

Bakry-Émery 型微分評価と 関連する話題

栗田 和正

(お茶の水女子大学)

研究集会「確率解析とその周辺」@佐賀大学, 2011 年 11 月 11 日-13 日

1. Bakry-Émery の微分評価とは

P_t : 生成作用素 \mathcal{L} を持つ半群 (P_t “=” $e^{t\mathcal{L}}$)

Bakry-Émery の (L^p -) 微分評価:

$$|\nabla P_t f|(x) \leq e^{-Kt} P_t(|\nabla f|^p)(x)^{1/p} \quad (G_p)$$

$$(p \in [1, \infty), K \in \mathbb{R})$$

[D. Bakry & M. Émery, '84]

- P_t の定義される関数空間: 標準的には,
 P_t : 強連続縮小, 対称, Markov 的 on $L^2(\mu)$
 ($\Rightarrow \exists$ symm. contr. extension on $L^p(\mu)$
 $(1 \leq p \leq \infty)$)
- 微分 (の絶対値) $|\nabla f|$ の定義も必要

- P_t の定義される関数空間: 標準的には,
 P_t : 強連続縮小, 対称, Markov 的 on $L^2(\mu)$
 $\left(\begin{array}{l} \Rightarrow \exists \text{symm. contr. extension on } L^p(\mu) \\ (1 \leq p \leq \infty) \end{array} \right)$

- 微分 (の絶対値) $|\nabla f|$ の定義も必要

★ もし P_t : Markovian

(i.e. $0 \leq f \leq 1 \Rightarrow 0 \leq P_t f \leq 1$) なら,

$p' < p$ のとき 「 $(G_{p'}) \Rightarrow (G_p)$ 」

(\because Hölder ineq. for $(f, g) \mapsto P_t(fg)$)

例 (\mathbb{R}^m の熱半群, i.e. $\mathcal{L} = \Delta$)

$$f : \mathbb{R}^m \rightarrow \mathbb{R}$$

$$P_t f(x) := \int_{\mathbb{R}^m} p_t(x, y) f(y) dy,$$

$$p_t(x, y) := \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{|x - y|^2}{4t}\right)$$

例 (\mathbb{R}^m の熱半群, i.e. $\mathcal{L} = \Delta$)

$$f : \mathbb{R}^m \rightarrow \mathbb{R}$$

$$P_t f(x) := \int_{\mathbb{R}^m} p_t(x, y) f(y) dy,$$

$$p_t(x, y) := \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{|x - y|^2}{4t}\right)$$

$$\Rightarrow \partial_{x_i} P_t f(x) = P_t(\partial_{x_i} f)(x)$$

例 (\mathbb{R}^m の熱半群, i.e. $\mathcal{L} = \Delta$)

$$f : \mathbb{R}^m \rightarrow \mathbb{R}$$

$$P_t f(x) := \int_{\mathbb{R}^m} p_t(x, y) f(y) dy,$$

$$p_t(x, y) := \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{|x - y|^2}{4t}\right)$$

$$\Rightarrow \partial_{x_i} P_t f(x) = P_t(\partial_{x_i} f)(x)$$

(\because (可換な) 変数変換 (平行移動))

例 (\mathbb{R}^m の熱半群, i.e. $\mathcal{L} = \Delta$)

$$f : \mathbb{R}^m \rightarrow \mathbb{R}$$

$$P_t f(x) := \int_{\mathbb{R}^m} p_t(x, y) f(y) dy,$$

$$p_t(x, y) := \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{|x - y|^2}{4t}\right)$$

$$\Rightarrow \partial_{x_i} P_t f(x) = P_t(\partial_{x_i} f)(x)$$

(\because (可換な) 変数変換 (平行移動))

$$\Rightarrow |\nabla P_t f|(x) \leq P_t(|\nabla f|)(x),$$

i.e. (G_1) with $K = 0$

微分 $|\nabla f|$ について:

★ 平方場作用素 (carré du champ)

$$\Gamma(f, g) := \frac{1}{2} (\mathcal{L}(fg) - g\mathcal{L}f - f\mathcal{L}g)$$

(例 : $\mathcal{L} = \Delta$ on $\mathbb{R}^m \Rightarrow \Gamma(f, f) = |\nabla f|^2$)

- $\Gamma(f, f) = \lim_{t \rightarrow 0} \frac{1}{t} (P_t(f^2) - (P_t f)^2) \geq 0$
- $P_t \leftrightarrow (\mathcal{E}, \mathcal{F})$: Dirichlet form on $L^2(\mu)$ なら,
$$\int h\Gamma(f, f)d\mu = \frac{1}{2} (2\mathcal{E}(hf, f) - \mathcal{E}(h, f^2))$$

($h, f \in \mathcal{F} \cap L^\infty$)

以下, この § では, 以下を仮定:

- $\exists \Gamma$

- $|\nabla f| = \sqrt{\Gamma(f, f)}$



以下, この § では, 以下を仮定:

- $\exists \Gamma$

- $|\nabla f| = \sqrt{\Gamma(f, f)}$



⇒ 例えば (G_2) は,

$$\Gamma(P_t f, P_t f) \leq e^{-2Kt} P_t(\Gamma(f, f))$$

以下, この § では, 以下を仮定:

- $\exists \Gamma$

- $|\nabla f| = \sqrt{\Gamma(f, f)}$



⇒ 例えば (G_2) は,

$$\Gamma(P_t f, P_t f) \leq e^{-2Kt} P_t(\Gamma(f, f))$$

★ 「“=” at $t = 0$ 」を利用して $t = 0$ で微分:

$$2\Gamma(f, \mathcal{L}f) \leq -2K\Gamma(f, f) + \mathcal{L}(\Gamma(f, f))$$

$$2\Gamma(f, \mathcal{L}f) \leq -2K\Gamma(f, f) + \mathcal{L}(\Gamma(f, f))$$

$$\Downarrow \Gamma_2(f, f) := \frac{1}{2} (\mathcal{L}(\Gamma(f, f)) - 2\Gamma(f, \mathcal{L}f))$$

