Invariance principle for time-inhomogeneous geodesic random walks

Kazumasa Kuwada

(Ochanomizu University)

1. Introduction

M: m-dim. manifold, $-\infty < T_1 < T_2 \leq \infty$,
$(g(t))_{t\in[T_1,T_2]}$: smooth complete Riemannian metrics on M

★ Difficulty: Almost every geometric structure but the topology depends on time

Examples

- $\partial_t g(t) = \mathrm{Ric}_{g(t)}$ (backward Ricci flow)
- $\partial_t g(t) \leq \mathrm{Ric}_{g(t)} - K g(t)$ (time-dependent extension of "$\mathrm{Ric} \geq K$")

$\mathcal{A}_t := \frac{1}{2}\Delta_t + Z_t$ time-inhomogeneous generator

$(X_t)_{t \in [T_1, T_2]}$: $\mathcal{A}_t$-diffusion, i.e.

$$f(t, X_t) - f(T_1, X_{T_1}) - \int_{T_1}^{t} \left(\frac{\partial}{\partial s} + \mathcal{A}_s\right) f(s, X_s)\, ds$$

is a local martingale for $\forall f$: smooth

Goal

Approximating the $\mathcal{A}_t$-diffusion
by "nice" geodesic random walks

Purpose

Application to coupling methods
(especially, coupling by reflection)

2. Framework and the main result

Discrete time geodesic random walk Y^ε

- $(\xi_n)_{n\in\mathbb{N}}$: $\mathbb{R}^d$-valued i.i.d., unif. dist. on a centered disk, $\mathrm{Var}(\xi_n) = I$
- $t_{n+1} := t_n + \varepsilon^2$, $t_0 = T_1$
- $\Phi_t : M \to \mathcal{O}_{g(t)}(M)$ m'ble orth. frame

Given $Y^\varepsilon_{t_n}$,

$$\xi^\dagger_{n+1} := \Phi_{t_n}(Y^\varepsilon_{t_n})\xi_{n+1},$$

$$Y^\varepsilon_{t_{n+1}} := \exp^{g(t_n)}_{Y^\varepsilon_{t_n}}(\varepsilon\xi^\dagger_{n+1} + \varepsilon^2 Z_{t_n}(Y^\varepsilon_{t_n}))$$

Two continuous-time interpolations

(i) Linear interpolation of tangent vector: X^{ε}

$\Rightarrow$ Trajectory of X^{ε} is a piecewise geodesic

$\Rightarrow$ X^{ε} has continuous sample path

$\Rightarrow$ X^{ε}: not Markov

(ii) Poisson subordination: $\tilde{X}^{\varepsilon}$

$\Rightarrow$ $\tilde{X}^{\varepsilon}$ has cadlag sample path

$\Rightarrow$ $\tilde{X}^{\varepsilon}$: Markov

Assumption

$\exists b : [0, \infty) \longrightarrow \mathbb{R}$ sufficiently regular s.t.

(i) $\partial_t g(t) \leq \mathrm{Ric}_t^Z + b(d(o, \cdot))g(t)$

$\left(\mathrm{Ric}_t^Z := \mathrm{Ric}_{g(t)} - 2(\nabla Z_t)^{\mathrm{sym}} \right)$

(ii) $\forall C > 0$, ρ_t cannot go to ∞,

where $d\rho_t = d\beta_t + \left(C + \frac{\Psi(\rho_t)}{2} \right) dt$,

$\left(\beta_t\text{: } \mathrm{BM}^1, \ \Psi(r) := \int_0^r b(s)ds \right)$

★ No time-uniform bounds are necessary!

Thm 1 [K.] (invariance principle)

Under Assumptions (i)(ii),

$\boldsymbol{X}^{\varepsilon} \xrightarrow{d} \boldsymbol{X}$ as $\boldsymbol{\varepsilon} \to \mathbf{0}$ in $\boldsymbol{\mathcal{C}}([\boldsymbol{T_1}, \boldsymbol{T_2}] \to \boldsymbol{M})$.

Rem

(i) Convergence of X^ε $\Leftrightarrow$ Convergence of $\tilde{X}^\varepsilon$

(ii) When $\partial_t g(t) \equiv 0$, our assumption is a well-known sufficient condition for conservativity

(iii) Under the same assumption, X_t is conservative [K.-Philipowski]

Known results (When $\partial_t g(t) \equiv 0$)

- [Jørgensen '75]
  IP under analytic assumptions
  (Spacially inohomogeneous noise, with killing)

- [M.Pinsky '76] / [Blum '84]
  Conv. of C_0-semigr. under Feller / $\mathbf{Ric} \geq K$
  (isotropic noise)

- [von Renesse '04, K. '10]
  Coupling via geodesic RWs

3. Sketch of the proof

3.1. Overview

Structure of the proof

(i) Tightness for X^ε

$\Leftarrow$ Estimate of the modulus of continuity

(curvature bound & unif. error est.)

(ii) Identification of the (subsequential) limit for $\tilde{X}^\varepsilon$

$\Leftarrow$! of the $\mathcal{A}_t$-martingale problem

(cpt unif. convergence of generators)

★ Crucial to localize the problem!

Lem 1 (localization of underlying geometry)

For $\forall R > 0$, $\exists M_0 \subset M$: cpt s.t.

$$\left\{ p \in M \;\middle|\; \inf_{t \in [T_1, T_2]} d_t(o, p) \le R \right\} \subset M_0$$

$\Downarrow$

Localization by

$$\sigma_R := \inf \{ t \mid d_t(o, X_t^\varepsilon) \ge R \}$$

Thm 2 [K.] (unif. non-explosion)

$$\lim_{R\uparrow\infty} \overline{\lim_{\varepsilon\downarrow 0}} \mathbb{P}_x \left[\sup_{0\le s\le t} d_s(o, X_s^{\varepsilon}) > R \right] = 0$$

Thm 2 [K.] (unif. non-explosion)

$$\lim_{R\uparrow\infty} \overline{\lim_{\varepsilon\downarrow 0}} \mathbb{P}_x \left[\sup_{0\le s\le t} d_s(o, X_s^\varepsilon) > R \right] = 0$$

Idea of the proof: Follow [K.-Philipowski]

3.2. Conservativity of $\mathcal{A}_t$-diffusions
(under Assumptions(i)(ii))

Strategy

(i) Consider $r_t(X_t)$, $r_t(x) := d_t(o, x)$

(ii) Apply the Itô formula:
$dr_t(X_t) \le \sqrt{2} d\beta_t + (\partial_t + \mathcal{A}_t) r_t(X_t) dt$
(β_t becomes BM[1])

(iii) Comparison thm: $(\partial_t + \mathcal{A}_t) r_t \le C + \dfrac{\Psi(r_t)}{2}$

(iv) $r_t \le \rho_t$,
ρ_t solves $d\rho_t = d\beta_t + \left(C + \dfrac{\Psi(\rho_t)}{2} \right) dt$

More on comparison thm

$\gamma : [0, r_t(x)] \to M$: $g(t)$-geod. from o to x

- $$\partial_t r_t(x) = \frac{1}{2} \int_0^{r_t(x)} \partial_t g(t)(\dot{\gamma}_s, \dot{\gamma}_s) ds$$
- $$\mathcal{A}_t r_t(x) \le c + \frac{d-1}{2} \frac{G'(r_t(x))}{G(r_t(x))},$$

where $$G''(s) = -\frac{\mathrm{Ric}_t^Z(\dot{\gamma}_s, \dot{\gamma}_s)}{(d-1)} G(s),$$
$$G(0) = 0,\ G'(0) = 1$$

$\Downarrow$ Ass. (i)

$$(\partial_t + \mathcal{A}_t) r_t \le \frac{1}{2}\int_0^{r_t} \mathrm{Ric}_t^Z(\dot\gamma_s, \dot\gamma_s)\,ds + \frac{d-1}{2}\frac{G'(r_t)}{G(r_t)} + c + \Psi(r_t)$$

$$= [\text{non-incr. fn. along } \gamma](r_t) + \Psi(r_t)$$

Obstruction to the Itô formula:

Singularity of r_t at $\mathrm{Cut}_{g(t)}(o)$ ($g(t)$-cut locus)

⋆ Singular points depends on time!

