqueness of the gradient flow of Ent.

Claims

(i) $\partial_t \mathrm{Ent}(\mu_t) = -I(\mu_t)$

(ii) $|\nabla_- \mathrm{Ent}(\mu_t)|^2 \leq I(\mu_t)$

(iii) $|\dot{\mu}_t|^2 \leq I(\mu_t)$ a.e. t

$$\left(I(\mu_t) := \int_X \frac{|\nabla \rho_t|^2}{\rho_t} d\mathcal{H}^n \text{: Fisher information} \right)$$

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- Integration by parts $\Rightarrow$ (i)
- [Villani '09] & CD$(K, \infty) \Rightarrow$ (ii)
- Kantorovich duality & Calculus of Hamilton-Jacobi semigr. $\Big\} \Rightarrow$ (iii)

§4 Applications (under CD(K, ∞))

L^2-Wasserstein contraction for T_t

$$W_2(T_t\mu, T_t\nu) \leq \mathrm{e}^{-Kt} W_2(\mu, \nu)$$

$\boldsymbol{L^2}$-Wasserstein contraction for $\boldsymbol{T_t}$

$$\boldsymbol{W_2(T_t\mu, T_t\nu) \le \mathrm{e}^{-Kt} W_2(\mu, \nu)}$$

[K.'10] $\Downarrow$ $\boldsymbol{T_t}$: linear

$\boldsymbol{L^2}$-gradient estimate for $\boldsymbol{f}$: Lipschitz

$$\boldsymbol{|\mathrm{Lip}(T_t f)|^2(x) \le \mathrm{e}^{-2Kt} T_t(|\mathrm{Lip}(f)|^2)(x)}$$

L^2-Wasserstein contraction for T_t

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$\Downarrow$ $\exists p_t$: conti.

L^2-gradient estimate for $f \in W^{1,2}(X)$

$$|\mathrm{Lip}(T_t f)|^2(x) \le \mathrm{e}^{-2Kt} T_t(|\nabla f|^2)(x)$$

Theorem 2 [G.-K.-O.]

(i) $T_t f \in C^{\mathrm{Lip}}(X)$ for $f \in W^{1,2}(X)$

(ii) For $\forall f$: L^2-eigenfn. of Δ, $f \in C^{\mathrm{Lip}}(X)$

(iii) $p_t(x, \cdot) \in C^{\mathrm{Lip}}(X)$

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- Theorem 2(ii) $\Rightarrow$ Another proof to [Petrunin '03]

Theorem 3 [G.-K.-O.]

For $g \in D(\Delta) \cap L^\infty, g \geq 0, \Delta g \in L^\infty$ and $f \in D(\Delta) \cap L^\infty, \Delta f \in W^{1,2}(X)$,

$$\int_X \left(\frac{1}{2} \Delta g \langle \nabla f, \nabla f \rangle - g \langle \nabla f, \nabla \Delta f \rangle \right) d\mathcal{H}^n \geq K \int_X g \langle \nabla f, \nabla f \rangle d\mathcal{H}^n$$

This is a weak form of Bakry-Émery's Γ_2-condition:

$$\frac{1}{2} \Delta \langle \nabla f, \nabla f \rangle - \langle \nabla f, \nabla \Delta f \rangle \geq K \langle \nabla f, \nabla f \rangle$$

Recent developments

- Theorem 1

  $\rightsquigarrow$ [Ambrosio & Gigli & Savaré '11]

  (Exntension to general metric measure spaces)

- Theorem 2 (ii)(iii) and Theorem 3

  $\rightsquigarrow$ [Zhang & Zhu '10] (two papers)

  (stronger results,

  under stronger notion of "$\mathrm{Ric} \geq K$")

  $\rightsquigarrow$ [Qian & Zhang & Zhu '10]

  (further applications)