Γ_2 -条件:

$$\Gamma_2(f, f) \geq K\Gamma(f, f) \quad (\Gamma_2(K))$$

命題

$$(G_2) \Leftrightarrow (\Gamma_2(K))$$

命題

$$(G_2) \Leftrightarrow (\Gamma_2(K))$$

“証明”

「 $(\Gamma_2(K)) \Rightarrow (G_2)$ 」のみ示せば良い.

$$\Phi(s) := P_s(\Gamma(P_{t-s}f, P_{t-s}f))$$

- $\Phi(0) = \Gamma(P_t f, P_t f), \Phi(t) = P_t(\Gamma(f, f))$
- $(\Gamma_2(K)) \Rightarrow \Phi'(s) \geq 2K\Phi(s)$
 $\Rightarrow \Phi(0) \leq e^{-2Kt}\Phi(t) \quad \blacksquare$

つまり、時間変数 t に関して、

$$\begin{array}{ccc} & \text{微分} & \\ & \Rightarrow & \\ (G_2) & & (\Gamma_2(K)) \\ & \Leftarrow & \\ & \text{積分} & \end{array}$$

注

- (G_2) の weak formulation:

$$\int g \Gamma(P_t f, P_t f) d\mu \leq e^{-2Kt} \int g P_t(\Gamma(f, f)) d\mu$$

($\forall g \geq 0$: “nice” test function)

に依れば, 弱い仮定で厳密に証明可能

[cf. Gigli-K.-Ohta '10, Theorem 4.8]

- この議論に, Γ や \mathcal{L} の derivation property (or $(\mathcal{E}, \mathcal{F})$ の strong locality) は不要

Γ_2 の計算例

(i) $V : \mathbb{R}^m \rightarrow \mathbb{R}$, $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ on \mathbb{R}^m

$$\Gamma(f, f) = |\nabla f|^2,$$

$$\Gamma_2(f, f) = \|\text{Hess } f\|_{\text{HS}}$$

$$+ \text{Hess } V(\nabla f, \nabla f)$$

Γ_2 の計算例

(i) $V : \mathbb{R}^m \rightarrow \mathbb{R}$, $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ on \mathbb{R}^m

$$\Gamma(f, f) = |\nabla f|^2,$$

$$\Gamma_2(f, f) = \|\text{Hess } f\|_{\text{HS}}$$

$$+ \text{Hess } V(\nabla f, \nabla f)$$

$$\Rightarrow \boxed{\Gamma_2(K) \Leftrightarrow \text{Hess } V \geq K}$$

Γ_2 の計算例

(i) $V : \mathbb{R}^m \rightarrow \mathbb{R}$, $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ on \mathbb{R}^m

$$\Gamma(f, f) = |\nabla f|^2,$$

$$\Gamma_2(f, f) = \|\text{Hess } f\|_{\text{HS}}$$

$$+ \text{Hess } V(\nabla f, \nabla f)$$

$$\Rightarrow \boxed{\Gamma_2(K) \Leftrightarrow \text{Hess } V \geq K}$$

特に, Ornstein-Uhlenbeck 過程の場合 :

$$\mathcal{L} = \Delta - Kx \cdot \nabla \quad (\Leftrightarrow V(x) = \frac{K}{2}|x|^2)$$

$$\Rightarrow (\Gamma_2(K))$$

(ii) $\mathcal{L} = \Delta$ on a Riemannian manifold

$$\begin{aligned}\Gamma_2(f, f) &= \frac{1}{2} (\Delta(|\nabla f|^2) - 2\langle \nabla f, \nabla \Delta f \rangle) \\ &= \|\text{Hess } f\|_{\text{HS}} + \text{Ric}(\nabla f, \nabla f)\end{aligned}$$

(Bochner-Weitzenböck の公式)

(ii) $\mathcal{L} = \Delta$ on a Riemannian manifold

$$\begin{aligned}\Gamma_2(f, f) &= \frac{1}{2} (\Delta(|\nabla f|^2) - 2\langle \nabla f, \nabla \Delta f \rangle) \\ &= \|\text{Hess } f\|_{\text{HS}} + \text{Ric}(\nabla f, \nabla f)\end{aligned}$$

(Bochner-Weitzenböck の公式)

$$\Rightarrow (\Gamma_2(K)) \Leftrightarrow \text{Ric} \geq K$$

(ii) $\mathcal{L} = \Delta$ on a Riemannian manifold

$$\begin{aligned}\Gamma_2(f, f) &= \frac{1}{2} (\Delta(|\nabla f|^2) - 2\langle \nabla f, \nabla \Delta f \rangle) \\ &= \|\text{Hess } f\|_{\text{HS}} + \text{Ric}(\nabla f, \nabla f)\end{aligned}$$

(Bochner-Weitzenböck の公式)

$$\Rightarrow (\Gamma_2(K)) \Leftrightarrow \text{Ric} \geq K$$

★ 微分幾何で「 $\text{Ric} \geq K$ 」から従う事の多くは、 $(\Gamma_2(K))$ からも (抽象的な枠組で) 従う

(G_2) の応用

- Poincaré inequality (for P_t)

$$P_t(f^2) - (P_t f)^2 \leq \frac{1 - e^{-2Kt}}{K} P_t(\Gamma(f, f))$$

- Reversed Poincaré inequality (for P_t)

$$P_t(f^2) - (P_t f)^2 \geq \frac{e^{2Kt} - 1}{K} \Gamma(P_t f, P_t f)$$

$$\left(\Rightarrow |\nabla P_t f| \leq \sqrt{\frac{K}{e^{2Kt} - 1}} \text{osc } f \right)$$