★ Establish the Itô formula for r_t

involving "local time at $g(t)$-cut locus"

([Kendall '86, Hsu '02] when $\partial_t g(t) \equiv 0$)

Obstruction to the Itô formula:

Singularity of r_t at $\mathrm{Cut}_{g(t)}(o)$ ($g(t)$-cut locus)

⋆ Singular points depends on time!

★ Establish the Itô formula for r_t

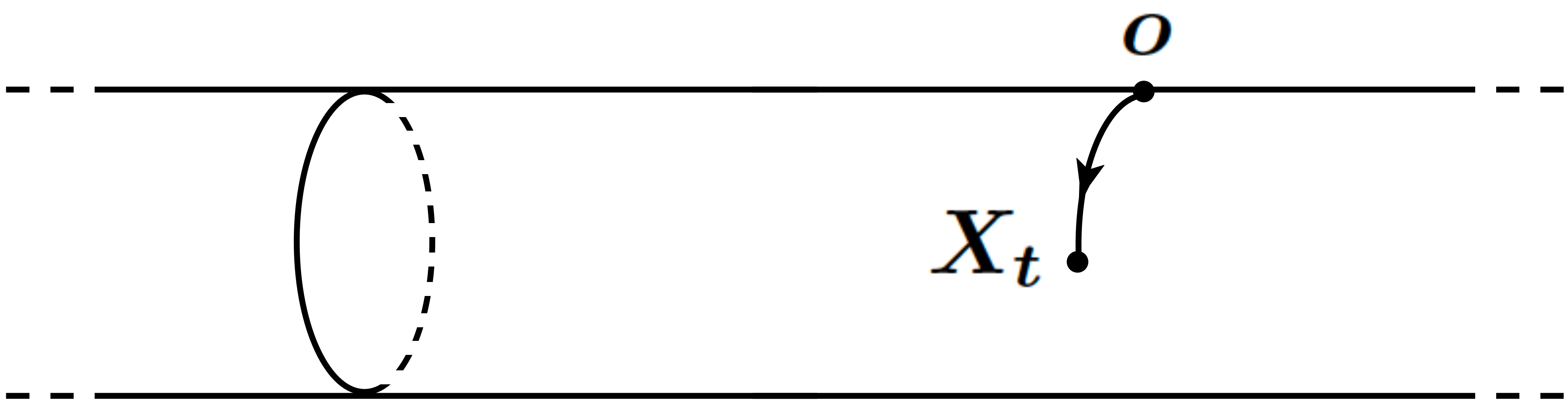
involving "local time at $g(t)$-cut locus"

([Kendall '86, Hsu '02] when $\partial_t g(t) \equiv 0$)

Obstruction to the Itô formula:

Singularity of r_t at $\mathrm{Cut}_{g(t)}(o)$ ($g(t)$-cut locus)

⋆ Singular points depends on time!



★ Establish the Itô formula for r_t

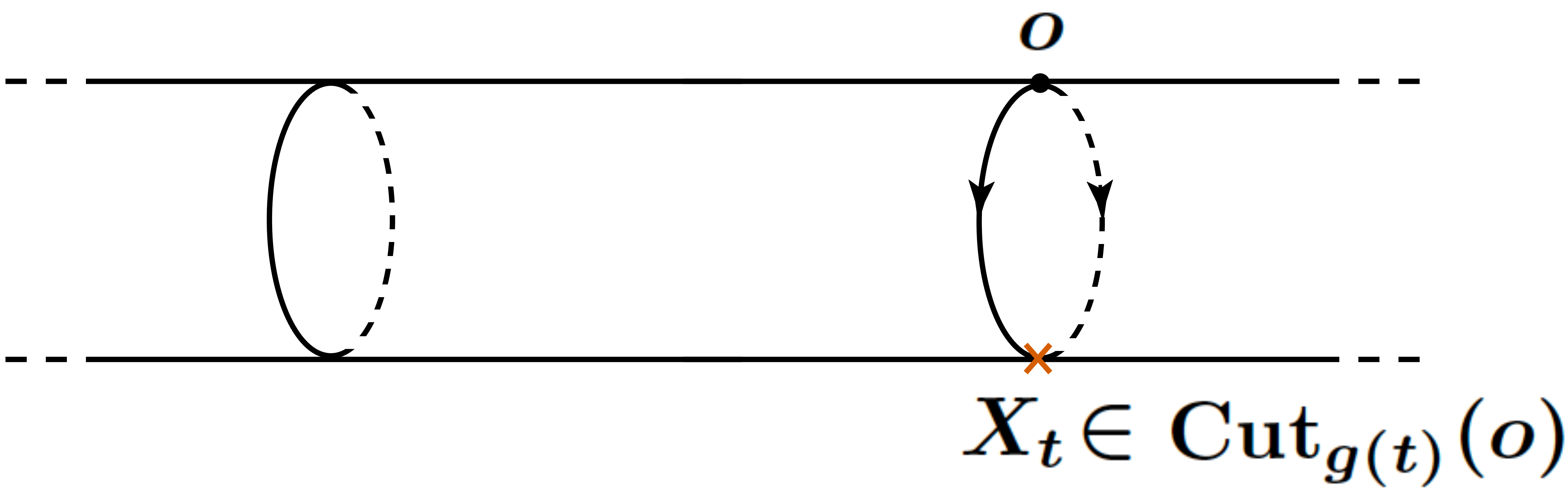
involving "local time at $g(t)$-cut locus"

([Kendall '86, Hsu '02] when $\partial_t g(t) \equiv 0$)

Obstruction to the Itô formula:

Singularity of r_t at $\mathrm{Cut}_{g(t)}(o)$ ($g(t)$-cut locus)

⋆ Singular points depends on time!



★ Establish the Itô formula for r_t

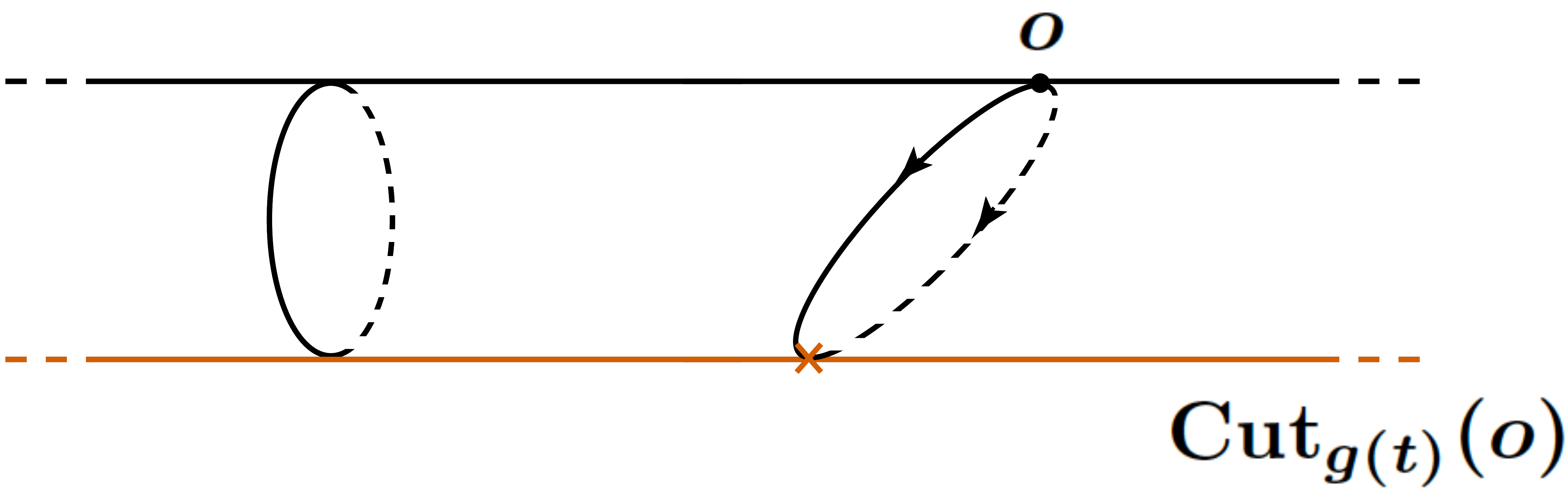
involving "local time at $g(t)$-cut locus"

([Kendall '86, Hsu '02] when $\partial_t g(t) \equiv 0$)

Obstruction to the Itô formula:

Singularity of r_t at $\mathrm{Cut}_{g(t)}(o)$ ($g(t)$-cut locus)

⋆ Singular points depends on time!



