Coupling of Brownian motions associated with a time-dependent metric

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1. Introduction

M: m-dim. manifold, $0 \leq T_1 < T_2 \leq \infty$,
$(g(t))_{t \in [T_1, T_2]}$: smooth complete Riemannian metrics on M.

<u>Examples</u>

- $\partial_t g(t) = 2\,\mathrm{Ric}_{g(t)}$ (backward Ricci flow)
- $\partial_t g(t) \leq 2\,\mathrm{Ric}_{g(t)} - 2K g(t)$ $\quad (*)$
  (time-dependent extension of "$\mathrm{Ric} \geq K$")
- Given g_0 with $\mathrm{Ric}_{g_0} \geq 0$,
  $g(t) := \mathrm{e}^{-2K(t-T_1)} g_0$ satisfies $(*)$

<u>Q.</u> Does $(*)$ work as an analogue of $\mathrm{Ric} \geq K$?

$(X_t)_{t\in[T_1,T_2]}$: $g(t)$-Brownian motion on M, i.e.

$$P_{T_1\to t}f(x) := \mathbb{E}[f(X_t)|X_{T_1} = x]$$

solves the heat equation

$$\begin{cases} \partial_t u = \Delta_{g(t)} u, & (t > T_1), \\ u(T_1, \cdot) = f \end{cases}$$

[Arnaudon, Coulibaly & Thalmaier '08/Coulibaly]: constructions via SDE on the frame bundle

Rem X_t cannot explode under $(*)$ [K.-Philipowski]

Coupling:

A pair of particles (X_t, Y_t) each of which moves as an $g(t)$-Brownian motion on M

- ⋆ Behavior of Y_t can depend on that of X_t

  $\rightsquigarrow$ Useful for studying variation of the sol. to the heat eq.

- ⋆ When $\partial_t g(t) \equiv 0$,

  $\mathrm{Ric} \geq K$ yields "nice" couplings

What we will consider:

(i) Coupling by parallel transport

(ii) Coupling by reflection

(iii) Coupling by spacetime parallel transport

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Goal

A nice control of "distance" between X_t and Y_t

$\Rightarrow$ (Sharper) monotonicity of transportation costs

$\Rightarrow$ Functional inequalities (gradient estimates)

2. Coupling by parallel transport

Transportation cost

$c : M \times M \to \mathbb{R}$: cost function

($c(x, y)$: cost of bringing a unit mass from x to y)

For $\mu, \nu \in \mathcal{P}(M)$,

$$\Pi(\mu, \nu) := \left\{ \pi \;\middle|\; \begin{array}{l} \pi(E \times M) = \mu(E), \\ \pi(M \times E) = \nu(E) \end{array} \right\}$$

(set of all couplings between μ and ν),

$$\mathcal{T}_c(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{M \times M} c(x, y) \pi(dxdy)$$

(Minimal total transportation cost from μ to ν)

Equivalent conditions for $\mathrm{Ric} \geq K$ when $\partial_t g(t) \equiv 0$, $T_1 = 0$

[von Renesse & Sturm '05, etc.]

$P_t := P_{0\to t} = \mathrm{e}^{t\Delta}$: the heat semigroup

(i) $\mathrm{Ric} \geq K$

(ii) $\mathcal{T}_{(\mathrm{e}^{Kt} d)^p}(P_t^* \mu, P_t^* \nu) \searrow$ for some $p \in [1, \infty]$

(iii) $|\nabla P_t f|(x)^q \leq \mathrm{e}^{-qKt} P_t(|\nabla f|^q)(x)$ for some $q \in [1, \infty]$

The case $\partial_t g(t) \not\equiv 0$

[McCann & Topping '10]:

Suppose M: cpt.

$\Rightarrow$ The following are equivalent:

(i) $\partial_t g(t) \le 2\,\mathrm{Ric}_{g(t)}$

(ii) $\forall \mu_t, \nu_t$: heat distributions, $\mathcal{T}_{d^2_{g(t)}}(\mu_t, \nu_t) \searrow$

(iii) $\mathrm{Lip}_{g(s)}(P_{s \to t} f) \le \mathrm{Lip}_{g(t)}(f)$

⋆ They used techniques in optimal transport

An extension via stochastic analysis

[Arnaudon & Coulibaly & Thalmaier '09]:

Suppose

$$\partial_t g(t) \le 2\,\mathrm{Ric}_{g(t)} - 2K g(t)$$

$\Rightarrow$ For $\forall \mu, \nu \in \mathcal{P}(M)$, $\forall \pi \in \Pi(\mu, \nu)$,

$\exists$ (X_t, Y_t): coupled $g(t)$-BMs

with the initial distribution π s.t.

$$\mathrm{e}^{K(t-T_1)} d_{g(t)}(X_t, Y_t) \searrow \quad \text{a.s.}$$

Rem

Heuristically,

$$\partial_t \Big(d_{g(t)}(X_t, Y_t) \Big) \leq -K$$

$$\Downarrow$$

$$e^{K(t-T_1)} d_{g(t)}(X_t, Y_t) \searrow$$

Why ACT is an extension of MT?

$\star$ $\mathrm{e}^{K(t-T_1)} d_{g(t)}(X_t, Y_t) \searrow$

$$\Rightarrow \boxed{\forall \varphi \nearrow, \mathcal{T}_{\varphi(\mathrm{e}^{K(t-T_1)} d_{g(t)})}(\mu_t, \nu_t) \searrow}$$

Why ACT is an extension of MT?

$\star\ \mathrm{e}^{K(t-T_1)} d_{g(t)}(X_t, Y_t) \searrow$

$\Rightarrow$ $\forall \varphi \nearrow, \mathcal{T}_{\varphi(\mathrm{e}^{K(t-T_1)} d_{g(t)})}(\mu_t, \nu_t) \searrow$

$\because$ Let π be a minimizer of $\mathcal{T}_{\varphi(d_{T_1})}(\mu_{T_1}, \nu_{T_1})$

$$\Downarrow$$

$$\begin{aligned}
&\mathcal{T}_{\varphi(\mathrm{e}^{K(t-T_1)} d_{g(t)})}(\mu_t, \nu_t) \\
&\quad \le \mathbb{E}[\varphi(\mathrm{e}^{K(t-T_1)} d_{g(t)}(X_t, Y_t))] \\
&\quad \le \mathbb{E}[\varphi(d_{g(T_1)}(X_{T_1}, Y_{T_1}))] \\
&\quad = \mathcal{T}_{\varphi(d_{g(T_1)})}(\mu_{T_1}, \nu_{T_1})
\end{aligned}$$

Bakry-Émery's gradient estimate

⋆ $\mathrm{e}^{K(t-T_1)} d_{g(t)}(X_t, Y_t) \searrow$

$$\Rightarrow \ |\nabla P_{s\to t} f|_{g(s)}(x) \leq \mathrm{e}^{-K(t-s)} P_{s\to t}(|\nabla f|_{g(t)})(x)$$

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$$\Rightarrow |\nabla P_{s\to t} f|_{g(s)}(x) \le \mathrm{e}^{-K(t-s)} P_{s\to t}(|\nabla f|_{g(t)})(x)$$

$\because$ Let $\mu_{T_1} = \delta_x$ and $\nu_{T_1} = \delta_y$

$$|P_{s\to t} f(x) - P_{s\to t} f(y)| \le \mathbb{E}[|f(X_t) - f(Y_t)|]$$

$$\le \mathbb{E}\left[\left|\frac{f(X_t) - f(Y_t)}{d_{g(t)}(X_t, Y_t)}\right|\right] \mathrm{e}^{-K(t-T_1)} d_{g(T_1)}(x, y)$$

3. Coupling by reflection

Thm 1 [K.]

Suppose $\exists K \in \mathbb{R}$ s.t.

$$\partial_t g(t) \leq 2\,\mathrm{Ric}_{g(t)} - 2K g(t)$$

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$$\partial_t g(t) \leq 2\,\mathrm{Ric}_{g(t)} - 2K g(t)$$

$\Rightarrow \forall x, y \in M$, $\exists (X_t, Y_t)$: Markovian coupling of $g(t)$-BMs from (x, y) s.t.

$$\mathbb{P}\Big[\inf_{T_1 \leq s \leq t} d_{g(s)}(X_s, Y_s) > 0\Big] \leq \mathbb{P}\Big[\inf_{T_1 \leq s \leq t} \rho_s > 0\Big]$$

where ρ_t solves $\rho_{T_1} = d_{g(T_1)}(x, y)$ and

$$d\rho_t = 2\sqrt{2}d\beta_t - K\rho_t dt$$

What is ρ_t?

- Heuristically, "$d_{g(t)}(X_t, Y_t) \leq \rho_t$" $\Rightarrow$ Thm 1
- ρ_t can be realized as $\rho_t := |\hat{X}_t - \hat{Y}_t|$, where $\hat{X}_t$ solves $\hat{X}_{T_1} = d_{g(T_1)}(x, y)/2$ and

$$d\hat{X}_t = \sqrt{2} d\beta_t - K\hat{X}_t dt$$

(Ornstein-Uhlenbeck process)

and $\hat{Y}_t = -\hat{X}_t$

($(\hat{X}_t, \hat{Y}_t)$: coupling by reflection of O-U proc.)

