Optimal transport and coupled diffusion by reflection

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[joint work with K.-Th. Sturm (Bonn)]

1. Introduction

M: complete Riemannian manifold, $\dim M \geq 2$

$(X(t), \mathbb{P}_x)$: Brownian motion on M

P_t: heat semigroup

$$\mathrm{Ric} \geq K$$

$$\Downarrow$$

Good control of

- Coupling by parallel transport
- Coupling by reflection

of two BMs $(X(t), \mathbb{P}_{x_1})$ and $(X(t), \mathbb{P}_{x_2})$

- **Coupling by parallel transport**

- **Coupling by reflection**

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$\Rightarrow$ (monotonicity of a transportation cost)

$\Rightarrow$ $\|\nabla P_t f\|_\infty \le C_{N,K}(t)\,\mathrm{osc}(f)$

Motivation: Stochastic analysis on singular spaces

- Construction of coupling by reflection

  $\Leftarrow$ differentiable structure on M

  (How do we formulate it on singular sp.'s?)

- "Monotonicity of transportation cost" is robust

  $\Rightarrow$ Stable under Gromov-Hausdorff conv.

2. Main results

Assumption

$\dim M \le N$ and $\mathrm{Ric} \ge K$

$(N \in [2, \infty]$ and $K \in \mathbb{R})$

- $\bar{R} := \sqrt{\dfrac{N-1}{K \vee 0}}\pi$

⋆ $\mathrm{diam}(M) \le \bar{R}$ [Bonnet-Myers thm]

Theorem 1 [K. & Sturm]

$(X_1(t), X_2(t))$: a coupling by refl. of two BMs.

$\Rightarrow \exists \varphi = \varphi^{N,K} : [0,\infty) \times \overline{[0,\bar{R})} \to [0,1]$

s.t. for $t > 0$,

$$\mathbb{E}[\varphi_{t-s}(d(X_1(s), X_2(s)))] \searrow$$

in $s \in [0, t]$

Optimal transportation cost

For $c : M \times M \to \mathbb{R}$, $\mu_1, \mu_2 \in \mathcal{P}(M)$,

$$\mathcal{T}_c(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_2)} \int_{M \times M} c \, d\pi$$

($\Pi(\mu_1, \mu_2) \subset \mathcal{P}(M^2)$: couplings of μ_1 & μ_2)

Theorem 2 [K. & S.]

For $t > 0$, $\mu_1, \mu_2 \in \mathcal{P}(M)$,

$$\mathcal{T}_{\varphi_{t-s}(d)}(P_s^* \mu_1, P_s^* \mu_2) \searrow \text{ in } s \in [0, t]$$

Definition of $\varphi_t^{K,N}(a)$ (for $N \in \mathbb{N}$)

$$\varphi_t^{K,N}(a) := \frac{1}{2} \left\| \tilde{P}_t^* \delta_{\tilde{x}} - \tilde{P}_t^* \delta_{\tilde{y}} \right\|_{\mathrm{TV}}$$

- $\tilde{P}_t^*$: heat semigr. on the spaceform $\mathbb{M}_{N,K}$ ($\mathbb{M}_{N,K}$: sphere, Euclidean sp. or hyperbolic sp.)
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⋆ We can define φ_t even when $N \notin \mathbb{N}$

⋆ More explicit expression is possible but complicated when $N < \infty$ or $K \neq 0$

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  $(\Rightarrow \mathcal{T}_{\varphi_0(d)}(\mu_1, \mu_2) = \frac{1}{2}\|\mu_1 - \mu_2\|_{\mathrm{TV}})$
- $N < N' \Rightarrow \varphi_t^{K,N}(a) \leq \varphi_t^{K,N'}(a)$
- $\partial^+\varphi_t(0) \leq \frac{1}{2\sqrt{2\pi}}\left(\frac{\mathrm{e}^{Kt} - 1}{K}\right)^{-1/2}$

3. Applications

Theorem 2

$$\Downarrow$$

$$\mathcal{T}_{\varphi_0(d)}(P_t^*\delta_x, P_t^*\delta_y) \leq \mathcal{T}_{\varphi_t(d)}(\delta_x, \delta_y)$$

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Corollary 1 (Comparison thm for total variations)

$$\left\| P_t^* \delta_x - P_t^* \delta_y \right\|_{\mathrm{TV}} \leq \left\| \tilde{P}_t^* \delta_{\tilde{x}} - \tilde{P}_t^* \delta_{\tilde{y}} \right\|_{\mathrm{TV}}$$

Corollary 2 (Gradient estimate)

For any bounded measurable f on M,

$$\|\nabla P_t f\|_\infty \leq \partial^+ \varphi_t(0)\, \mathbf{osc}(f)$$

Remark

Corollary 2 directly follows from

$$\mathcal{T}_{\varphi_0(d)}(P_t \delta_x, P_t \delta_y) \leq \mathcal{T}_{\varphi_t(d)}(\delta_x, \delta_y)$$

Remark

These results hold in more general framework:

(i) Diffsions associated with $\Delta/2 + Z$,

$$\mathrm{Ric}^{Z,N} \geq K$$

($\mathrm{Ric}^{Z,N}$: Bakry-Émery Ricci tensor)

⋆ Z can be of non-gradient type

(ii) Riemannian metric g depends on time,

$$\mathrm{Ric}^{Z,\infty}_{g(t)} \geq \partial_t g(t) + K$$

(a generalization of super Ricci flow)

Stability under GH-convergence

(M_m, g_m): n-dim. cpt. Riem. mfds, $\mathrm{Ric}_{g_m} \ge K$

Suppose

$$(M_m, d_m, \mathrm{vol}_{g_m}) \xrightarrow{\text{mGH}} (M_\infty, d_\infty, v_\infty)$$

$\Downarrow$

For $\mu^{(m)} \in \mathcal{P}(M_m)$

with $\mu^{(m)} \to \mu^{(\infty)} \in \mathcal{P}(M_\infty)$,

$P_t \mu^{(m)} \to$ a "heat distribution" μ_t^∞ on M_∞

[Gigli '10, Ambrosio, Gigli & Savaré '11]

Theorem 3 [K. & S.]

$(M_\infty, d_\infty, v_\infty)$: as above, $N \geq n$

$\mu_1(t), \mu_2(t)$: heat distributions on M_∞

$\Rightarrow$ For $t > 0$,

$$\mathcal{T}_{\varphi_{t-s}^{K,N}(d)}(\mu_1(s), \mu_2(s)) \searrow$$

in $s \in [0, t]$