証明

$$\Psi(s) := P_s((P_{t-s}f)^2)$$

- $\Psi(0) = (P_t f)^2, \Psi(t) = P_t(f^2)$
- $\Psi'(s) = 2P_s(\Gamma(P_{t-s}f, P_{t-s}f))$

By (G_2) ,

$$\dagger \Psi'(s) \leq 2e^{-2K(t-s)} P_s P_{t-s}(\Gamma(f, f))$$

$$\dagger \Psi'(s) \geq 2e^{2Ks} \Gamma(P_s P_{t-s}f, P_s P_{t-s}f)$$

\Rightarrow conclusion by integration in s ■

注

- 形式的には,

Poincaré or rev. Poincaré ineq. $\Rightarrow (\Gamma_2(K))$

(両辺を $t = 0$ で 2 次まで Taylor 展開)

- これらの議論には, Γ および \mathcal{L} の derivation property は不要

条件 (G_1) について

- 強い仮定のもと, $(\Gamma_2(K)) \Rightarrow (G_1)$

$\uparrow \exists \mathcal{C} \subset \text{Dom}(\mathcal{L})$ dense s.t.

- \mathcal{C} は積で閉じている線形空間

- $P_t \mathcal{C} \subset \mathcal{C}, \mathcal{L} \mathcal{C} \subset \mathcal{C}$

- $\forall \varphi: C^\infty(\mathbb{R}), f \in \mathcal{C} \Rightarrow \varphi(f) \in \mathcal{C}$

- $(G_1) \Rightarrow$ **log-Sobolev ineq.** (for P_t):

$$P_t(f^2 \log f^2) - P_t(f^2) \log(P_t(f^2))$$

$$\leq \frac{2(1 - e^{-2Kt})}{K} P_t(\Gamma(f, f))$$

Derivation property

$$\varphi \in C^2(\mathbb{R}), \varphi(0) = 0$$

- $f \in \text{Dom}(\mathcal{L})$

$$\Rightarrow \varphi(f) \in \text{Dom}(f),$$

$$\mathcal{L}(\varphi(f)) = \varphi'(f)\mathcal{L}f + \varphi''(f)\Gamma(f, f)$$

- $f \in \text{Dom}(\Gamma)$

$$\Rightarrow \varphi(f) \in \text{Dom}(\Gamma),$$

$$\Gamma(\varphi(f), g) = \varphi'(f)\Gamma(f, g)$$

Proof of $\Gamma(G_1) \Rightarrow \text{log-Sobolev}$

(P_t : Markov は仮定)

$$\Psi(s) := P_s(\varphi(P_{t-s}f)) \quad (\varphi \in C^2(\mathbb{R}))$$

\Rightarrow By the derivation property,

$$\Psi'(s) = P_s(\varphi''(P_{s-t}f)\Gamma(P_{s-t}f, P_{s-t}f))$$

• $\varphi(u) := u \log u \Rightarrow \varphi''(u) = \frac{1}{u}$

\Rightarrow Goal: For $f \geq 0$,

$$\Psi'(s) \leq 4e^{-2K(t-s)} P_t(\Gamma(\sqrt{f}, \sqrt{f}))$$

By (G_1) & Schwarz ineq. for $(g, h) \mapsto P_{t-s}(gh)$,

$$\begin{aligned}
 \Psi'(s) &= P_s \left(\frac{\Gamma(P_{t-s}f, P_{t-s}f)}{P_{t-s}f} \right) \\
 &\leq e^{-2K(t-s)} P_s \left(\frac{\left(P_{t-s}(\sqrt{\Gamma(f, f)}) \right)^2}{P_{t-s}f} \right) \\
 &\leq e^{-2K(t-s)} P_s P_{t-s} \left(\sqrt{\frac{\Gamma(f, f)}{f}} \right) \\
 &= 4e^{-2K(t-s)} P_t \left(\Gamma(\sqrt{f}, \sqrt{f}) \right) \quad \blacksquare
 \end{aligned}$$

大域的 Poincaré/log-Sobolev

もし, $K > 0$ & $\lim_{t \rightarrow \infty} P_t f \equiv \int f d\mu$ ならば,

$$P_t(f^2) - (P_t f)^2 \leq \frac{1 - e^{-2Kt}}{K} P_t(\Gamma(f, f))$$

\Downarrow

$$\int f^2 d\mu - \left(\int f d\mu \right)^2 \leq \frac{1}{K} \int \Gamma(f, f) d\mu$$

大域的 Poincaré/log-Sobolev

もし, $K > 0$ & $\lim_{t \rightarrow \infty} P_t f \equiv \int f d\mu$ ならば,

$$P_t(f^2 \log f^2) - P_t(f^2) \log P_t(f^2)$$

$$\leq \frac{2(1 - e^{-2Kt})}{K} P_t(\Gamma(f, f))$$

\Downarrow

$$\int f^2 \log f^2 d\mu - \int f^2 d\mu \log \left(\int f^2 d\mu \right)$$

$$\leq \frac{2}{K} \int \Gamma(f, f) d\mu$$

(G_1) への確率論的アプローチ ... 結合法

例

$$\mathcal{L} := \Delta - \nabla V \cdot \nabla \text{ on } \mathbb{R}^m$$

\updownarrow

$$dX_t^x = \sqrt{2}dW_t - \nabla V(X_t^x)dt, \quad X_0^x = x$$

仮定: $\forall x, y \in \mathbb{R}^m,$

$$(\nabla V(x) - \nabla V(y)) \cdot (x - y) \geq K|x - y|^2$$

(\uparrow Hess $V \geq K$)