Rem (Coalescence)

- We may assume $Y_t = X_t$ when $t > \inf\{s > T_1 \mid X_s = Y_s\}$
- The same is true for $(\hat{X}_t, \hat{Y}_t)$

Rem (Maximality)

$$\mathbb{P}\Big[\inf_{T_1 \le s \le t} |\hat{X}_s - \hat{Y}_s| > 0\Big] = \varphi_{t-T_1}(|\hat{X}_{T_1} - \hat{Y}_{T_1}|)$$

where

$$\varphi_{s-T_1}(a) := \|\mathbb{P}_{a/2}\hat{X}_s^{-1} - \mathbb{P}_{-a/2}\hat{X}_s^{-1}\|_{\mathrm{var}}$$

$$\left(\begin{aligned} \varphi_s(a) &= \sqrt{\frac{2}{\pi}} \int_0^{\frac{a}{2\sqrt{2}\sqrt{\beta(s)}}} \mathrm{e}^{-x^2/2} dx, \\ \beta(s) &= \frac{\mathrm{e}^{Ks} - 1}{K} \end{aligned}\right)$$

$$\star\ \mathbb{E}[\varphi_s(|\hat{X}_t - \hat{Y}_t|)] = \varphi_{s+t-T_1}(|\hat{X}_{T_1} - \hat{Y}_{T_1}|)$$

Thm 2 [K. & Sturm]

Let $T_1 < T \leq T_2$. $\forall \mu_t, \nu_t$: heat distributions,

$$\mathcal{T}_{\varphi_{T-t} \circ d_{g(t)}}(\mu_t, \nu_t) \searrow$$

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$\because$ Suppose $\mu_{T_1} = \delta_x$, $\nu_{T_1} = \delta_y$

$d_{g(t)}(X_t, Y_t) \le |\hat{X}_t - \hat{Y}_t|$ & $\varphi_{T-t} \nearrow$

$$\Downarrow$$

$$\begin{aligned} \mathbb{E}[\varphi_{T-t}(d_{g(t)}(X_t, Y_t))] &\le \mathbb{E}[\varphi_{T-t}(|\hat{X}_t - \hat{Y}_t|)] \\ &\le \mathbb{E}[\varphi_{T-T_1}(|\hat{X}_{T_1} - \hat{Y}_{T_1}|)] \\ &= \mathcal{T}_{\varphi_{T-T_1} \circ d_{g(T_1)}}(\delta_x, \delta_y) \end{aligned}$$

Cor 1 [K.]

$$\left\| |\nabla P_{T_1 \to t} f|_{g(T_1)} \right\|_\infty \le \frac{1}{2\sqrt{\pi\beta(t - T_1)}} \operatorname{osc} f$$

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$$\left\| |\nabla P_{T_1 \to t} f|_{g(T_1)} \right\|_\infty \leq \frac{1}{2\sqrt{\pi\beta(t - T_1)}} \operatorname{osc} f$$

$$\because \lim_{y \to x} \frac{P_{T_1 \to t} f(x) - P_{T_1 \to t} f(y)}{d_{g(T_1)}(x, y)}$$

$$= \lim_{y \to x} \frac{\mathbb{E}[f(X_t) - f(Y_t)]}{d_{g(T_1)}(x, y)}$$

$$= \operatorname{osc} f \lim_{y \to x} \frac{\mathbb{P}[\inf_{T_1 \leq s \leq t} d_{g(s)}(X_s, Y_s) > 0]}{d_{g(T_1)}(x, y)}$$

$$\leq \varphi'_{t-T_1}(0) \operatorname{osc} f$$

4. Sketch of the proof of Thm 1

(Suppose $T_1 = 0$ here, for simplicity)

4.1. Heuristic idea

Construct (X_t, Y_t)

where $dY_t =$ "(local) reflection" of dX_t

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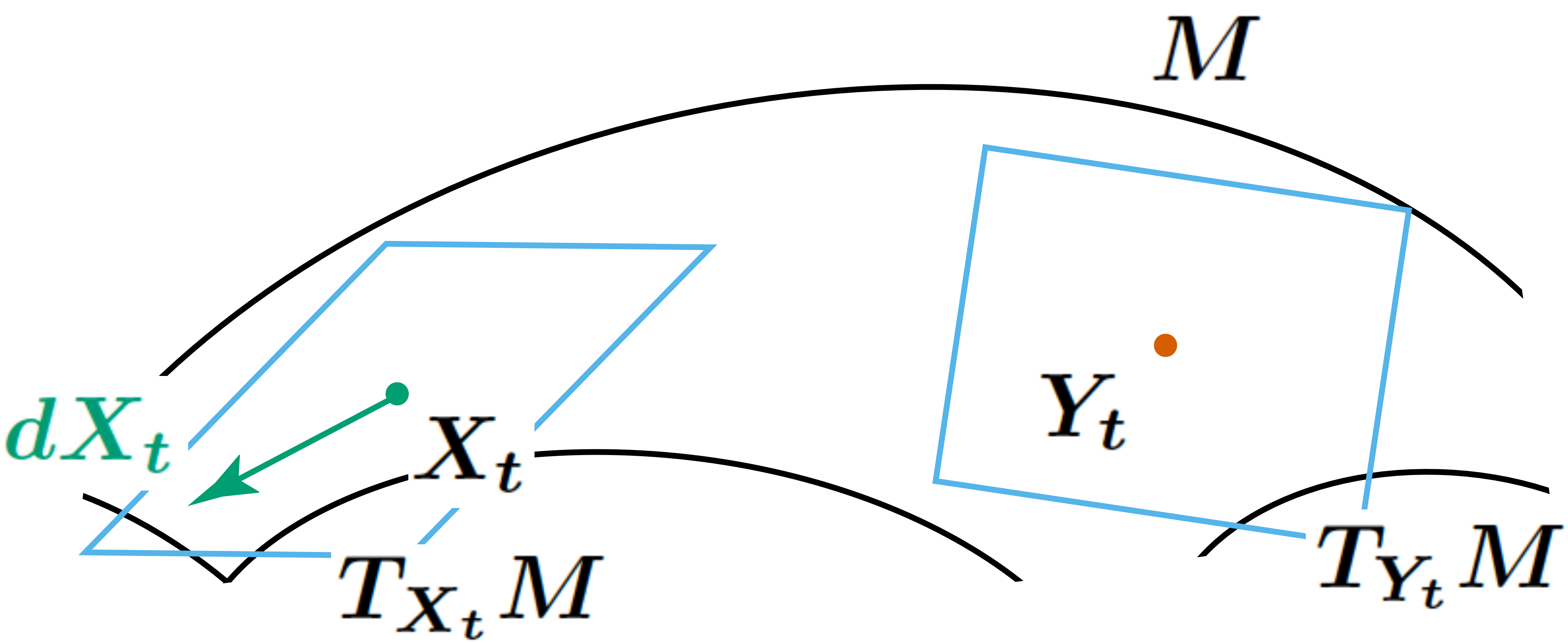

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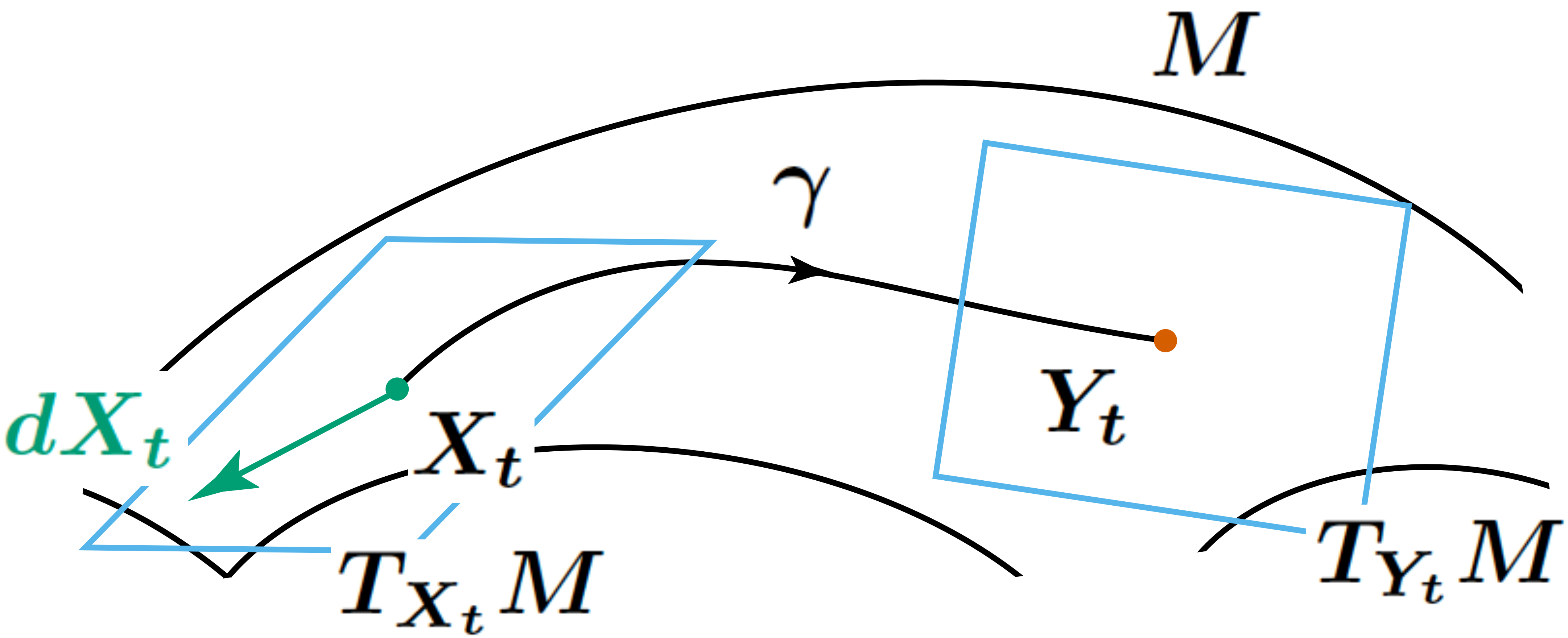

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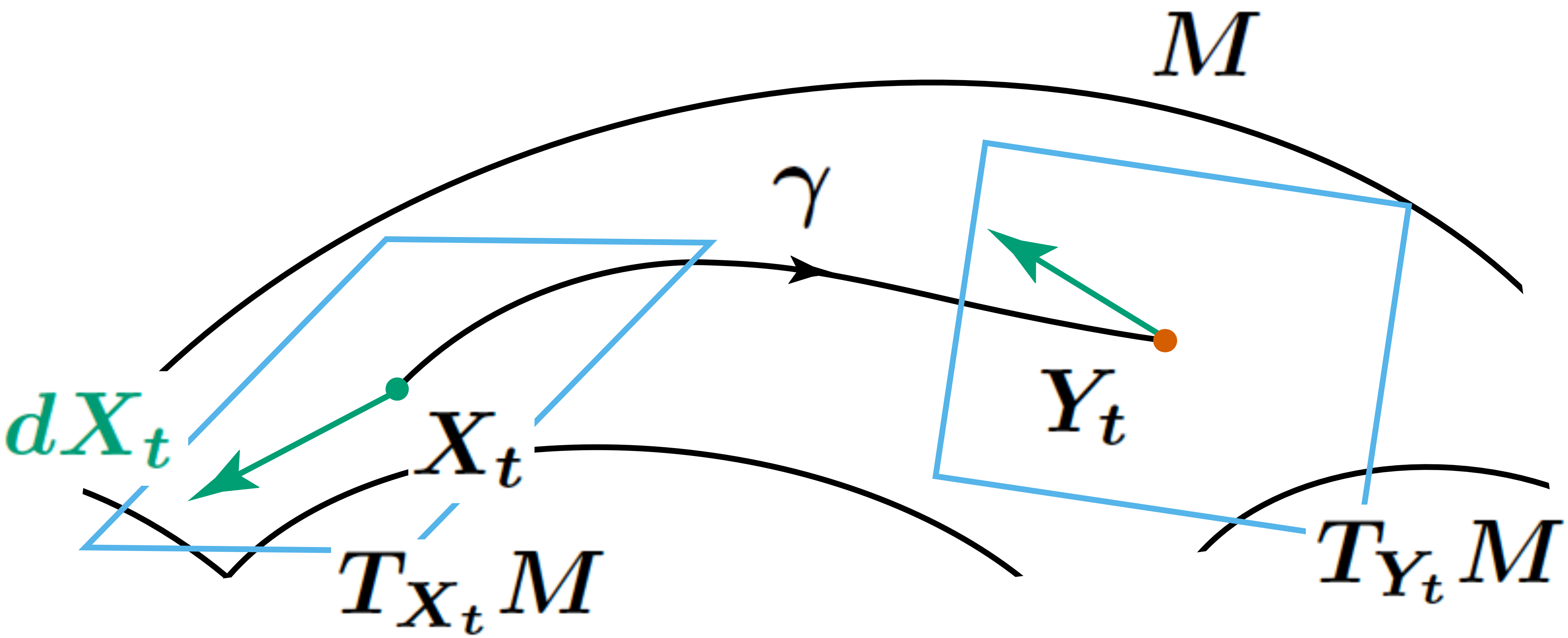