★ Establish the Itô formula for r_t

involving "local time at $g(t)$-cut locus"

([Kendall '86, Hsu '02] when $\partial_t g(t) \equiv 0$)

$$\mathbf{Cut}_{\mathbf{ST}} := \{(t, x, y) \mid y \in \mathbf{Cut}_{g(t)}(x)\}$$

Few facts on $g(t)$-cut locus

- $\mathbf{vol}_{g(t)}(\mathbf{Cut}_{g(t)}(x)) = 0$
- $\mathbf{Cut}_{\mathbf{ST}}$: closed

Our approach: follow [Kendall '86]

$\tau_0 := 0 < \tau_1 < \cdots < \tau_n < \cdots$: stopping times

- τ_{2n+1}: 1st visit to $\mathrm{Cut}(o)$ after τ_{2n}
- τ_{2n}: 1st exit from δ-nbd of $X(\tau_{2n-1})$ after τ_{2n-1}

★ $\sum_n |\tau_{2n} - \tau_{2n-1}| \to 0$ a.s. as $\delta \downarrow 0$

★ When $t \in [\tau_{2n-1}, \tau_{2n}]$, use an alternative ref. pt o_n instead of o

3.3. Uniform non-explosion bound of geodesic RWs

Discrete Itô formula (Taylor expansion)

$$
\begin{aligned}
& r_{t_{n+1}}(X^{\varepsilon}_{t_{n+1}}) - r_{t_n}(X^{\varepsilon}_{t_n}) \\
& \quad \boxed{\leq} \varepsilon g(t_n)(\nabla r_{t_n}, \xi^{\dagger}_{n+1}) + \varepsilon^2 \partial_t r_{t_n}(X^{\varepsilon}_{t_n}) \\
& \qquad + \varepsilon^2 \mathrm{Z}_{t_n} r_{t_n}(X^{\varepsilon}_{t_n}) + \frac{\varepsilon^2}{2} \mathrm{Hess}\, r_{t_n}(\xi^{\dagger}_n, \xi^{\dagger}_n) \\
& \qquad + \delta + o(\varepsilon^2)
\end{aligned}
$$

★ "$\leq$" without extracting "local time at $\mathrm{Cut}_{\mathrm{ST}}$"

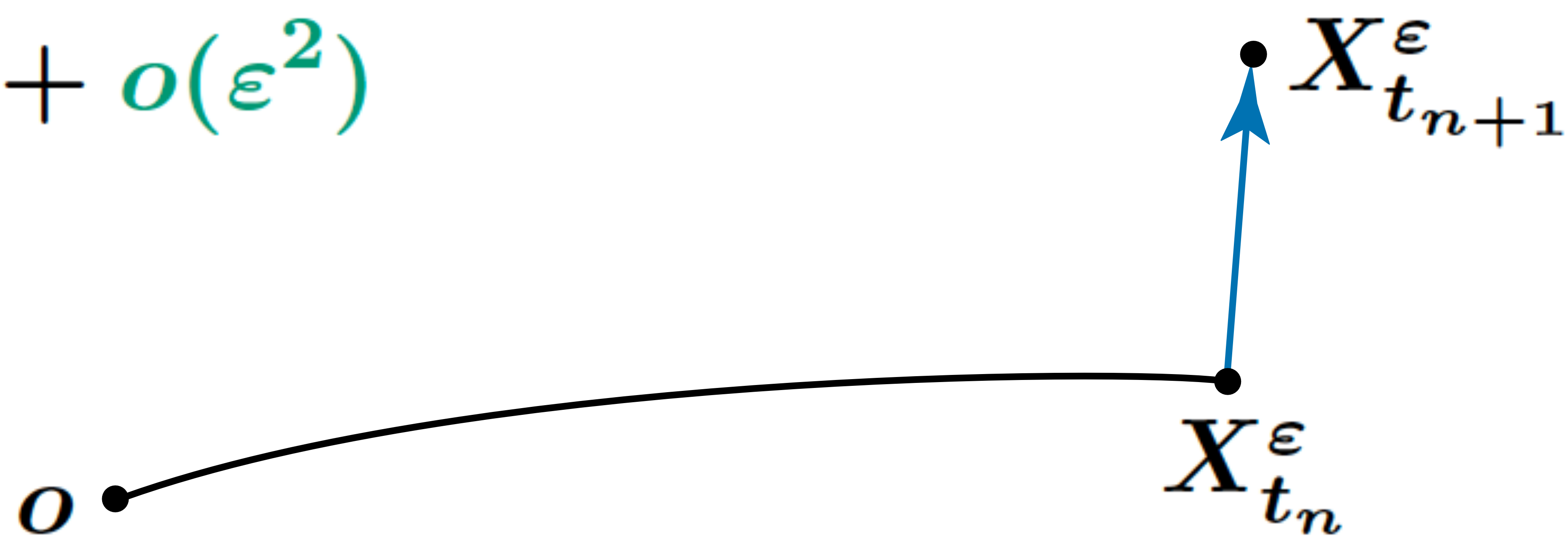
Discrete Itô formula (Taylor expansion)

$$
\begin{aligned}
&r_{t_{n+1}}(X^{\varepsilon}_{t_{n+1}}) - r_{t_n}(X^{\varepsilon}_{t_n}) \\
&\quad \leq \varepsilon g(t_n)(\nabla r_{t_n}, \xi^{\dagger}_{n+1}) + \varepsilon^2 \partial_t r_{t_n}(X^{\varepsilon}_{t_n}) \\
&\qquad + \varepsilon^2 Z_{t_n} r_{t_n}(X^{\varepsilon}_{t_n}) + \frac{\varepsilon^2}{2} \mathrm{Hess}\, r_{t_n}(\xi^{\dagger}_{n}, \xi^{\dagger}_{n}) \\
&\qquad + \delta + o(\varepsilon^2)
\end{aligned}
$$

★ "$\leq$" without extracting "local time at $\mathrm{Cut}_{\mathrm{ST}}$"

Discrete Itô formula (Taylor expansion)

$$
\begin{aligned}
r_{t_{n+1}}(X^{\varepsilon}_{t_{n+1}}) - r_{t_n}(X^{\varepsilon}_{t_n})
&\leq \varepsilon g(t_n)(\nabla r_{t_n}, \xi^{\dagger}_{n+1}) + \varepsilon^2 \partial_t r_{t_n}(X^{\varepsilon}_{t_n}) \\
&\quad + \varepsilon^2 Z_{t_n} r_{t_n}(X^{\varepsilon}_{t_n}) + \frac{\varepsilon^2}{2} \operatorname{Hess} r_{t_n}(\xi^{\dagger}_n, \xi^{\dagger}_n) \\
&\quad + \delta + o(\varepsilon^2)
\end{aligned}
$$



★ "$\leq$" without extracting "local time at $\mathrm{Cut}_{\mathrm{ST}}$"