X_t^x, X_t^y : 前掲の SDE の (共通の W_t に対する) 解

⇓

$$d(X_t^x - X_t^y) = -(\nabla V(X_t^x) - \nabla V(X_t^y)) dt$$

⇓

$$d|X_t^x - X_t^y|^2 \leq -2K|X_t^x - X_t^y|^2 dt$$

⇓

$$|X_t^x - X_t^y|^2 \leq e^{-2Kt} |x - y|^2$$

\Downarrow

$$\begin{aligned} |P_t f(x) - P_t f(y)| &= |\mathbb{E}[f(X_t^x) - f(X_t^y)]| \\ &\leq \mathbb{E}[|\nabla f|(X_t^x) |X_t^x - X_t^y| + o(|X_t^x - X_t^y|)] \\ &\leq e^{-Kt} \mathbb{E}[|\nabla f|(X_t^x)] |x - y| + o(|x - y|) \end{aligned}$$

 \Downarrow

$$\begin{aligned} |\nabla P_t f|(x) &= \limsup_{y \rightarrow x} \frac{|P_t f(x) - P_t f(y)|}{|x - y|} \\ &\leq e^{-Kt} \mathbb{E}[|\nabla f|(X_t^x)] \\ &= e^{-Kt} P_t(|\nabla f|)(x) \end{aligned}$$

注

- 結合法の方法は，Riemann 多様体上や無限次元空間上にも拡張されている。
- 設定によっては，解析は単純ではない (拡散係数が定数でない場合の SDE 等)。
- それらの拡張では，Bakry-Émery の議論「 $(\Gamma_2(K)) \Rightarrow (G_1)$ 」で用いた，**強い仮定は要らない**。

2. Wasserstein 距離の Lipschitz 型評価との 双対性

(X, d) : ポーランド空間, d : 測地距離

• \tilde{d} : X 上の連続な測地距離関数

(例: $\tilde{d} = e^{-Kt}d$)

• $(P(x, \cdot))_{x \in X} \subset \mathcal{P}(X)$: Markov 核

$$Pf(x) := \int_X f(y) dP(x, dy),$$

$$P^* \mu(A) := \int_X P(x, A) \mu(dx)$$

(例: $P(x, dy) = p_t(x, dy)$ 推移確率)

L^p -Wasserstein 距離

For $p \in [1, \infty]$, $\mu, \nu \in \mathcal{P}(X)$

$$W_{d,p}(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \|d\|_{L^p(\pi)} \in [0, \infty],$$

$$\Pi(\mu, \nu) \subset \mathcal{P}(X \times X),$$

$$\Pi(\mu, \nu) := \left\{ \pi \left| \begin{array}{l} \pi(A \times X) = \mu(A), \\ \pi(X \times A) = \nu(A) \end{array} \right. \right\}$$

$((\mu, \nu)$ のカップリング全体)

“微分” $|\nabla f|$

$$|\nabla_d f|(x) := \limsup_{y \rightarrow x} \left| \frac{f(x) - f(y)}{d(x, y)} \right|,$$

$$\|\nabla_d f\|_\infty := \sup_{x \in X} |\nabla_d f|(x)$$

- $|\nabla_d f|(x)$: x での d -局所 Lipschitz 定数
- $\|\nabla_d f\|_\infty$: d -Lipschitz 定数, i.e.
 $|f(x) - f(y)| \leq \|\nabla_d f\|_\infty d(x, y)$

$$W_{d,q}(P^*\mu, P^*\nu) \leq W_{\tilde{d},q}(\mu, \nu) \quad (W_q)$$

$$|\nabla_{\tilde{d}} P f|(x) \leq P(|\nabla_d f|^p)(x)^{1/p} \quad (G'_p)$$

$$(\mu, \nu \in \mathcal{P}(X), f \in C_b^{\text{Lip}}(X))$$

定理[cf. K. '10]

For $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$(W_q) \Leftrightarrow (G'_p)$$

注

- For $p' < p$,
 $(G'_{p'}) \Rightarrow (G'_p)$ and $(W_{q'}) \Rightarrow (W_q)$
- $(W_1) \Leftrightarrow (G'_\infty)$ は既知
(Kantorovich-Rubinstein formula による)
- $(W_\infty) \Rightarrow (G'_1)$ は (本質的に) 既知
(結合法)
- [K. '10] で課していた, **技術的な仮定は不要**

証明の概略

(for $p, q \in (1, \infty)$)

[A] $(W_q) \Rightarrow (G'_p)$:

$$W_{d,q}(P^*\mu, P^*\nu) \leq W_{\tilde{d},q}(\mu, \nu) \quad (W_q)$$

$$|\nabla_{\tilde{d}} P f|(x) \leq P(|\nabla_d f|^p)(x)^{1/p} \quad (G'_p)$$

Goal:

$$|P f(x) - P f(y)|$$

$$\leq P(|\nabla_d f|^p)(x)^{1/p} \tilde{d}(x, y) + o(\tilde{d}(x, y))$$

π : $W_{d,p}(P^*\delta_x, P^*\delta_y)$ の minimizer とする

$$Pf(x) - Pf(y)$$

$$= \int_{\mathbf{X}} f dP^* \delta_x - \int_{\mathbf{X}} f dP^* \delta_y$$

$$= \int_{\mathbf{X} \times \mathbf{X}} (f(z) - f(w)) \pi(dzdw)$$

$$= \int_{\mathbf{X} \times \mathbf{X}} \mathbf{1}_{\{d \leq r\}} \left(\frac{f(z) - f(w)}{d} \right) \pi(dzdw)$$