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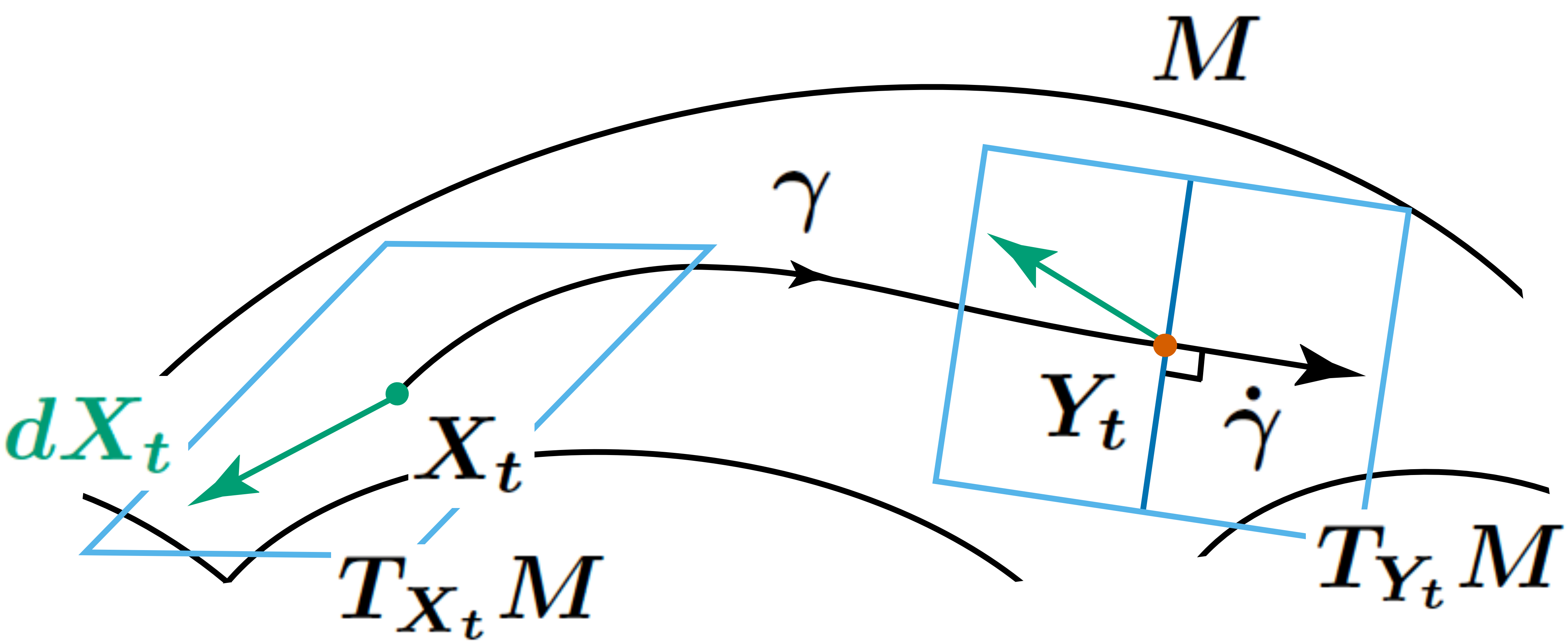

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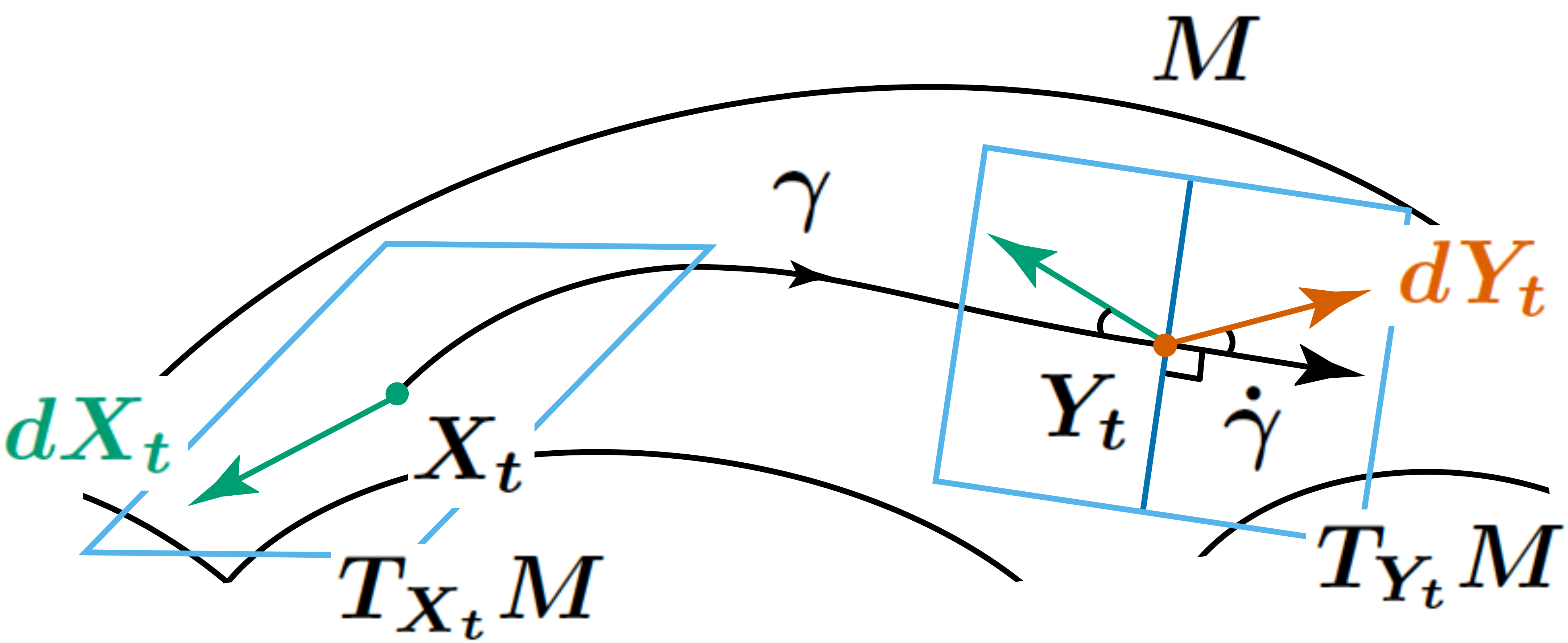

$$\sigma_t := d_{g(t)}(X_t, Y_t)$$

$\Downarrow$ the "Itô formula"

$$
\begin{aligned}
d\sigma_t = & \sqrt{2}\langle \nabla d_{g(t)}(X_t, Y_t), dX_t \otimes dY_t \rangle \\
& + (\partial_t d_{g(t)})(X_t, Y_t)dt \\
& + \sum_{i=2}^{m} \langle \mathrm{Hess}\, d_{g(t)}(X_t, Y_t), e_i \otimes /\!/_\gamma e_i \rangle dt \\
& - L_t \\
= & \text{(1st)} + \text{(T)} + \text{(2nd)} + \text{(CL)}
\end{aligned}
$$

($(e_i)_{i=1}^m$: ONB of $T_{X_t}M$, $e_1 = \dot{\gamma}$)

$$d\sigma_t = (\mathsf{1st}) + (\mathsf{T}) + (\mathsf{2nd}) + (\mathsf{CL})$$

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$$\Downarrow$$

1st order:

$$\sqrt{2}\langle \nabla d_{g(t)}(X_t, Y_t), dX_t \otimes dY_t \rangle_{g(t)}$$

$$= \sqrt{2}\left(\langle \dot{\gamma}, dY_t \rangle - \langle \dot{\gamma}, dX_t \rangle\right)$$

$$= -2\sqrt{2}\langle \dot{\gamma}, dX_t \rangle =: 2\sqrt{2}\beta_t$$

($\Rightarrow$ β_t: 1-dim. BM)

Recall: $d\rho_t = 2\sqrt{2}d\beta_t - K\rho_t dt$

$$d\sigma_t = \text{(1st)} + \text{(T)} + \text{(2nd)} + \text{(CL)}$$

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$\Downarrow$

(T):

$$\partial_t d_{g(t)}(X_t, Y_t) = \frac{1}{2}\int_\gamma \partial_t g(t)(\dot\gamma(s), \dot\gamma(s))ds$$

Recall: $d\rho_t = 2\sqrt{2}d\beta_t - K\rho_t dt$

$d\sigma_t =$ (1st) + (T) + (2nd) + (CL)