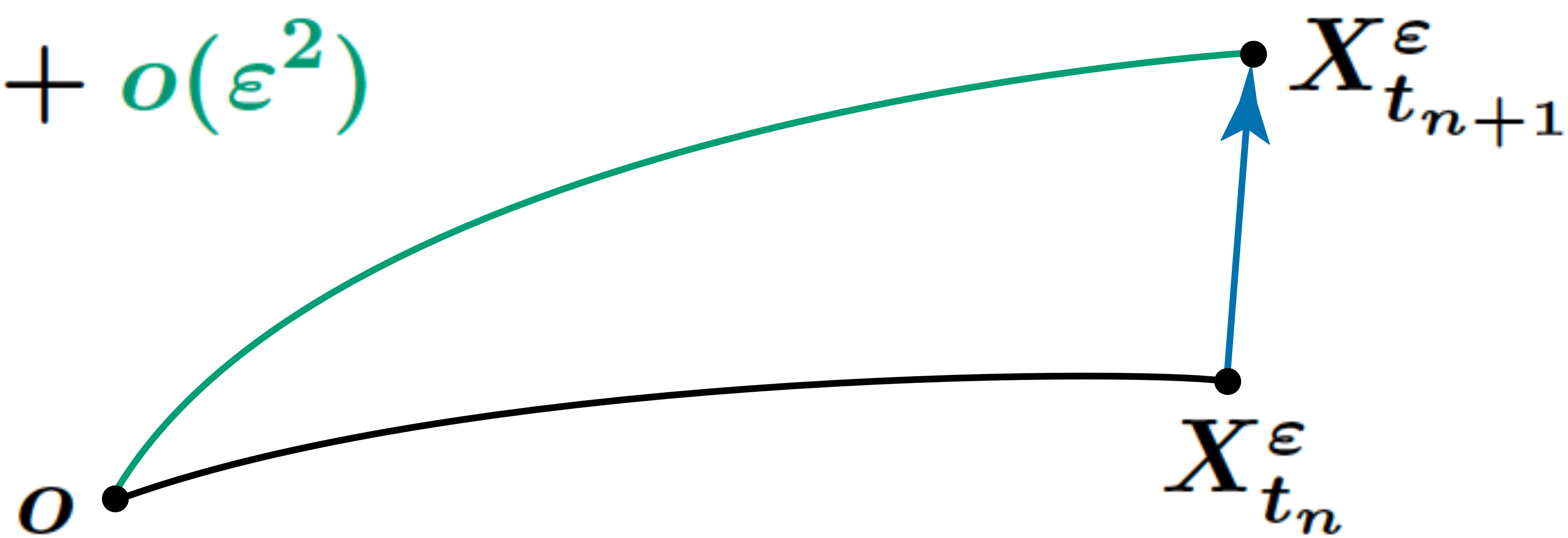
Discrete Itô formula (Taylor expansion)

$$r_{t_{n+1}}(X^{\varepsilon}_{t_{n+1}}) - r_{t_n}(X^{\varepsilon}_{t_n})$$

$$\le \varepsilon g(t_n)(\nabla r_{t_n}, \xi^{\dagger}_{n+1}) + \varepsilon^2 \partial_t r_{t_n}(X^{\varepsilon}_{t_n})$$

$$+ \varepsilon^2 Z_{t_n} r_{t_n}(X^{\varepsilon}_{t_n}) + \frac{\varepsilon^2}{2} \mathrm{Hess}\, r_{t_n}(\xi^{\dagger}_n, \xi^{\dagger}_n)$$

$$+ \delta + o(\varepsilon^2)$$



★ "$\le$" without extracting "local time at $\mathrm{Cut}_{\mathrm{ST}}$"

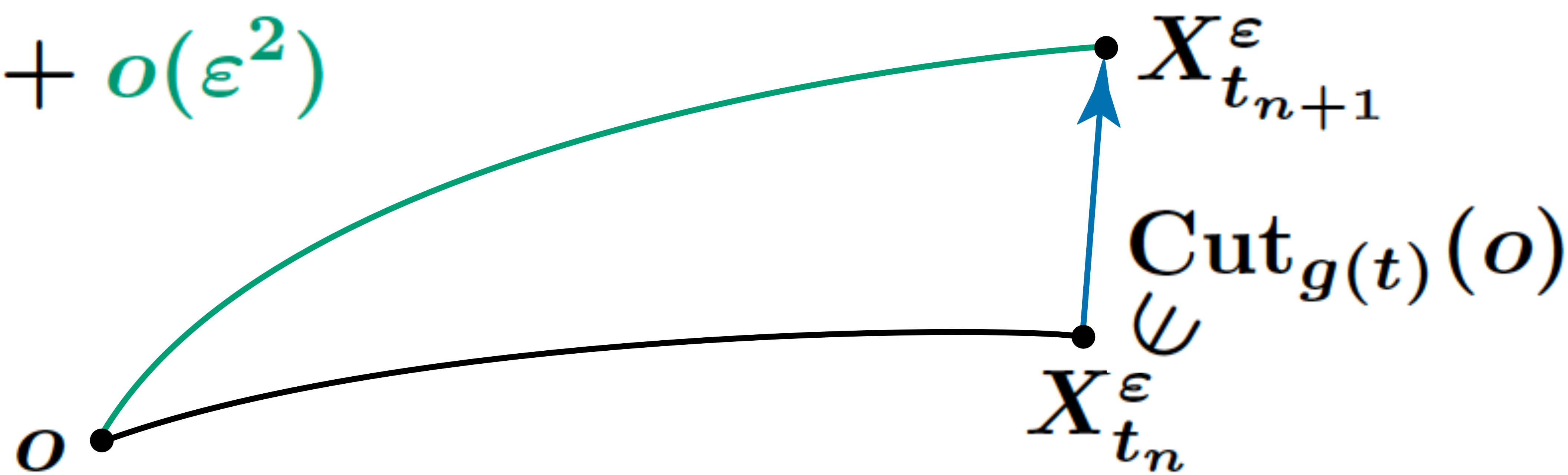
Discrete Itô formula (Taylor expansion)

$$r_{t_{n+1}}(X^{\varepsilon}_{t_{n+1}}) - r_{t_n}(X^{\varepsilon}_{t_n})$$

$$\boxed{\leq} \varepsilon g(t_n)(\nabla r_{t_n}, \xi^{\dagger}_{n+1}) + \varepsilon^2 \partial_t r_{t_n}(X^{\varepsilon}_{t_n})$$

$$+ \varepsilon^2 Z_{t_n} r_{t_n}(X^{\varepsilon}_{t_n}) + \frac{\varepsilon^2}{2} \mathrm{Hess}\, r_{t_n}(\xi^{\dagger}_n, \xi^{\dagger}_n)$$

$$+ \delta + o(\varepsilon^2)$$



★ "$\leq$" without extracting "local time at $\mathrm{Cut}_{\mathrm{ST}}$"

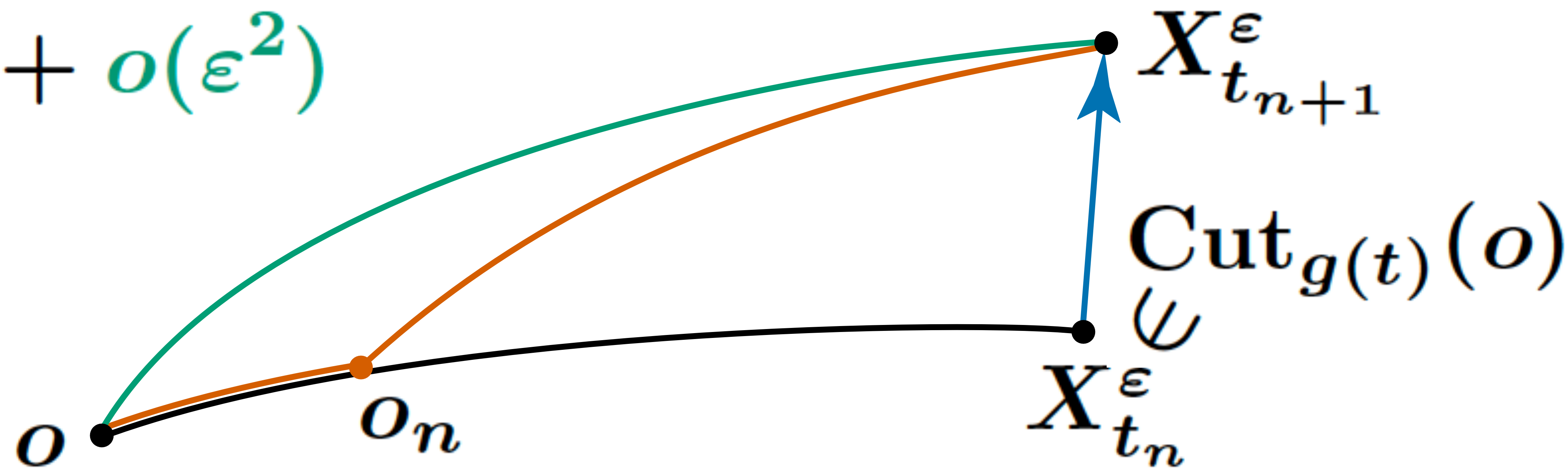
Discrete Itô formula (Taylor expansion)