$$+ \int_{\mathbf{X} \times \mathbf{X}} \mathbf{1}_{\{d > r\}} \left(\frac{f(z) - f(w)}{d} \right) \pi(dzdw)$$

$$= (I) + (II)$$

$$G_r f(z) := \sup_{d(w,z) \leq r} \frac{|f(z) - f(w)|}{d(z,w)}$$

⇓

$$\begin{aligned} (I) &\leq \int_{X \times X} G_r f(z) d(z,w) \pi(dzdw) \\ &\leq P(|G_r f|^p)(x)^{1/p} W_{d,q}(P^* \delta_x, P^* \delta_y) \end{aligned}$$

$$\begin{aligned} (W_q) &\boxed{\leq} P(|G_r f|^p)(x)^{1/p} \tilde{d}(x,y) \\ &= P(|\nabla_d f|^p)(x)^{1/p} \tilde{d}(x,y) + o(\tilde{d}(x,y)) \end{aligned}$$

if $r = o(1)$ as $\tilde{d}(x,y) \rightarrow 0$

$$\begin{aligned}
(II) &\leq 2\|f\|_\infty \pi(d > r) \\
&\leq \frac{1}{r^q} W_{d,q}(P^* \delta_x, P^* \delta_y)^q \\
&\leq \left(\frac{\tilde{d}(x, y)}{r} \right)^q = o(\tilde{d}(x, y))
\end{aligned}$$

for a suitable choice of r

(i.e. $\tilde{d}(x, y)^{1/p} = o(r)$) //

[B] $(G'_p) \Rightarrow (W_q)$:

$$|\nabla_{\tilde{d}} P f|(x) \leq P(|\nabla_d f|^p)(x)^{1/p} \quad (G'_p)$$

$$W_{d,q}(P^* \mu, P^* \nu) \leq W_{\tilde{d},q}(\mu, \nu) \quad (W_q)$$

[B] $(G'_p) \Rightarrow (W_q)$:

$$|\nabla_{\tilde{d}} P f|(x) \leq P(|\nabla_d f|^p)(x)^{1/p} \quad (G'_p)$$

$$W_{d,q}(P^* \mu, P^* \nu) \leq W_{\tilde{d},q}(\mu, \nu) \quad (W_q)$$

- (W_q) を $\mu = \delta_x, \nu = \delta_y$ の場合のみ示す

$$\Rightarrow \underline{\text{Goal:}} \quad \frac{1}{q} W_{d,q}(P^* \delta_x, P^* \delta_y)^q \leq \frac{1}{q} \tilde{d}(x, y)^q$$

Kantorovich duality

$$\frac{W_{d,q}(\mu, \nu)^q}{q} = \sup_{f \in C_b^{\text{Lip}}(X)} \left[\int_X \hat{f} d\mu - \int_X f d\nu \right],$$

$$\hat{f}(x) := \inf_{y \in X} \left[f(y) + \frac{d(x, y)^q}{q} \right]$$

Kantorovich duality

$$\frac{W_{d,q}(\mu, \nu)^q}{q} = \sup_{f \in C_b^{\text{Lip}}(X)} \left[\int_X \hat{f} d\mu - \int_X f d\nu \right],$$

$$\hat{f}(x) := \inf_{y \in X} \left[f(y) + \frac{d(x, y)^q}{q} \right]$$

$$\star \frac{1}{q} W_{d,q}(P^* \delta_x, P^* \delta_y)^q = \sup_f \left[P \hat{f}(x) - P f(y) \right]$$

Kantorovich duality

$$\frac{W_{d,q}(\mu, \nu)^q}{q} = \sup_{f \in C_b^{\text{Lip}}(X)} \left[\int_X \hat{f} d\mu - \int_X f d\nu \right],$$

$$\hat{f}(x) := \inf_{y \in X} \left[f(y) + \frac{d(x, y)^q}{q} \right]$$

$$\star \frac{1}{q} W_{d,q}(P^* \delta_x, P^* \delta_y)^q = \sup_f \left[P \hat{f}(x) - P f(y) \right]$$

$$\Rightarrow \underline{\text{Goal:}} \sup_f \left[P \hat{f}(x) - P f(y) \right] \leq \frac{\tilde{d}(x, y)^q}{q}$$

Kantorovich duality の “ \geq ” の証明

$$\pi \in \Pi(\mu, \nu),$$

$$g, f \in C_b^{\text{Lip}}(X), \quad g(x) - f(y) \leq \frac{d(x, y)^q}{q}$$

\Downarrow

$$\int_X g \, d\mu - \int_X f \, d\nu \leq \frac{1}{q} \|d\|_{L^q(\pi)}^q$$

\Downarrow maximize

\Downarrow minimize

$$\sup_f \left[\int_X \hat{f} \, d\mu - \int_X f \, d\nu \right] \leq \frac{W_{d,q}(\mu, \nu)^q}{q} //$$

Hopf-Lax 半群 (Hamilton-Jacobi 半群)

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + \frac{d(x, y)^q}{qt^{q-1}} \right],$$

$$Q_0 f(x) := f(x)$$

★ $Q_1 f = \hat{f}$

Hopf-Lax 半群 (Hamilton-Jacobi 半群)

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + \frac{d(x, y)^q}{qt^{q-1}} \right],$$

$$Q_0 f(x) := f(x)$$

★ $Q_1 f = \hat{f}$

Goal: $\sup_f \left[P \hat{f}(x) - P f(y) \right] \leq \frac{\tilde{d}(x, y)^q}{q}$

Hopf-Lax 半群 (Hamilton-Jacobi 半群)