$\Downarrow$

<u>2nd order</u>:

$$\sum_{i=2}^{m} \langle \mathrm{Hess}\, d_{g(t)}(X_t, Y_t), e_i \otimes /\!/_{\gamma} e_i \rangle$$

$$\leq - \int_{\gamma} \mathrm{Ric}_{g(t)}(\dot{\gamma}(s), \dot{\gamma}(s))ds$$

by the index lemma

Recall: $d\rho_t = 2\sqrt{2}d\beta_t - K\rho_t dt$

$$d\sigma_t = \text{(1st)} + \text{(T)} + \text{(2nd)} + \text{(CL)}$$

$$\Downarrow$$

$\Rightarrow$ (T) + (2nd)

$$\leq \int_\gamma \left(\partial_t g(t) - \frac{1}{2}\mathrm{Ric}_{g(t)}\right)(\dot\gamma(s), \dot\gamma(s))\,ds\,dt$$

$$\leq -K\sigma_t dt$$

by our assumption

Recall: $d\rho_t = 2\sqrt{2}d\beta_t - K\rho_t dt$

$$d\sigma_t = \text{(1st)} + \text{(T)} + \text{(2nd)} + \text{(CL)}$$

$$\Downarrow$$

(CL) $(= -L_t)$:

L_t comes from the effect of singularity of $d_{g(t)}$ at $g(t)$-cut locus

$\Rightarrow L_t \geq 0$

Recall: $d\rho_t = 2\sqrt{2}d\beta_t - K\rho_t dt$

$$d\sigma_t = \text{(1st)} + \text{(T)} + \text{(2nd)} + \text{(CL)}$$

$$\Downarrow$$

$$d\sigma_t \leq 2\sqrt{2}\beta_t - K\sigma_t dt$$

$$\Downarrow$$

$$\sigma_t \leq \rho_t$$

by a comparison argument

4.2. Toward a rigorous argument

Difficulty:
$$\begin{cases} \text{Solvability of SDE} \\ \text{Extraction of } L_t \\ \{t \mid (X_t, Y_t) \in \mathrm{Cut}_{g(t)}\}\text{: small?} \end{cases}$$

$$\left(\begin{array}{l} \text{When } \partial_t g(t) \equiv 0, \\ \text{[Kendall '86, Cranston '91, F.-Y. Wang '94/'05]} \end{array} \right)$$

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Our method: Random walk approximation

Show "$\sigma_t \le \rho_t$" in a weak sense

without extracting L_t

(When $\partial_t g(t) \equiv 0$, [von Renesse '04, K.'10])

Approximation by coupled RWs $(X_t^\varepsilon, Y_t^\varepsilon)$

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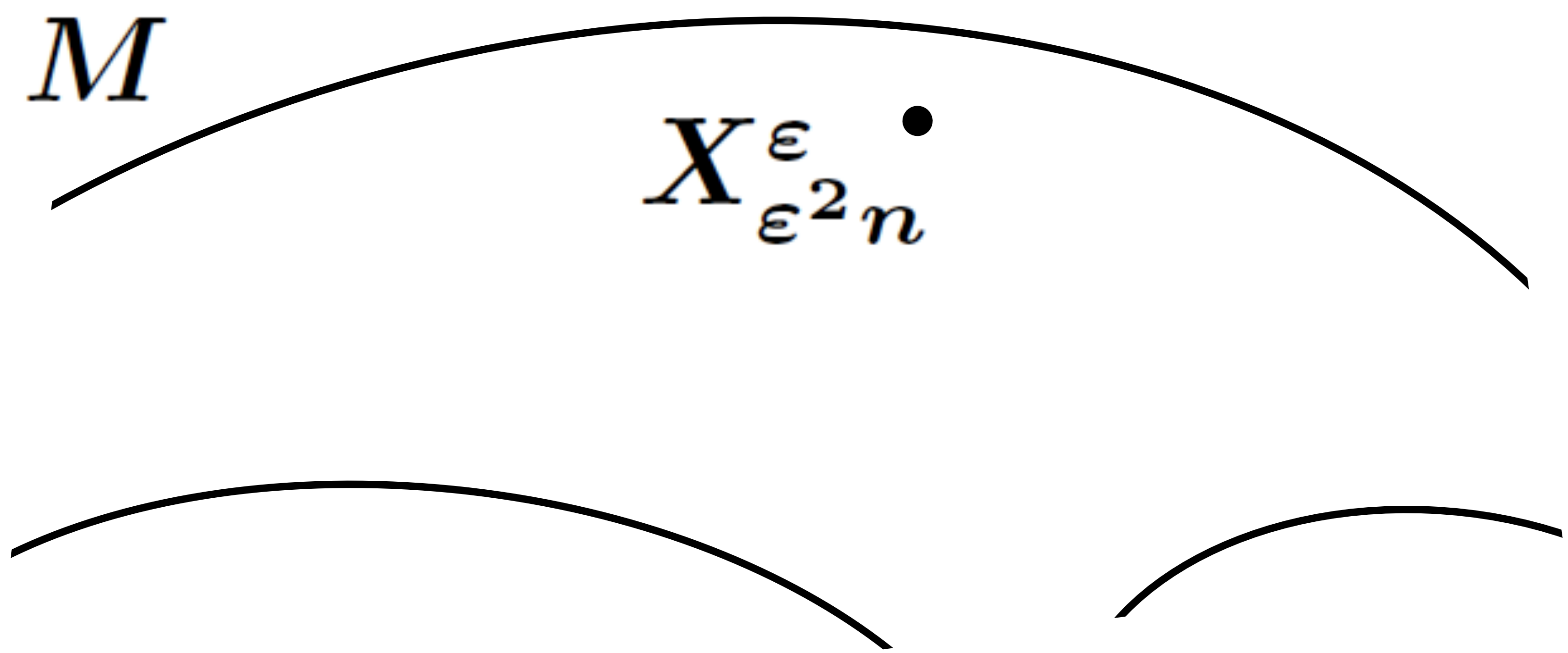

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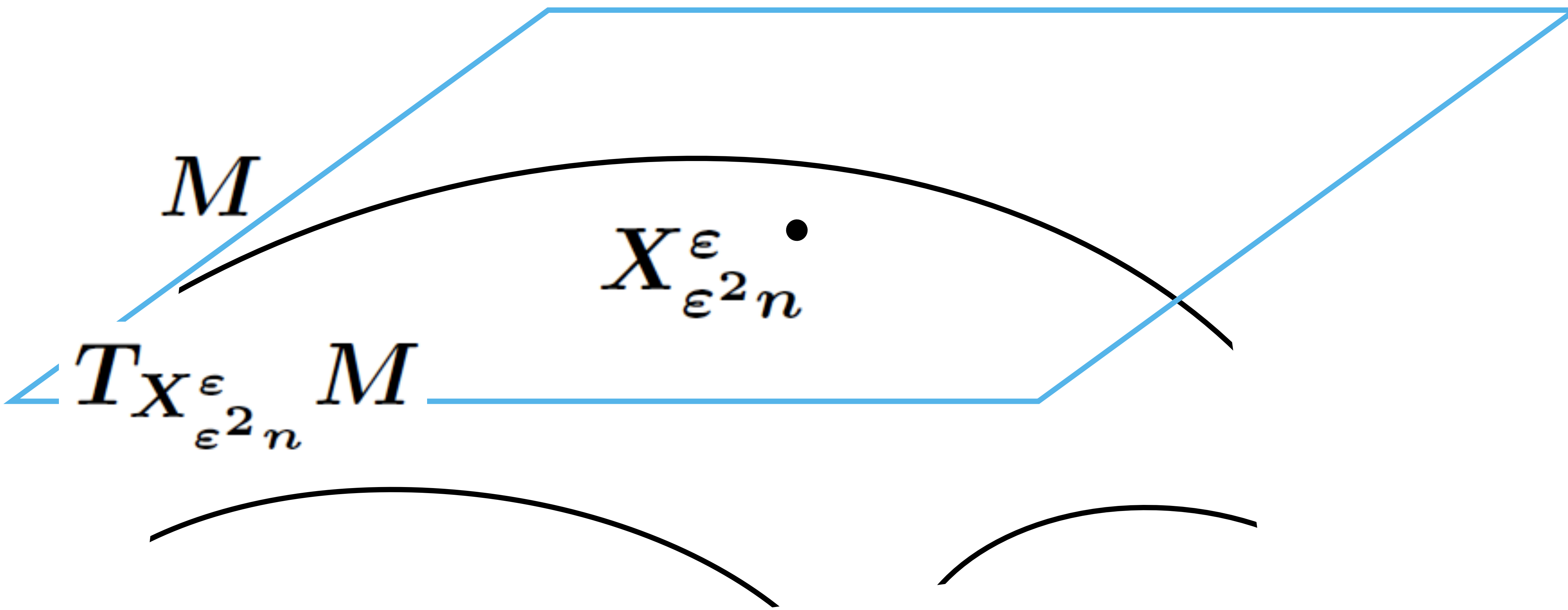

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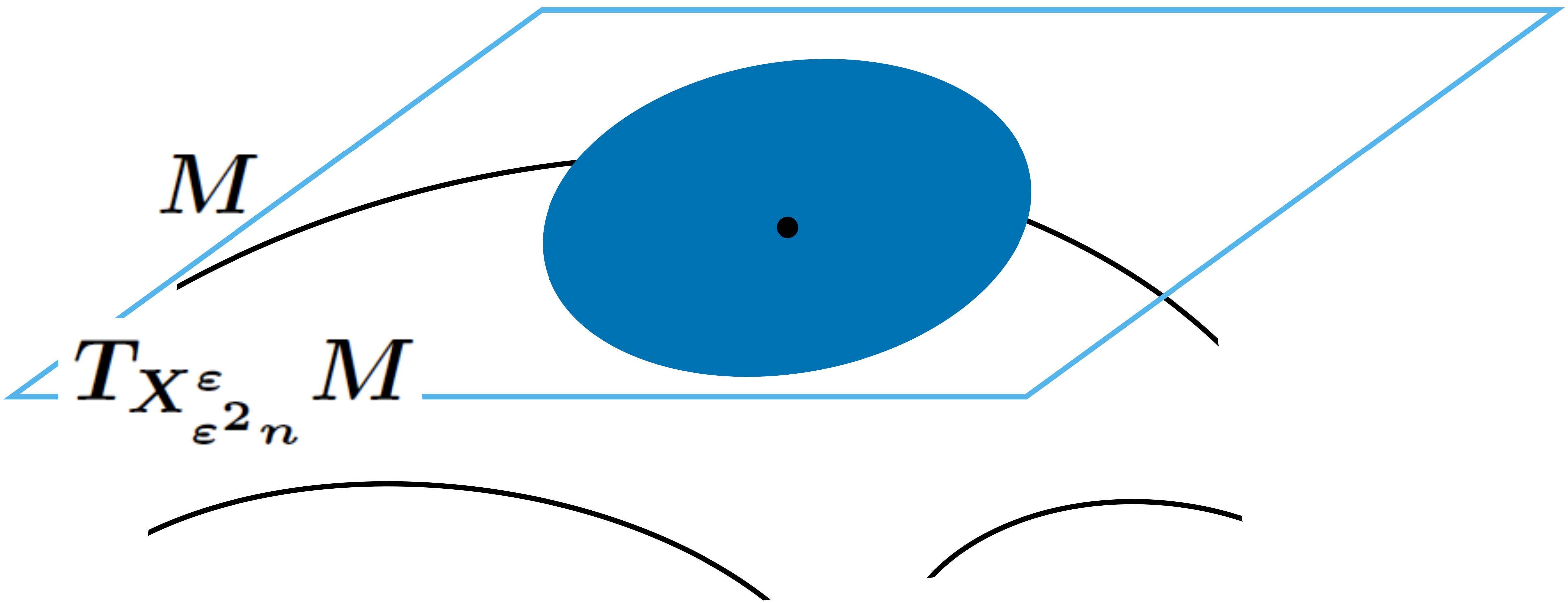