$$r_{t_{n+1}}(X^\varepsilon_{t_{n+1}}) - r_{t_n}(X^\varepsilon_{t_n})$$

$$\boxed{\leq} \varepsilon g(t_n)(\nabla r_{t_n}, \xi^\dagger_{n+1}) + \varepsilon^2 \partial_t r_{t_n}(X^\varepsilon_{t_n})$$

$$+ \varepsilon^2 \mathbf{Z}_{t_n} r_{t_n}(X^\varepsilon_{t_n}) + \frac{\varepsilon^2}{2} \mathbf{Hess}\, r_{t_n}(\xi^\dagger_n, \xi^\dagger_n)$$

$$+ \delta + o(\varepsilon^2)$$



★ "$\leq$" without extracting "local time at $\mathrm{Cut}_{\mathrm{ST}}$"

Obstructions to follow [K.-Philipowski]

(i) Singularity of r_t at o

(ii) (Local) uniform estimate of $o(\varepsilon^2)$

($\Leftarrow$ Localization + $\mathrm{Cut}_{\mathrm{ST}}$: closed)

(iii) Treatment of the 2nd order term:

$$\mathbb{E}\left[\mathrm{Hess}\, r_{t_n}(\xi_{n+1}^\dagger, \xi_{n+1}^\dagger) \middle| \mathcal{F}_n \right] = \Delta_{t_n} r_{t_n}$$

(iv) Different scalings

- 1st order: scale for CLT
- 2nd order: scale for LLN

(iii) Treatment of the 2nd order term

$$\Lambda^{\varepsilon}_{n+1} := \mathrm{Hess}\, r_{t_n}(\xi^{\dagger}_{n+1}, \xi^{\dagger}_{n+1}) \\ + Z_{t_n} r_{t_n}(X^{\varepsilon}_{t_n}) + \partial_t r_{t_n}(X^{\varepsilon}_{t_n})$$

Lem 2 (Martingale LLN for Λ^{ε}_n)

As $\varepsilon \longrightarrow 0$,

$$\sup_{t<\sigma_R} \left| \varepsilon^2 \sum_{t_n \le t} \left(\Lambda^{\varepsilon}_n - \mathbb{E}[\Lambda^{\varepsilon}_n | \mathcal{F}_{n-1}] \right) \right| \to 0$$

in probability

Idea for (i)(iv): Comparison before scaling limit

Discrete Comparison process ρ^ε

- $\lambda^\varepsilon_{n+1} := g(t_n)(\nabla r_{t_n}, \xi^\dagger_{n+1})$: i.i.d.
- β^ε_t: piecewise linear interpolation of $\varepsilon \sum \lambda^\varepsilon_n$
- ρ^ε_t solves $d\rho^\varepsilon_t = d\beta^\varepsilon_t + \tilde{\Psi}(\rho^\varepsilon_t)dt$,
  where

  $\tilde{\Psi} := \Psi + \Psi_0 +$ ("error"),
  $\Psi_0 \geq 0$: auxiliary drift (explained below)

(i) Singularity of r_t at o

Take Ψ_0 so that $0 < \exists a \leq \inf_t \rho_t^\varepsilon$

$$\Downarrow$$

$r_t(X_t^\varepsilon) \leq \rho_t^\varepsilon$ when $X_t^\varepsilon \approx o$

(iv) Different scalings

What we need:

Smallness of $\mathbb{P}\left[\sup_{t\le T} r_t(X_t^\varepsilon) > R\right]$ unif. in ε

Lem 2 + Discrete comparison thm

$$\Downarrow$$

$$\mathbb{P}\left[\sup_{t\le T} r_t(X_t^\varepsilon) > R\right] \le \mathbb{P}\left[\sup_{t\le T} \rho_t^\varepsilon > R\right]$$

$\therefore$ Studying the scaling limit of $\rho^\varepsilon \Rightarrow$ Thm 2

(Note: ρ^ε has unbounded drift)

4. Coupling by reflection

Thm 3 [K.]

Suppose $\exists K \in \mathbb{R}$ s.t. $\partial_t g(t) \leq \mathrm{Ric}_t^Z - K g(t)$

Thm 3 [K.]

Suppose $\exists K \in \mathbb{R}$ s.t. $\partial_t g(t) \leq \mathrm{Ric}_t^Z - K g(t)$

$\Rightarrow \forall x, \hat{x} \in M$,

$\exists (X_t, \hat{X}_t)$: coupled $\mathcal{A}_t$-diff. from $(x, \hat{x})$ s.t.

$$\mathbb{P}\Big[\inf_{T_1 \leq s \leq t} d_{g(s)}(X_s, \hat{X}_s) > 0\Big] \leq \mathbb{P}\Big[\inf_{T_1 \leq s \leq t} \rho_s > 0\Big]$$

where $\hat{\rho}_t$ solves $\hat{\rho}_{T_1} = d_{g(T_1)}(x, y)$ and

$$d\hat{\rho}_t = 2 dB_t - \frac{K}{2} \hat{\rho}_t dt$$

Rem

- Heuristically,
  "$d_{g(t)}(X_t, \hat{X}_t) \leq \hat{\rho}_t$" $\Rightarrow$ Thm 3

- (RHS) $= \varphi_{t-T_1}(d_{g(T_1)}(x, y))$,
  where

$$\varphi_s(a) := \sqrt{\frac{2}{\pi}} \int_0^{\frac{a}{2\sqrt{\beta(s)}}} \mathrm{e}^{-x^2/2} dx,$$

$$\beta(s) := \frac{\mathrm{e}^{Ks} - 1}{K}$$

Cor 1 [K. & Sturm]

Let $T_1 < T \le T_2$. $\forall \mu_t, \nu_t$: heat distributions,

$$\inf_{\pi \in \Pi(\mu_t, \nu_t)} \int \varphi_{T-t} d\pi \searrow$$

Cor 2 [K.]

$$\left\| |\nabla P_{T_1,t} f|_{g(T_1)} \right\|_\infty \le \frac{1}{\sqrt{2\pi\beta(t - T_1)}} \operatorname{osc} f$$

(cf. [Coulibaly] via stoch. diff. geom.)