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + \frac{d(x, y)^q}{qt^{q-1}} \right],$$

$$Q_0 f(x) := f(x)$$

$$\star Q_1 f = \hat{f}$$

$$\underline{\text{Goal:}} \sup_f \left[P \hat{f}(x) - P f(y) \right] \leq \frac{\tilde{d}(x, y)^q}{q}$$

- $\tilde{\gamma} : [0, 1] \rightarrow X$: \tilde{d} -測地線, $\tilde{\gamma}_0 = y$, $\tilde{\gamma}_1 := x$

Hopf-Lax 半群 (Hamilton-Jacobi 半群)

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + \frac{d(x, y)^q}{qt^{q-1}} \right],$$

$$Q_0 f(x) := f(x)$$

★ $Q_1 f = \hat{f}$

⇒ Goal: $\forall f \in C_b^{\text{Lip}}(X),$

$$PQ_1 f(\tilde{\gamma}_1) - PQ_0 f(\tilde{\gamma}_0) \leq \frac{\tilde{d}(x, y)^q}{q}$$

- $\tilde{\gamma} : [0, 1] \rightarrow X$: \tilde{d} -測地線, $\tilde{\gamma}_0 = y, \tilde{\gamma}_1 := x$

Key idea:

$$PQ_1 f(\tilde{\gamma}_1) - PQ_0 f(\tilde{\gamma}_0) \\ = \int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t)) dt$$

と変形 (補間)

Key idea:

$$PQ_1 f(\tilde{\gamma}_1) - PQ_0 f(\tilde{\gamma}_0) \\ = \int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t)) dt$$

と変形 (補間)

実現には,

$t \mapsto PQ_t f(\tilde{\gamma}_t)$: Lipschitz

を示せば充分

補題

For $f \in C_b^{\text{Lip}}(X)$,

$(t, z) \mapsto PQ_t f(z)$: Lipschitz w.r.t $|\cdot|$ & \tilde{d}

証明

[Balogh & Engoulatov & Hunziker & Maasalo '09]:

$$|Q_t f(z) - Q_t f(w)| \leq \|\nabla f\|_{\infty} d(z, w),$$

$$|Q_t f(z) - Q_s f(z)| \leq \frac{\|\nabla f\|_{\infty}^p}{p} |t - s|$$

$\therefore (G'_p) \Rightarrow$ conclusion \blacksquare



Goal: For $\forall f \in C_b^{\text{Lip}}(X)$,

$$\int_0^1 \partial_t(PQ_t f(\tilde{\gamma}_t)) dt \leq \frac{\tilde{d}(x, y)^q}{q}$$

★ 「 $\partial_t(PQ_t f(\tilde{\gamma}_t)) \leq \frac{\tilde{d}(x, y)^q}{q}$ for a.e. t 」

を示せば充分

By [Ambrosio & Gigli & Savaré '08, Lemma 4.3.4],

$$\partial_t P Q_t f(\tilde{\gamma}_t)$$

$$\leq \overline{\lim}_{h \downarrow 0} \frac{P Q_{t+h} f(\tilde{\gamma}_t) - P Q_t f(\tilde{\gamma}_t)}{h} \\ + \overline{\lim}_{h \downarrow 0} \frac{P Q_t f(\tilde{\gamma}_t) - P Q_t f(\tilde{\gamma}_{t-h})}{h}$$

$$= (I) + (II)$$

(Leibniz 則)

By the def. of $|\nabla_{\tilde{d}} \cdot|$,

$$(II) = \overline{\lim}_{h \downarrow 0} \frac{PQ_t f(\tilde{\gamma}_t) - PQ_t f(\tilde{\gamma}_{t-h})}{h}$$

$$\leq \tilde{d}(x, y) |\nabla_{\tilde{d}} PQ_t f|(\tilde{\gamma}_t)$$

$$(G'_p) \leq \tilde{d}(x, y) P(|\nabla_d Q_t f|^p)(\tilde{\gamma}_t)^{1/p}$$

$(\tilde{d}(x, y): \text{speed of } \tilde{\gamma})$

★ By [Ambrosio & Gigli & Savaré '11],

For $\forall t > 0, \forall z \in X, \exists \partial_t^+ Q_t f(z),$

$$\partial_t^+ Q_t f(z) + \frac{1}{p} |\nabla_d Q_t f|(z)^p \leq 0 \quad (*)$$

(*) が正しい (intuitive な) 理由

$$\lceil \exists y_0 \text{ s.t. } Q_s f(z) = f(y_0) + \frac{d(z, y_0)^q}{qs^{q-1}} \rceil$$

for $\forall s \approx t, \forall z \approx x$ と仮定

⇓

$$\partial_t Q_t f(x) = -\frac{q-1}{q} \left(\frac{d(x, y_0)}{t} \right)^q$$

$$\frac{1}{p} |\nabla_d Q_t f|^p = \frac{1}{p} \left(\frac{d(x, y_0)}{t} \right)^{p(q-1)}$$

★ By [Ambrosio & Gigli & Savaré '11],

For $\forall t > 0, \forall z \in X, \exists \partial_t^+ Q_t f(z),$

$$\partial_t^+ Q_t f(z) + \frac{1}{p} |\nabla_d Q_t f|(z)^p \leq 0 \quad (*)$$

\Downarrow

$$(I) = \overline{\lim}_{h \downarrow 0} \frac{P Q_{t+h} f(\tilde{\gamma}_t) - P Q_t f(\tilde{\gamma}_t)}{h}$$