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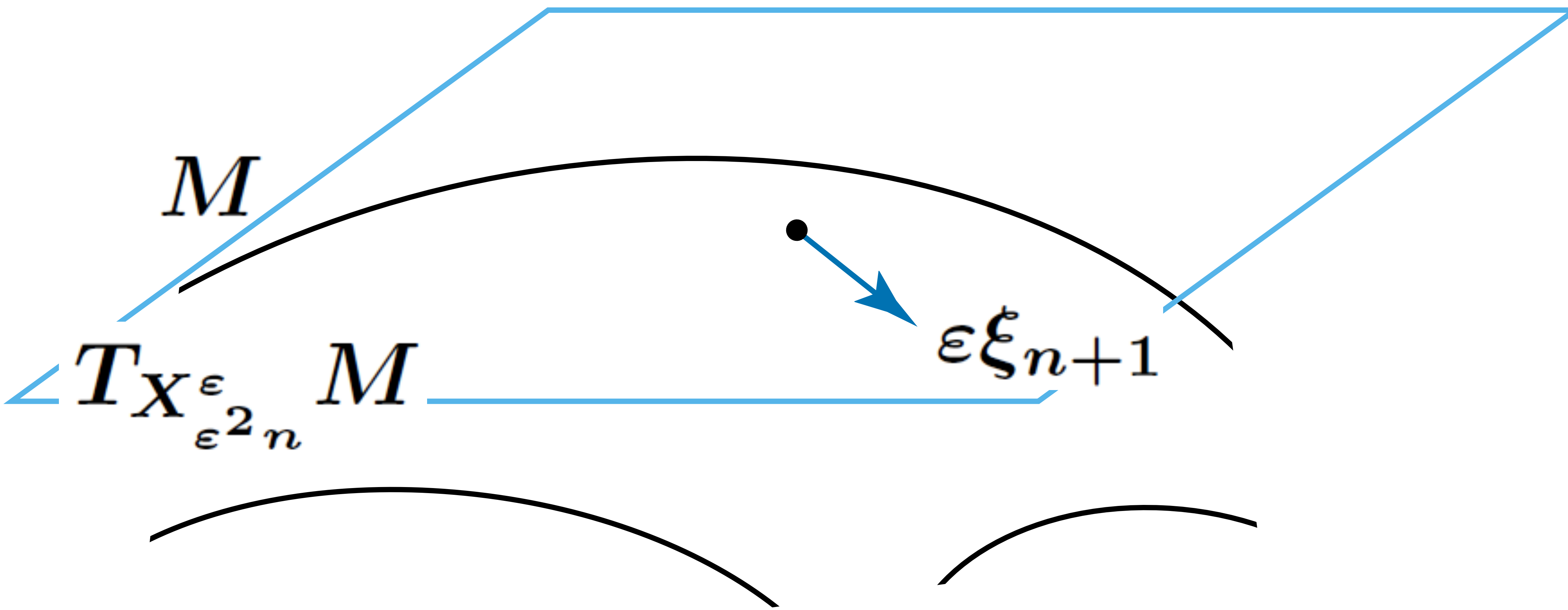

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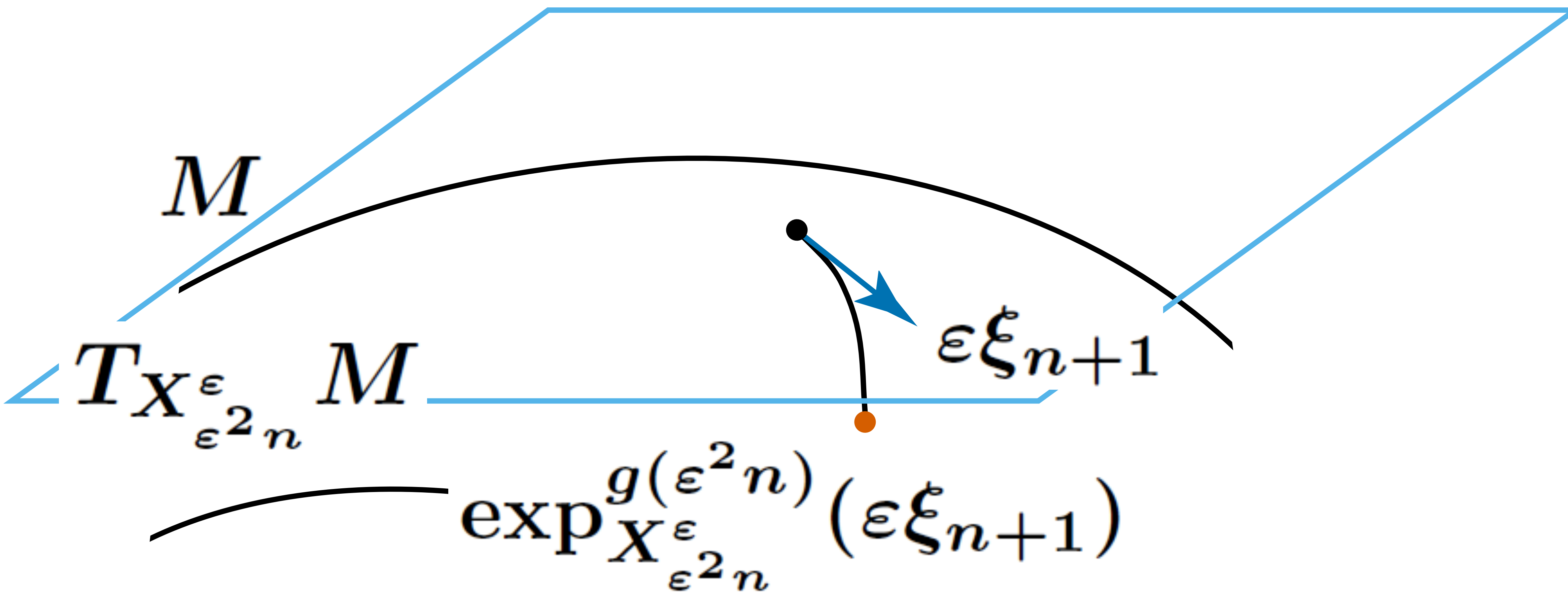

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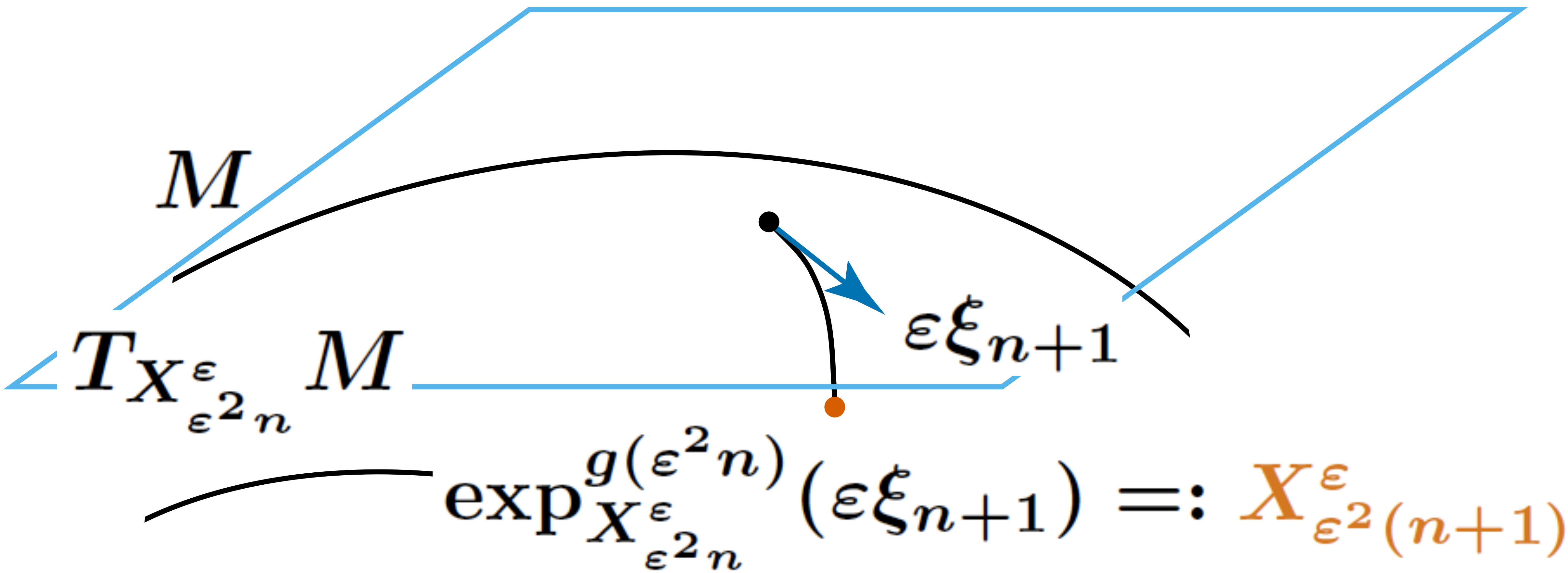