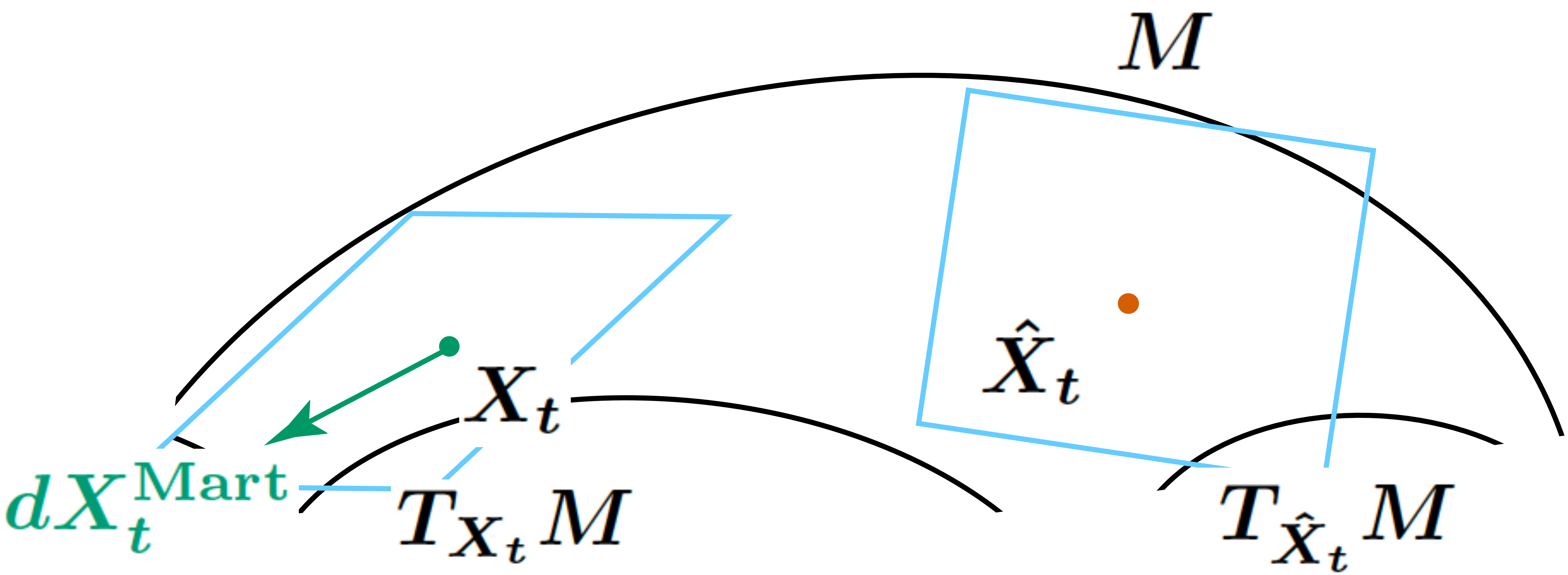
Idea of the proof of Thm 3

Construct $(X_t, \hat{X}_t)$

where $d\hat{X}_t^{\mathrm{Mart}} =$ "(local) reflection" of dX_t^{Mart}

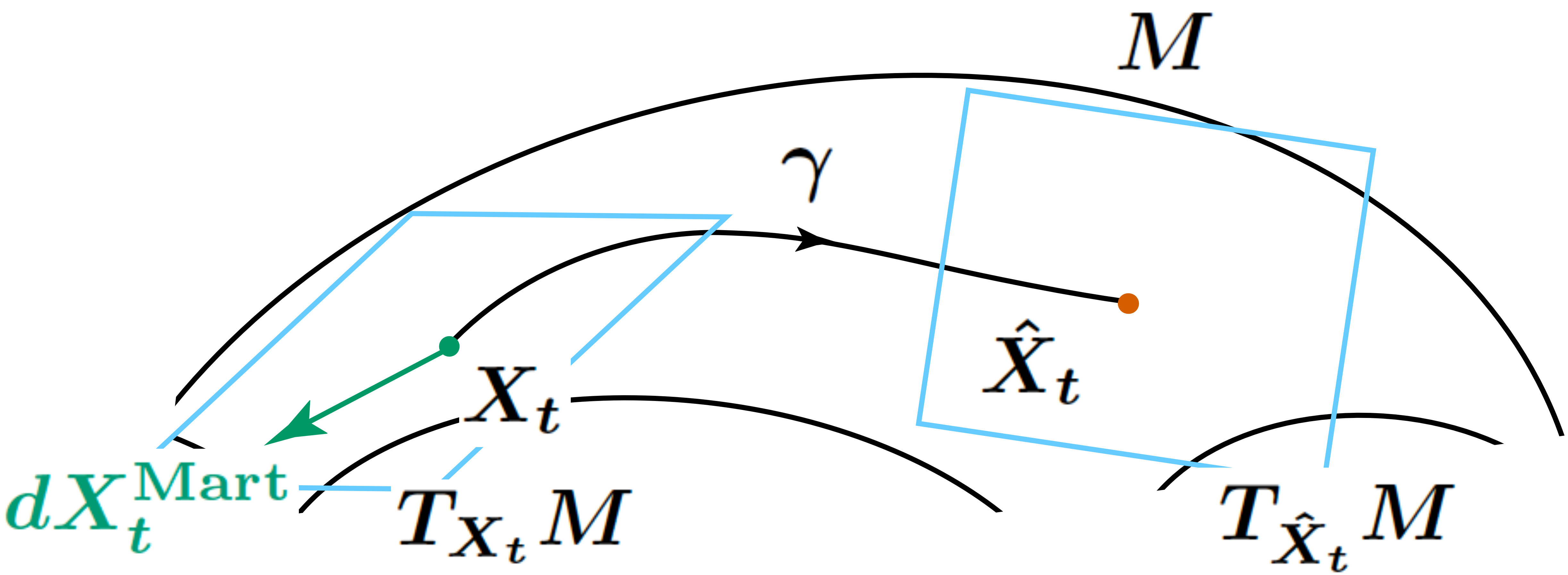
Construct $(X_t, \hat{X}_t)$

where $d\hat{X}_t^{\mathrm{Mart}} =$ "(local) reflection" of dX_t^{Mart}



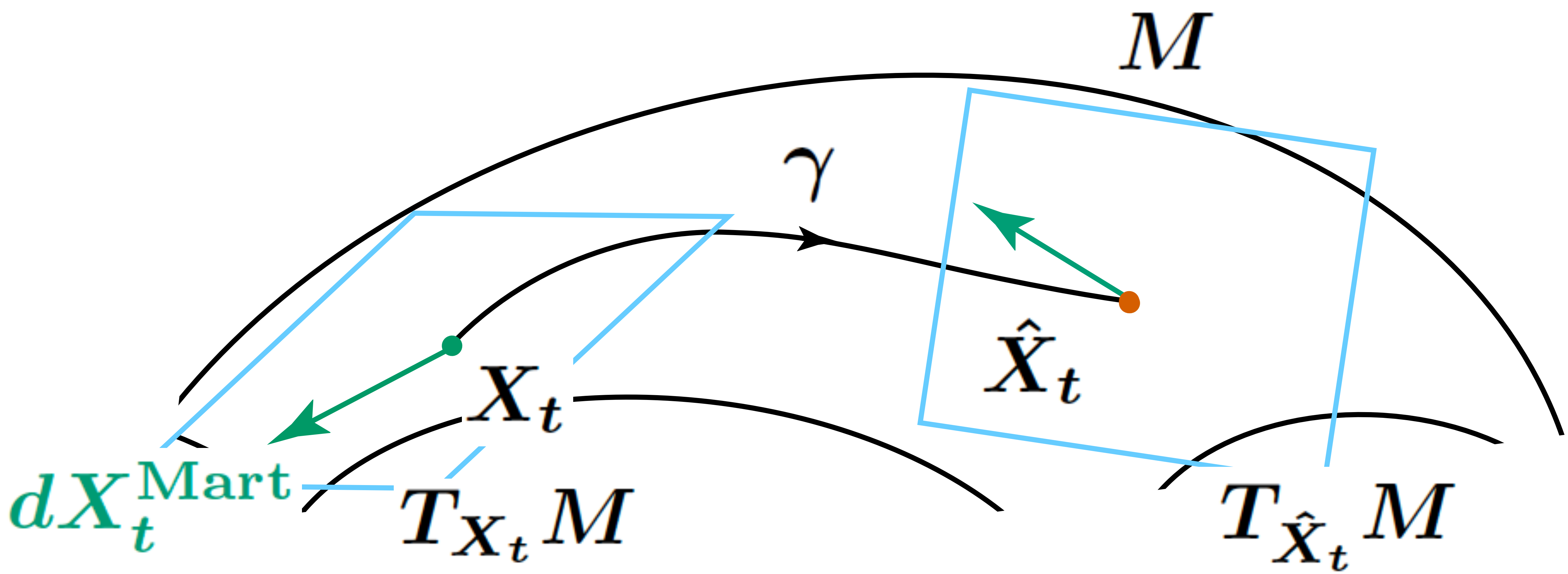
Construct $(X_t, \hat{X}_t)$

where $d\hat{X}_t^{\mathrm{Mart}}$ = "(local) reflection" of dX_t^{Mart}



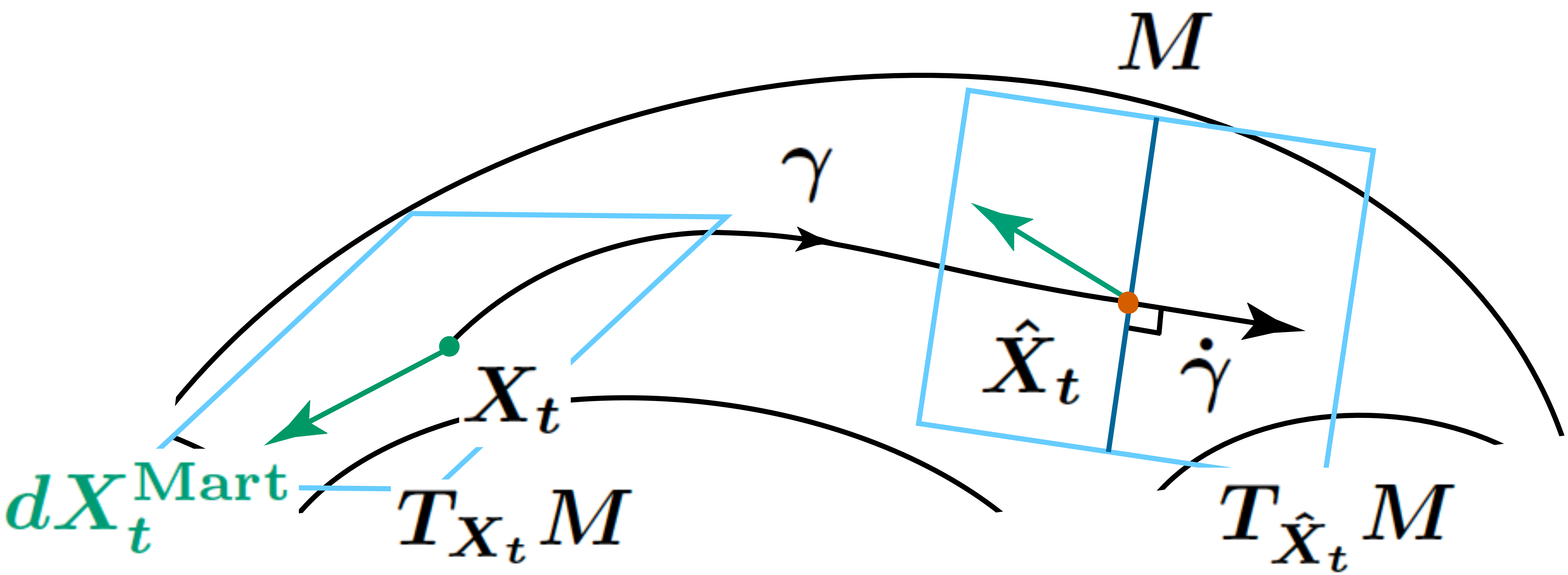
Construct $(X_t, \hat{X}_t)$

where $d\hat{X}_t^{\mathrm{Mart}} =$ "(local) reflection" of dX_t^{Mart}



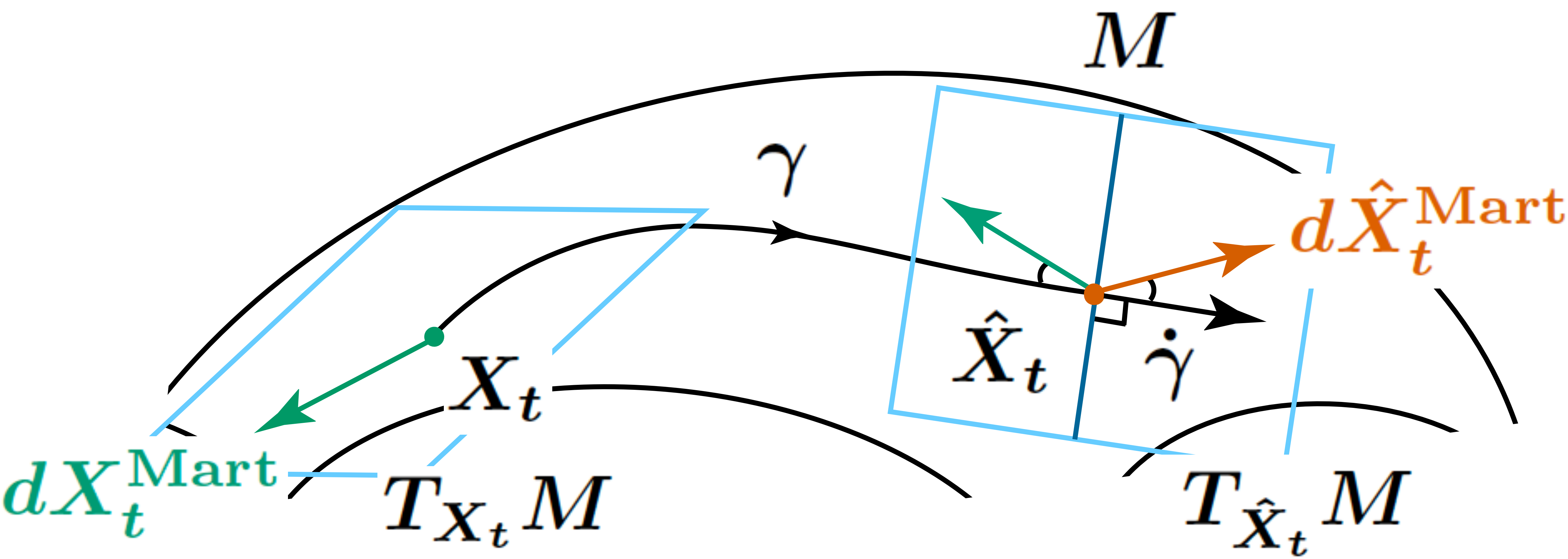