Fatou $\boxed{\leq}$ $P(\partial_t^+ Q_t f)(\tilde{\gamma}_t)$

$(*) \boxed{\leq}$ $-\frac{1}{p} P(|\nabla_d Q_t f|^p)(\tilde{\gamma}_t)$

$$(I) \leq \tilde{d}(x, y) P(|\nabla_d Q_t f|^p) (\tilde{\gamma}_t)^{1/p}$$

$$(II) \leq -\frac{1}{p} P(|\nabla_d Q_t f|^p) (\tilde{\gamma}_t)$$

⇓

$\sigma := P(|\nabla_d Q_t f|^p) (\tilde{\gamma}_t)^{1/p}$ とおくと,

$$\partial_t(PQ_t f(\tilde{\gamma}_t)) \leq \tilde{d}(x, y)\sigma - \frac{1}{p}\sigma^p$$

$$\leq \sup_{u \geq 0} \left[\tilde{d}(x, y)u - \frac{1}{p}u^p \right] = \frac{\tilde{d}(x, y)^q}{q} \quad \blacksquare$$

証明の概観

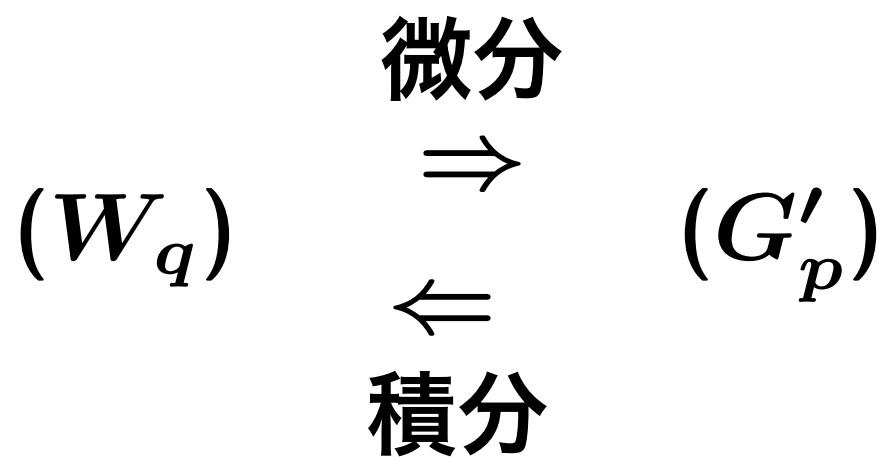
$$\frac{W_{d,q}(P^* \delta_x, P^* \delta_y)^q}{q}$$

Kant. duality $\boxed{=}$ $\sup_f [PQ_1 f(x) - PQ_0 f(y)]$

補間 $\boxed{=}$ $\sup_f \left[\int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t)) dt \right]$

Leibniz 則,
(G'_p), (*) $\boxed{\leq}$ $\frac{\tilde{d}(x, y)^q}{q}$

まとめると、空間変数に関して、



注

$(G'_p) \Rightarrow (W_q)$ の証明では、以下を要請:

(G'_p) が $\forall x \in X$ (or “on a. e. curves”) で成立.

しかし、関数解析的手法では、「 (G_p) a.e.」が普通.

一方、以下が成立すれば (G'_p) は v -a.e. で充分;

(i) $W_{d,p}$ が $\mathcal{P}_p(X)$ 上の測地距離

(ii) $\mu_0, \mu_1 \ll v \Rightarrow \forall \text{ geod. } (\mu_t)_{t \in [0,1]}, \mu_t \ll v$

($p = 2$ & MCP / Sturm-Lott-Villani 条件下で該当)

[cf. Gigli-K.-Ohta '10, Theorem 4.8(ii) \Rightarrow (i)]

応用

[A] 直積不変性

(X_i, d_i) : ポーランド空間, d_i : 測地距離,

\tilde{d}_i : 連続な測地距離関数,

P_i : X_i 上の Markov 核,

$$W_{d_i,2}(P_i^* \mu, P_i^* \nu) \leq W_{\tilde{d}_i,2}(\mu, \nu)$$

↓

$P := P_1 \times P_2$: $X_1 \times X_2$ 上の Markov 核,

$d := \sqrt{d_1^2 + d_2^2}$: 直積距離 (\tilde{d} も同様) で,

$$W_{d,2}(P^* \mu, P^* \nu) \leq W_{\tilde{d},2}(\mu, \nu)$$

証明

$$x^{(j)} = (x_1^{(j)}, x_2^{(j)}) \in X \times X,$$

$\mu = \delta_{x^{(1)}}$, $\nu = \delta_{x^{(2)}}$ で示せば充分

- π_i : minimizer of $W_{d_i, 2}(P^* \delta_{x_i^{(1)}}, P^* \delta_{x_i^{(2)}})$

⇓

$$\pi_1 \times \pi_2 \in \Pi(P^* \delta_{x^{(1)}}, P^* \delta_{x^{(2)}})$$

$$W_{d,2}(P^* \delta_{x^{(1)}}, P^* \delta_{x^{(2)}})^2$$

$$\leq \int_{X_1 \times X_1 \times X_2 \times X_2} d^2 d(\pi_1 \times \pi_2)$$

$$= \|d_1\|_{L^2(\pi_1)}^2 + \|d_2\|_{L^2(\pi_2)}^2$$

仮定 $\leq d_1(x_1^{(1)}, x_1^{(2)})^2 + d_2(x_2^{(1)}, x_2^{(2)})^2$

$$= d(x^{(1)}, x^{(2)})^2 \quad \blacksquare$$

[B] 従属操作

ξ_t : subordinator, i.e. $\xi_t \in \mathcal{P}([0, \infty))$,

$$\int_0^\infty e^{-\lambda s} \xi_t(ds) = e^{-t\phi(\lambda)}$$

P_t : X 上の推移確率族, $K \geq 0$,

$$W_{d,q}(P_t^* \mu, P_t^* \nu) \leq e^{-Kt} W_{d,q}(\mu, \nu)$$

と仮定. このとき $\hat{P}_t = \int_0^\infty P_s \xi_t(ds)$ について,

$$W_{d,q}(\hat{P}_t^* \mu, \hat{P}_t^* \nu) \leq e^{-\phi(qK)t/q} W_{d,q}(\mu, \nu)$$