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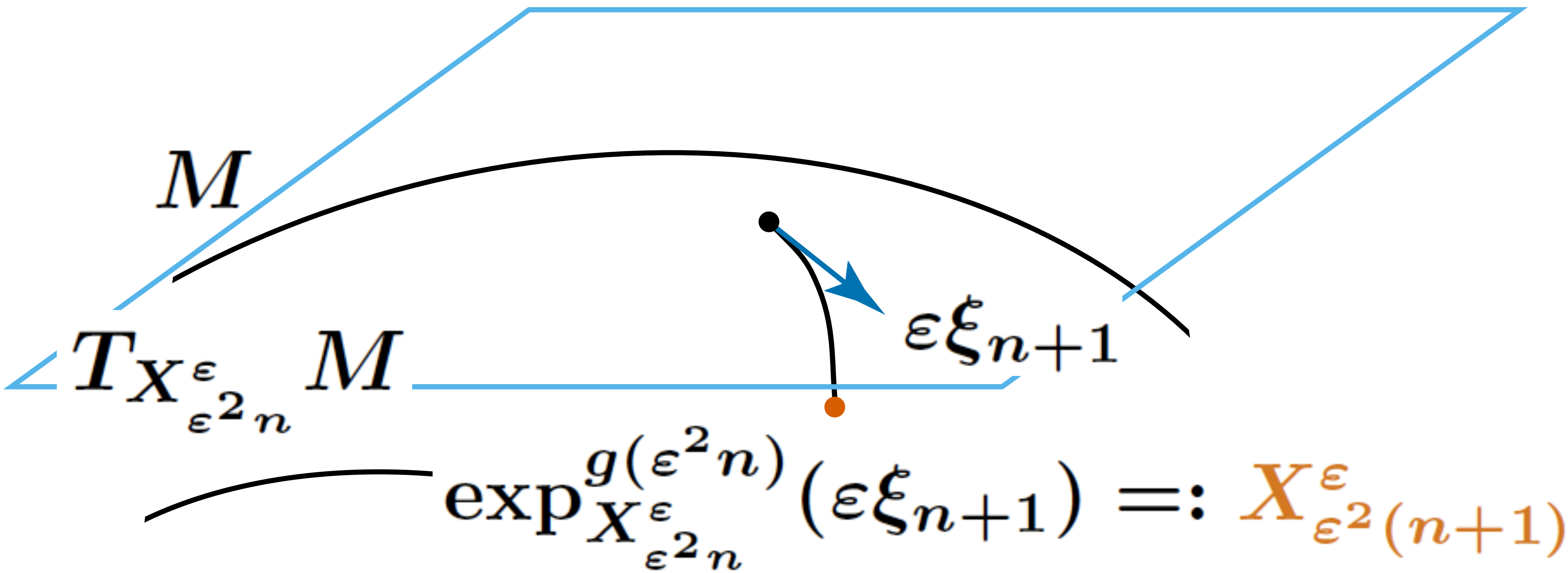

Thm 3 [K.] (invariance principle)

$X^\varepsilon \xrightarrow{d} X$ as $\varepsilon \to 0$.

Why does it works?

Discrete Itô formula (Taylor expansion):

$$\boldsymbol{\sigma}^{\varepsilon}_{\varepsilon^2(n+1)} = \boldsymbol{\sigma}^{\varepsilon}_{\varepsilon^2 n} + \varepsilon \boldsymbol{\lambda}^{\varepsilon}_{n+1} + \varepsilon^2 \boldsymbol{\Lambda}^{\varepsilon}_{n+1} + o(\varepsilon^2)$$

When $(X^{\varepsilon}_{\varepsilon^2 n}, Y^{\varepsilon}_{\varepsilon^2 n}) \in \mathrm{Cut}_{g(\varepsilon^2 n)}$,

Dividing a min. geod. into two pieces:

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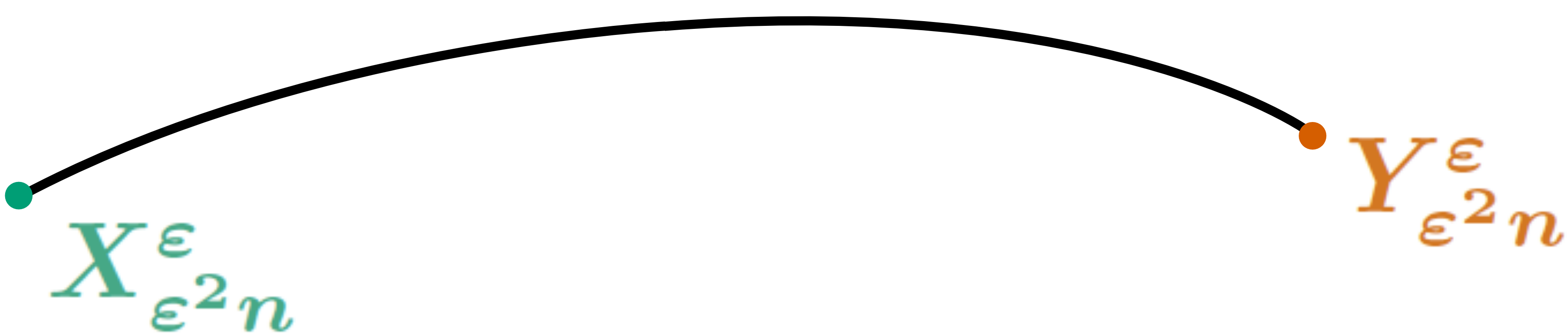

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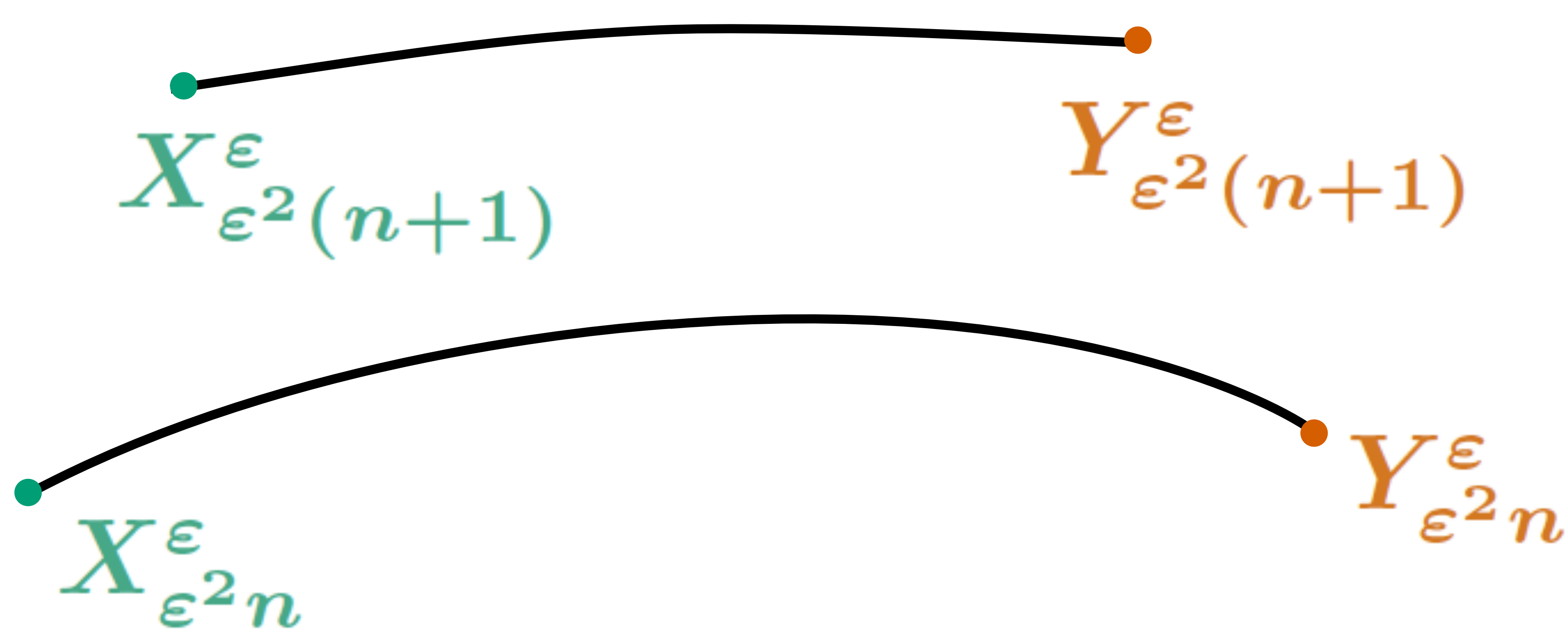

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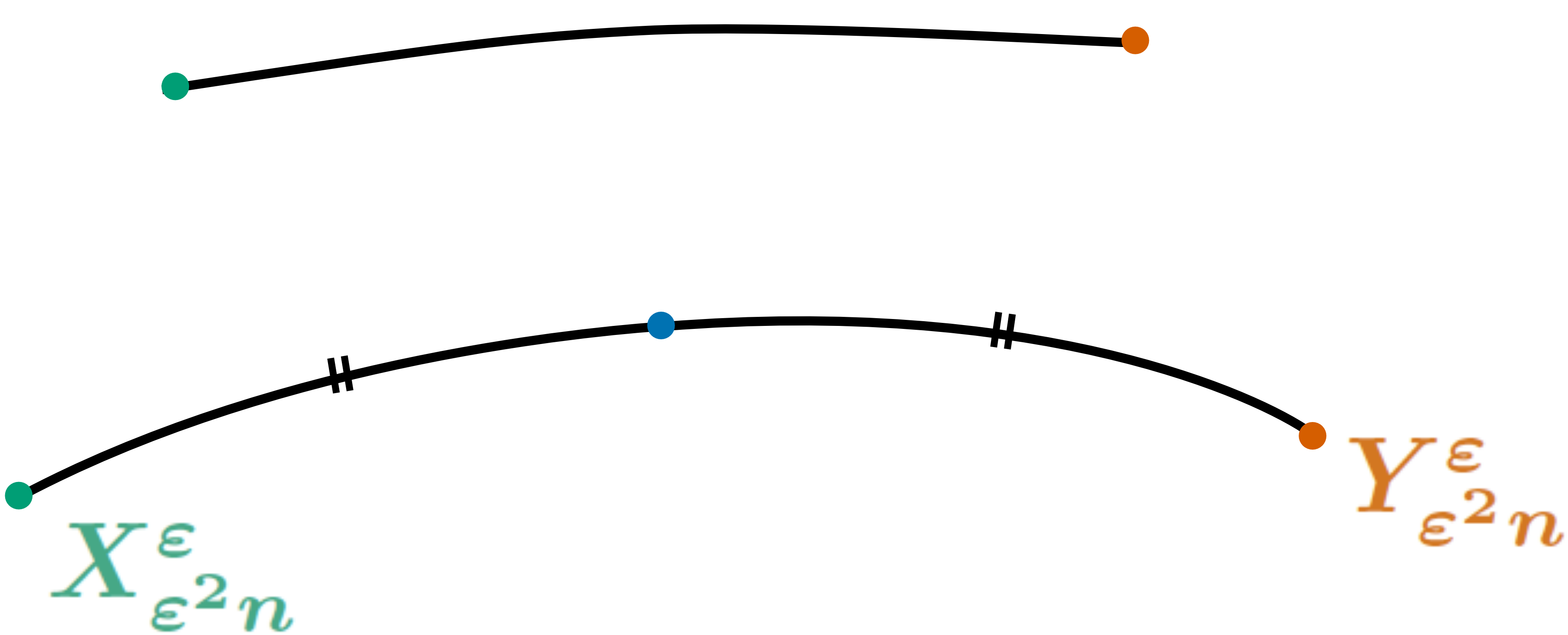