Construct $(X_t, \hat{X}_t)$

where $d\hat{X}_t^{\mathrm{Mart}} =$ "(local) reflection" of dX_t^{Mart}



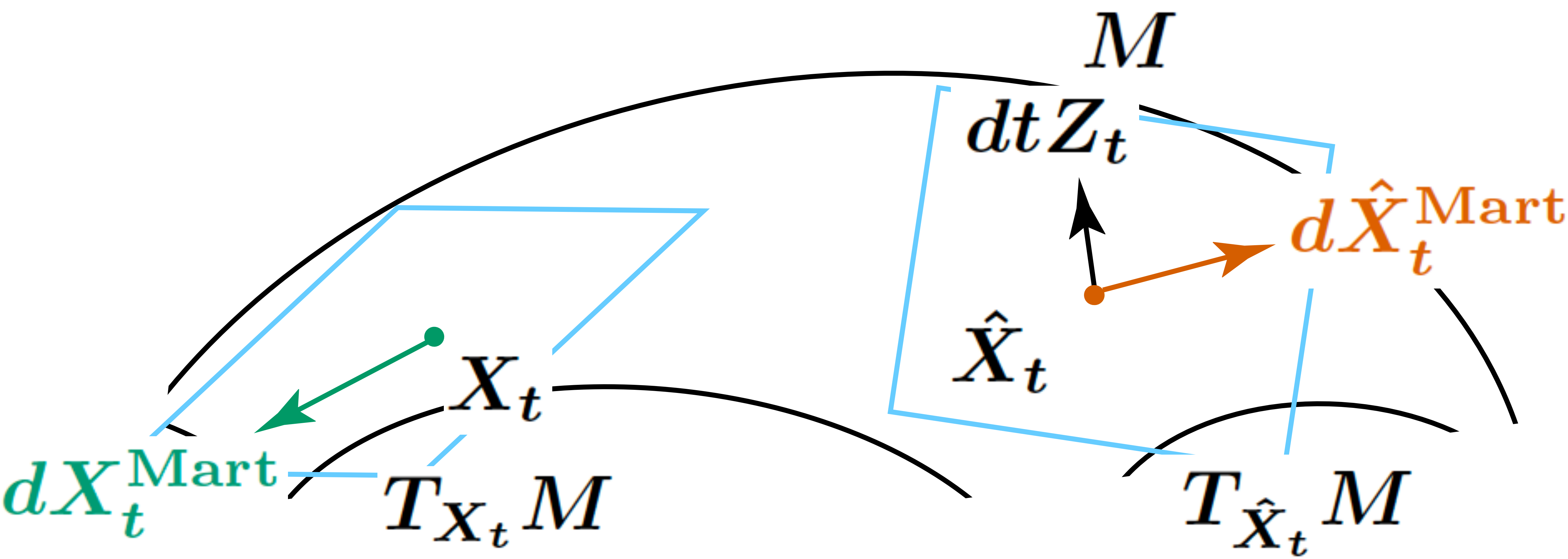
Construct $(X_t, \hat{X}_t)$

where $d\hat{X}_t^{\mathrm{Mart}}$ = "(local) reflection" of dX_t^{Mart}



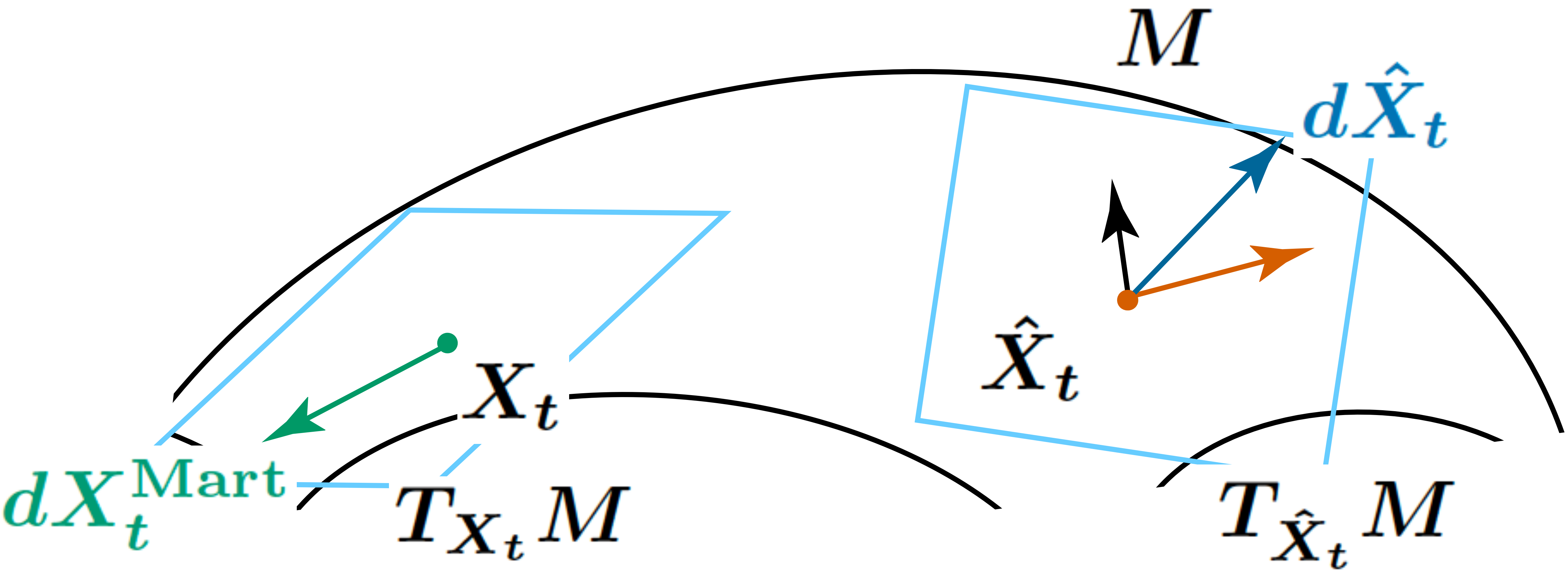
Construct $(X_t, \hat{X}_t)$

where $d\hat{X}_t^{\mathrm{Mart}} =$ "(local) reflection" of dX_t^{Mart}



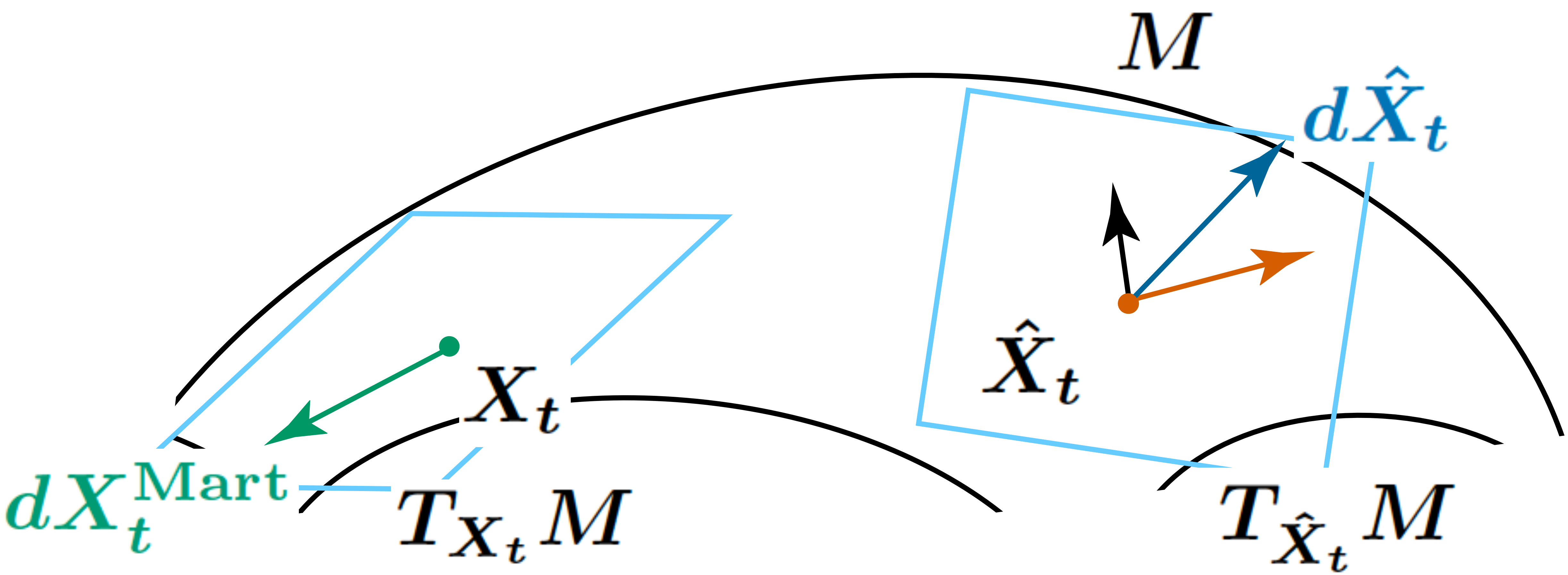
Construct $(X_t, \hat{X}_t)$

where $d\hat{X}_t^{\mathrm{Mart}} =$ "(local) reflection" of dX_t^{Mart}



Construct $(X_t, \hat{X}_t)$

where $d\hat{X}_t^{\mathrm{Mart}}$ = "(local) reflection" of dX_t^{Mart}



and apply the Itô formula

Obstruction

How frequently do $(X_t, \hat{X}_t)$ stay in $\mathrm{Cut}_{\mathrm{ST}}$?

★ $\mathrm{vol}_{g(t)}(\mathrm{Cut}_{g(t)}(x)) = 0$ is not sufficient

(When $\partial_t g(t) \equiv 0$, via SDE approach

[Kendall '86, Cranston '91, F.-Y. Wang '94/'05])

Obstruction

How frequently do $(X_t, \hat{X}_t)$ stay in $\mathrm{Cut}_{\mathrm{ST}}$?

$\bigstar$ $\mathrm{vol}_{g(t)}(\mathrm{Cut}_{g(t)}(x)) = 0$ is not sufficient

(When $\partial_t g(t) \equiv 0$, via SDE approach
[Kendall '86, Cranston '91, F.-Y. Wang '94/'05])

Advantage of our approach

"$\sigma_t \leq \hat{\rho}_t$" follows without extracting L_t