証明

$\pi_t \in \Pi(P_t^* \mu, P_t^* \nu)$: min. of $W_{d,q}(P_t^* \mu, P_t^* \nu)$

\Downarrow

$$\int_0^\infty \pi_s \xi_t(ds) \in \Pi(\hat{P}_t^* \mu, \hat{P}_t^* \nu)$$

$$W_{d,q}(\hat{P}_t^* \mu, \hat{P}_t^* \nu)^q$$

$$\leq \int_0^\infty \int_{X \times X} d(z, w)^q \pi_s(dzdw) \xi_t(ds)$$

$$\leq \int_0^\infty e^{-qKs} W_{d,q}(\mu, \nu)^q \xi_t(ds)$$

仮定 $\square \equiv e^{-\phi(qK)t} W_{d,q}(\mu, \nu)^q \quad \blacksquare$

[C] 熱半群での, (W_q) or (G'_p) の導出例

- Lie 群上, Hörmander 型作用素が生成する半群

- ▶ (G'_p) ($\forall p > 1$) or (G'_1)

- $((G_p)$ と異質な評価 / 解析的手法で導出)

- $\Rightarrow (W_q)$ ($1 \leq \forall q < \infty$) or (W_∞) [K. '10]

- (コンパクト) Alexandrov 空間

- ▶ (W_2) , $\tilde{d} = e^{-Kt} d$

- (特異空間での評価 / 最適輸送理論と融合)

- $\Rightarrow (G_2)$ [Gigli-K.-Ohta '10]

3. 関連する話題・課題

(1) “微分” について

- Bakry-Émery の元々の議論 (と結果)

$$\Leftarrow |\nabla f| = \sqrt{\Gamma(f, f)}$$

- (W_q) との双対性

$$\Leftarrow |\nabla f|: \text{局所 Lipschitz 定数}$$

Q.

d が intrinsic distance, i.e.

$$d(x, y) = \sup_f \{f(x) - f(y); \Gamma(f, f) \leq 1 \text{ a.e.}\}$$

ならば, 2つの微分は一致するか?

A. [Koskela & Zhou '11]

- 一般には, NO.
- (1,2)-local Poincaré ineq. および volume doubling property の下,

[Newtonian property] \Leftrightarrow YES.

(2) $(G_p) \Rightarrow (G_{p'})$ for $p > p'$

- $p' < p$ のとき, $(G_{p'}) \Rightarrow (G_p)$

Q.

逆はいつ成り立つか？

A.

- Bakry-Émery's argument:

$$(G_2) \Rightarrow (\Gamma_2(K)) \Rightarrow (G_1)$$

- [von Renesse & Sturm '05]

X : 完備 Riemann 多様体, P_t : 熱半群なら OK:

$$(G_\infty) \Rightarrow \text{Ric} \geq K \Rightarrow (W_\infty) \Rightarrow (G_1)$$

- [Wang '05]

一般には, $(G_2) \not\Rightarrow (G_1)$

(\mathbb{R}^m 上の SDE の解で反例が構成可?)

(3) 異質な評価

X : Lie 群

$\{X_i\}_{i=1}^n$: 左不変線型独立ベクトル場,

Hörmander 条件成立

$$\bullet P_t := e^{t\mathcal{L}}, \quad \mathcal{L} := \sum_{i=1}^n X_i^2$$

$$\bullet \Gamma(f, f) = \sum_{i=1}^n |X_i f|^2, \quad |\nabla f| = \sqrt{\Gamma(f, f)}$$

Bakry-Émery 型 (L^p -) 微分評価

$$|\nabla P_t f|(x) \leq C_p(t) P_t(|\nabla f|^p)(x)^{1/p} \quad (\tilde{G}_p)$$

既知の結果

- 3-dim. Heisenberg 群, $C_p(t) \equiv C_p > 1$
 - $p > 1$: [Driver & Melcher '05]
 - $p = 1$: [H.-Q. Li '06] /
[Bakry, Baudoin, Bonnefont & Chafaï '08]
- X : 一般, $p > 1$: [Melcher '08]
(X : 冪零 $\Rightarrow C_q(t) \equiv C_p$)
- X : group of type H, $p = 1$, $C_p(t) \equiv C_p$:
[Eldredge '10]
- $X = SU(2)$, $p > 1$, $C_p(t) = C_p e^{-t}$:
[Baudoin & Bonnefont '09]

注

- ★ \mathcal{L} に付随する距離 (Carnot-Carathéodory 距離) について,

$$\begin{aligned} & (\tilde{G}_p) \\ & \Updownarrow \\ & W_{d,q}(P_t^* \mu, P_t^* \nu) \leq C_p(t) W_{d,q}(\mu, \nu) \\ & \text{[K.'10]} \end{aligned}$$

- !! 定数が $C_p(t) = C_p e^{-Kt}$ with $C_p > 1$
 \Rightarrow 「 (\tilde{G}_p) \Rightarrow “ Γ_2 -条件”」にはならない
微分

Q.

\tilde{G}_p の (確認が容易な) 十分条件は？

- 結合法は困難 (Heisenberg 群の場合でも未知)
 - [Kendall '09]
Heisenberg 群上, 違う種類のカップリング
- 結合法による (W_q) の証明自体, 興味の対象
- [Baudoin, Garofalo, Bonnefont]
 Γ_2 -条件の一般化
ただし, (\tilde{G}_p) は導出できていない