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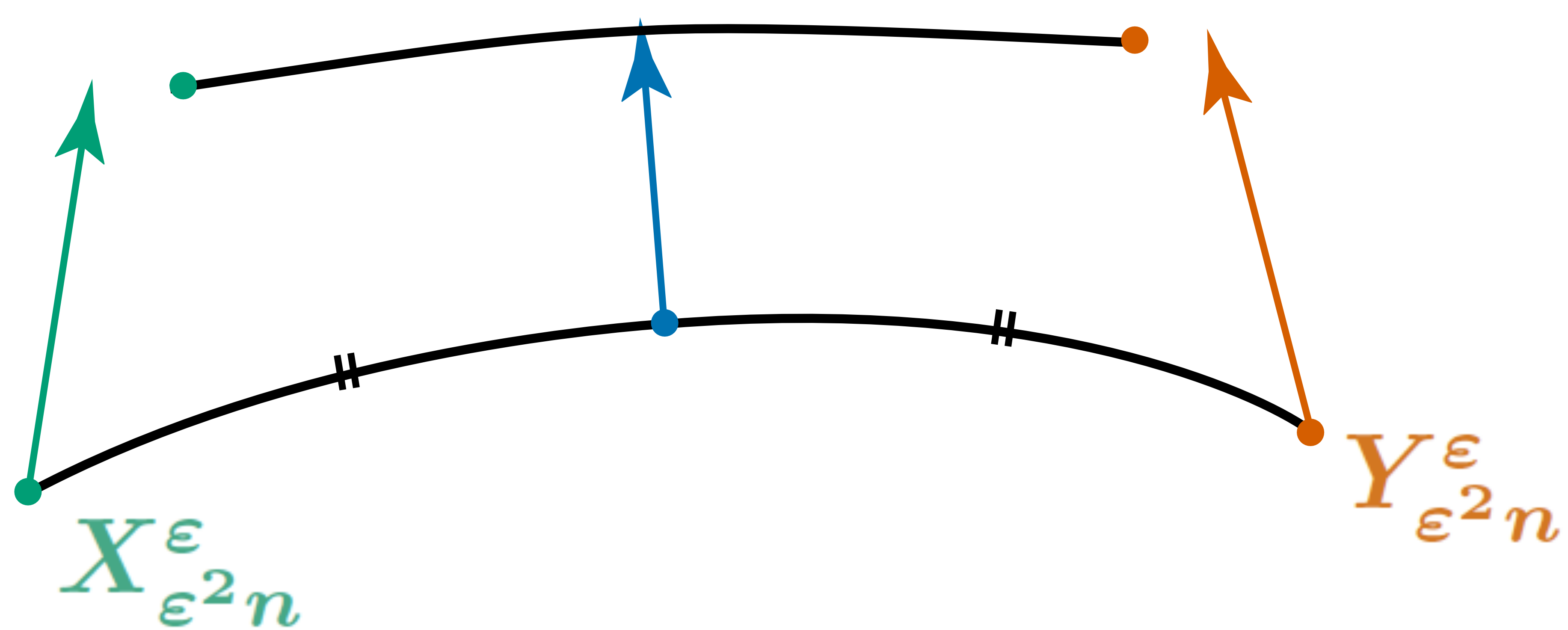

Why does it works?

Discrete Itô formula (Taylor expansion):

$$\sigma^{\varepsilon}_{\varepsilon^2(n+1)} = \sigma^{\varepsilon}_{\varepsilon^2 n} + \varepsilon\lambda^{\varepsilon}_{n+1} + \varepsilon^2\Lambda^{\varepsilon}_{n+1} + o(\varepsilon^2)$$

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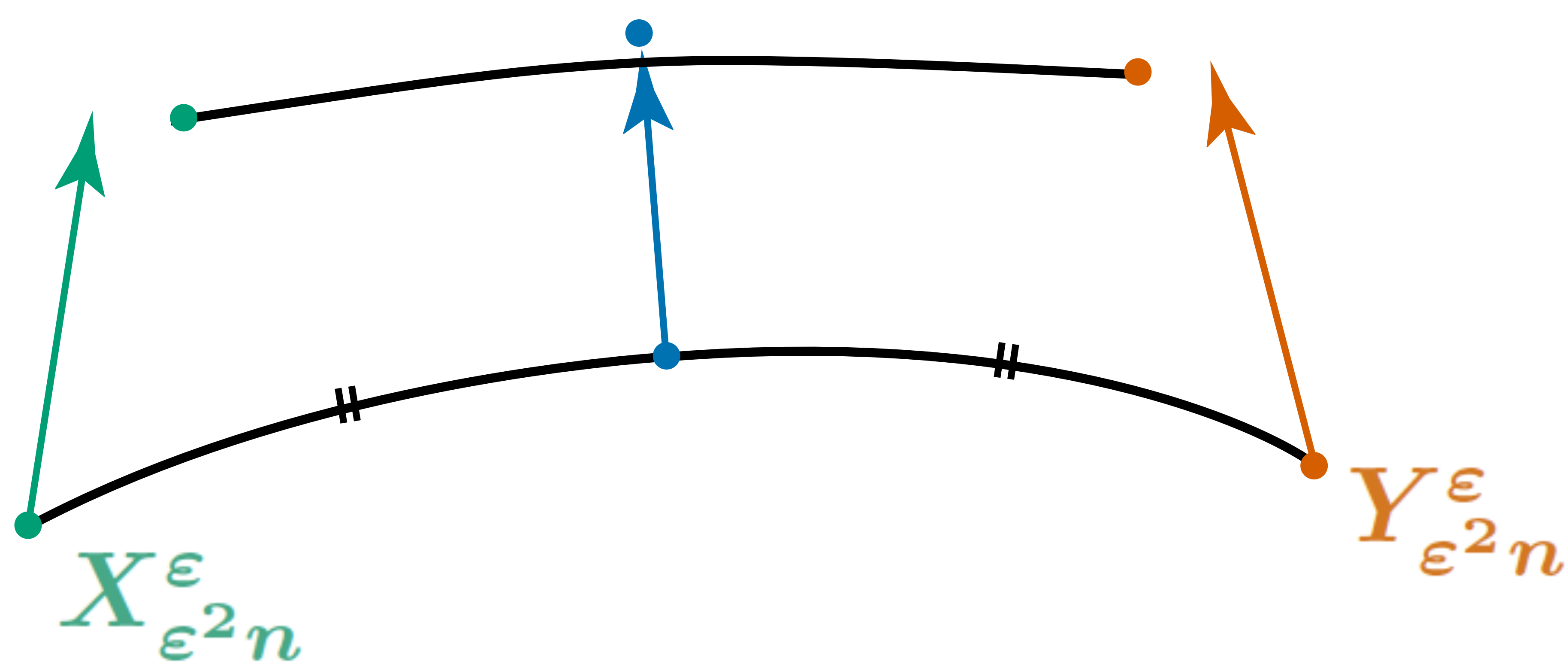

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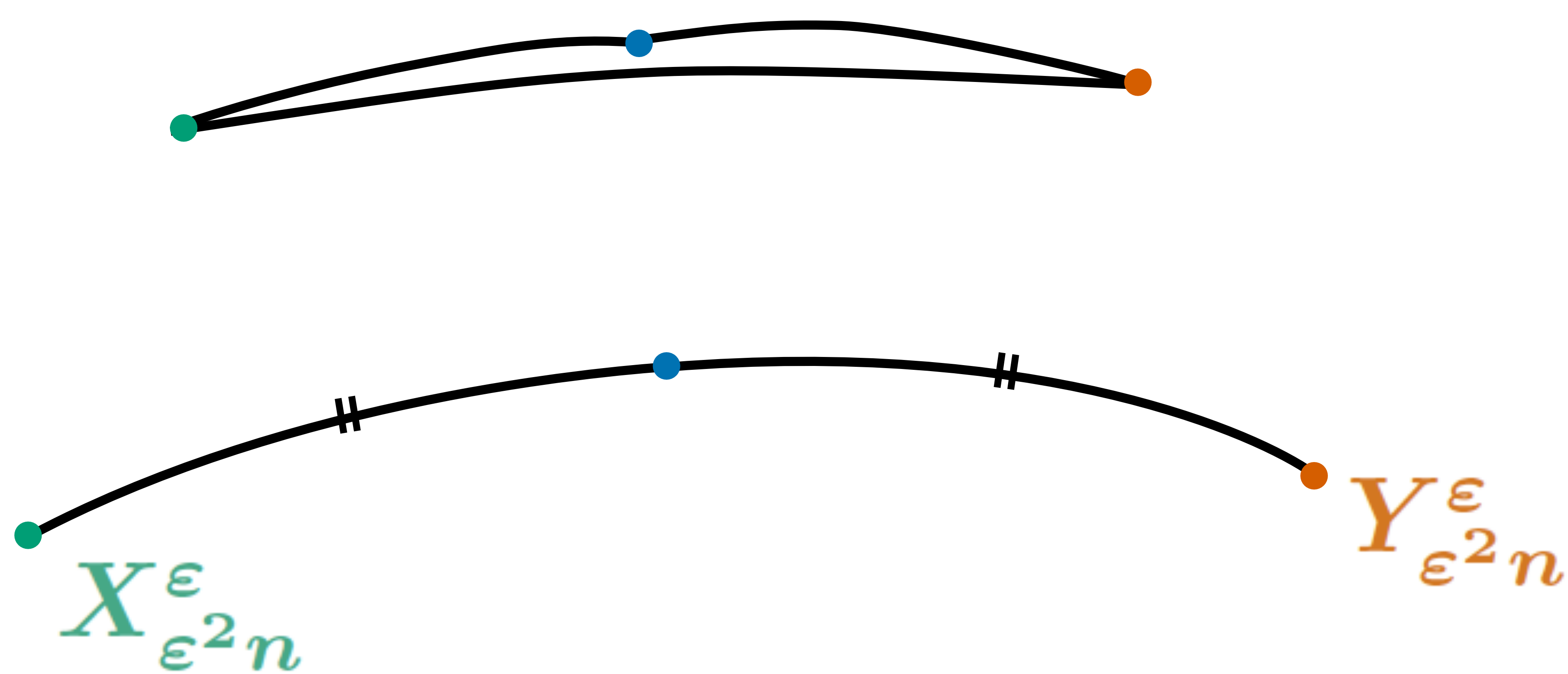

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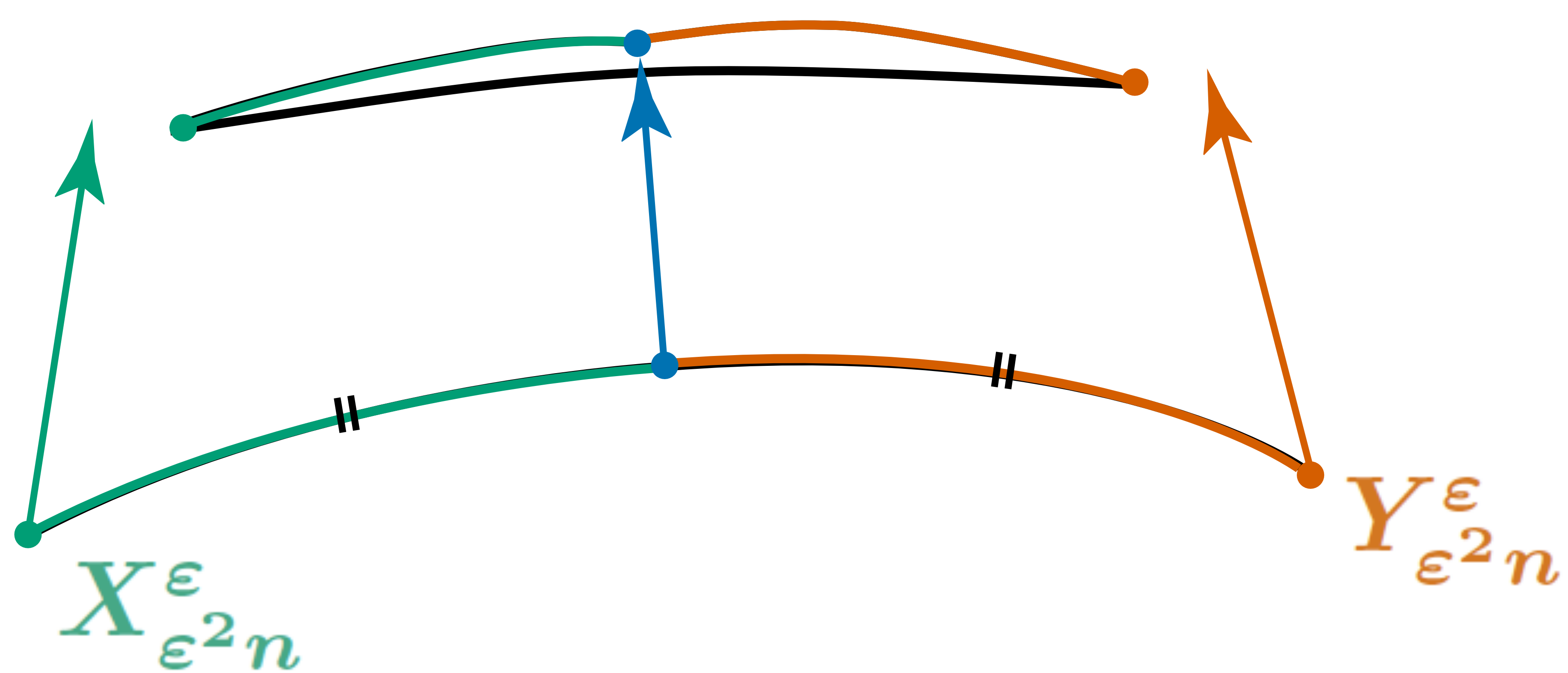

Obstructions: Discreteness of the Itô formula

- (Invariance principle)

  $$\varepsilon \sum_n \lambda_n^\varepsilon \to 2\sqrt{2}\beta. \text{ in law}$$

- (Law of large numbers)

  $$\overline{\lim_{\varepsilon \to 0}}\, \varepsilon^2 \sum_n \Lambda_n^\varepsilon \le \overline{\lim_{\varepsilon \to 0}} \left(-K\varepsilon^2 \sum_n \sigma_{\varepsilon^2 n}^\varepsilon \right)$$

(!) Λ_n^ε depends on λ_n^ε

(!) $\boxed{\Lambda_{n+1}^\varepsilon \le -K\sigma_{\varepsilon^2 n}^\varepsilon \text{ a.s.}}$ is NOT true

(!) Need a uniform bound for the error term $o(\varepsilon^2)$

Thm 4 [K.] (unif. nonexplosion/localization)

$$\sup_{\varepsilon} \mathbb{P}_x \left[\sup_{0 \le s \le t} d_{g(s)}(o, X^{\varepsilon}(s)) > R \right] \to 0$$

as $R \to \infty$

$\Rightarrow$ Local uniform control of $o(\varepsilon^2)$ is sufficient

$$\Rightarrow \varepsilon^2 \sum_n \Lambda_n^{\varepsilon} \approx \varepsilon^2 \sum_n \mathbb{E}[\Lambda_n^{\varepsilon} \mid \mathcal{F}_{n-1}]$$

with arbitrary high probability (as $\varepsilon \to 0$)

$$\star\ \mathbb{E}[\Lambda_n^{\varepsilon} \mid \mathcal{F}_{n-1}] \le -K\sigma^{\varepsilon}_{\varepsilon^2(n-1)}$$

$\Rightarrow \forall \delta > 0,$

$$\sigma^\varepsilon_{\varepsilon^2 n} \leq \sigma^\varepsilon_0 + \varepsilon \sum_{j \leq n} \lambda^\varepsilon_j - K\varepsilon^2 \sum_{j \leq n} \sigma^\varepsilon_{\varepsilon^2 j} + \delta$$

with arbitrary high probability (as $\varepsilon \to 0$).

$$\Downarrow$$

$$\mathbb{P}\left[\inf_{n \leq \varepsilon^{-2} T} \sigma^\varepsilon_{\varepsilon^2 n} > \delta\right] \leq \mathbb{P}\left[\inf_{n \leq \varepsilon^{-2} T} \rho^\varepsilon_{\varepsilon^2 n} > \delta'\right] + \text{(error)}$$

$\Rightarrow$ Thm 1 ($\varepsilon \to 0$, $\delta \to 0$)

5. Coupling by spacetime parallel transport

(joint work with R. Philipowski)

Perelman's $\mathcal{L}$-distance: Suppose $T_1 = 0$

$$\gamma : [\tau_1, \tau_2] \to M,\ [\tau_1, \tau_2] \subset [0, T_2]$$

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left(|\dot{\gamma}(\tau)|^2_{g(\tau)} + R_{g(\tau)}(\gamma(\tau)) \right) d\tau$$

$$L(\tau_1, x; \tau_2, y) := \inf \left\{ \mathcal{L}(\gamma) \;\middle|\; \begin{array}{l} \gamma(\tau_1) = x, \\ \gamma(\tau_2) = y \end{array} \right\}$$

Normalization

Given $0 \le \bar{\tau}_1 < \bar{\tau}_2 \le T_2$,

$$\Theta_t(x, y) := 2(\sqrt{\bar{\tau}_2 t} - \sqrt{\bar{\tau}_1 t}) L(\bar{\tau}_1 t, x; \bar{\tau}_2 t, y) - 2m(\sqrt{\bar{\tau}_2 t} - \sqrt{\bar{\tau}_1 t})^2$$

Thm 5 [K. & Philipowski]

Suppose

$$\begin{cases} \partial_t g(t) = 2\,\mathrm{Ric}_{g(t)}, \\ \inf_{\substack{X \in TM \\ t \in [T_1, T_2]}} \dfrac{\mathrm{Ric}_{g(t)}(X, X)}{g(t)(X, X)} > -\infty \end{cases}$$

$\Rightarrow$ $\exists (X_\tau, Y_\tau)$: coupling of $g(\tau)$-BMs

s.t. $(\Theta_t(X_{\bar{\tau}_1 t}, Y_{\bar{\tau}_2 t}))_{t \in [1, T_2/\bar{\tau}_2]}$

is a supermartingale

Cor 2 [K. & Philipowski]

$\forall \varphi$: $\nearrow$, concave & $\forall \mu_t, \nu_t$: heat distributions,

$\mathcal{T}_{\varphi(\Theta_t)}(\mu_{\bar{\tau}_1 t}, \nu_{\bar{\tau}_2 t}) \searrow$

- [Topping '09]: $\mathcal{T}_{\Theta_t}(\mu_{\bar{\tau}_1 t}, \nu_{\bar{\tau}_2 t}) \searrow$ when M:cpt, via optimal transport techniques ($\Rightarrow$ Monotonicity of Perelman's $\mathcal{W}$-entropy)

Strategy of the Proof

- Properties of $\mathcal{L}$-distance
  being analogous to the Riem. distance
  - $\mathcal{L}$-geodesic, 1st & 2nd variation of $\mathcal{L}$-length,
    $\mathcal{L}$-index lemma, $\mathcal{L}$-cut locus

- Approximation by random walks

- Coupling of $dX^{\varepsilon}_{\bar{\tau}_1 t}$ and $dY^{\varepsilon}_{\bar{\tau}_2 t}$ by
  spacetime-parallel transport along $\mathcal{L}$-geodesic

Spacetime parallel transport

For $\gamma : [s,t] \longrightarrow M$ & V: vector field along γ,

V: spacetime parallel

$$\overset{\text{def}}{\Leftrightarrow} \nabla^{g(u)}_{\dot\gamma(u)} V(u) = -\frac{1}{2}\partial_u g(u)^{\#} V(u)$$

$\mathcal{L}$-geodesic

$\gamma : [s,t] \longrightarrow M$: $\mathcal{L}$-geodesic

$$\overset{\text{def}}{\Leftrightarrow} \nabla^{g(u)}_{\dot\gamma_u} \dot\gamma_u = \frac{1}{2}\nabla^{g(u)} R_{g(u)} - 2\,\mathrm{Ric}^{\#}_{g(u)}(\dot\gamma_u) - \frac{1}{2u}\dot\gamma_u$$

$\sqrt{u}\dot\gamma_u$ is NOT spacetime parallel to γ!