Duality on gradient estimates and Wasserstein controls

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1. Introduction

Coupling by parallel transport & Bakry-Émery's gradient estimate

X: complete Riemannian manifold

$T_t = \mathrm{e}^{t\Delta}$: heat semigroup

Equivalent conditions [von Renesse & Sturm '05]:

(i) $\forall x, y \in X$, $\exists$ coupled B.m. (B_t^x, B_t^y) starting from (x, y) s.t.

$$d(B_t^x, B_t^y) \le \mathrm{e}^{-Kt} d(x, y)$$

(ii) $|\nabla T_t f|(x) \le \mathrm{e}^{-Kt} T_t(|\nabla f|)(x)$

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(ii) $|\nabla T_t f|(x) \le \mathrm{e}^{-Kt} T_t(|\nabla f|)(x)$

(iii) $\mathrm{Ric} \ge K$

A hypoelliptic diffusion on the Heisenberg group

$X = \mathbb{R}^3$, $\boldsymbol{B}_t := (B_t^1, B_t^2, B_t^3)$ from (x, y, z),

$$B_t^1 := W_t^1, \qquad B_t^2 := W_t^2,$$

$$B_t^3 := z + \frac{1}{2}\int_0^t W_s^1 dW_s^2 - W_s^2 dW_s^1,$$

where (W_t^1, W_t^2): 2-dim. BM starting from (x, y)

- Formally, "Ric" is unbounded from below
- $\exists$ B.-É. est. [Driver & Melcher '05, H.Q.Li '06, Bakry, Baudoin, Bonnefont & Chafaï '08]

Question

Does there exist a coupling corresponding to the Bakry-Émery estimate?

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Answer

Yes, in a weak sense.
(by a general duality result below)

2. Framework and the main result

(X, d): Polish space

- $(P(x, \cdot))_{x \in X} \subset \mathcal{P}(X)$: Markov kernel

$$Pf(x) := \int_X f(y)\, dP(x, dy),$$

$$P^* \mu(A) := \int_X P(x, A) \mu(dx)$$

(e.g. $P(x, dy) = p_t(x, dy)$: heat semigroup)

- $\tilde{d}$: continuous distance functions on X
(e.g. $\tilde{d} = \mathrm{e}^{-Kt} d$)

$$\Pi(\mu,\nu) := \left\{ \pi \;\middle|\; \begin{array}{l} \pi(A \times X) = \mu(A), \\ \pi(X \times A) = \nu(A) \end{array} \right\}$$

(couplings of $\mu, \nu \in \mathcal{P}(X)$)

L^p-Wasserstein distance

For $p \in [1, \infty]$,

$$d_p^W(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \|d\|_{L^p(\pi)} \in [0,\infty]$$

L^p-Wasserstein control

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu,\nu) \qquad (C_p)$$

Gradient

$$|\nabla_d f|(x) := \limsup_{y \to x} \left| \frac{f(x) - f(y)}{d(x, y)} \right|,$$

$$\|\nabla_d f\|_\infty := \sup_{x \in X} |\nabla_d f|(x)$$

L^q-gradient estimate

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

for $\forall f \in C_b \cap \mathrm{Lip}_d$ when $q \in [1, \infty)$,

$$\|\nabla_{\tilde{d}} Pf\|_\infty \le \|\nabla_d f\|_\infty \qquad (G_\infty)$$

when $q = \infty$

Theorem [K. '10]

For $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q)$

(ii) Under Assumptions 1-4 below, $(G_q) \Rightarrow (C_p)$

v: pos. Radon measure on X with $\mathrm{supp}(v) = X$

<u>Assumption 1</u>:

d: geodesic metric, (X, d): locally compact

<u>Assumption 2</u>:

- local (uniform) volume doubling condition
- $(1, \rho)$-local Poincaré inequality ($\exists \rho \geq 1$)

<u>Assumption 3</u>:

$\tilde{d}$: geodesic metric

<u>Assumption 4</u>:

$$P(x, \cdot) \ll v, \; x \mapsto \frac{dP(x, \cdot)}{dv}(y): \text{ conti.}$$

Examples satisfying Assumptions 1-4

A canonical heat semigroup on:

- Complete Riemannian manifold with $\mathbf{Ric} \geq K_0$ (metric can depend on time, e.g. Ricci flow)
- Carnot groups
- Alexandrov spaces

Remarks (without Assumptions)

- For $p' > p$,
  $(C_{p'}) \Rightarrow (C_p)$ and $(G_{q'}) \Rightarrow (G_q)$

- $(C_1) \Leftrightarrow (G_\infty)$ is well known
  (via Kantorovich-Rubinstein formula)

- $(C_\infty) \Rightarrow (G_1)$ is essentially well known
  (Coupling method)

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Most interesting part:

$(G_q) \Rightarrow (C_p)$ for $p \in (1, \infty]$

3. Sketch of the proof

Idea of the proof of $(C_p) \Rightarrow (G_q)$

Recall:

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu, \nu) \qquad (C_p)$$

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

Take π: minimizer of $d_p^W(P^*\delta_x, P^*\delta_y)$

Remark: $\tilde{d}_p^W(\delta_x, \delta_y) = \tilde{d}(x, y)$

$$\Rightarrow \left| \frac{Pf(x) - Pf(y)}{\tilde{d}(x,y)} \right|$$

$$= \frac{1}{\tilde{d}(x,y)} \left| \int_{X \times X} (f(z) - f(w)) \pi(dz dw) \right|$$

$$(C_p) \boxed{\leq} \left\{ P \left(\sup_{d(\cdot,w) \leq r} \left| \frac{f(\cdot) - f(w)}{d(\cdot,w)} \right|^q \right) (x) \right\}^{1/q}$$

$$+ o(1)$$

for a suitable choice of r with $\lim_{\tilde{d}(x,y)\to 0} r = 0$ ■

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- (C_p) for $\forall p < \infty \Rightarrow (C_\infty)$

$\rightsquigarrow$ We may assume $p \in (1, \infty)$

- (C_p) for $\mu = \delta_x, \nu = \delta_y \Rightarrow (C_p)$

$\rightsquigarrow$ We show $\dfrac{d_p^W(P^*\delta_x, P^*\delta_y)^p}{p} \le \dfrac{\tilde{d}(x,y)^p}{p}$

Kantorovich duality

$$\frac{d_p^W(\mu,\nu)^p}{p} = \sup_{f \in C_b \cap \mathrm{Lip}_d} \left[\int_X Q_1 f \, d\mu - \int_X f \, d\nu \right]$$

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + \frac{t}{p} \left(\frac{d(x,y)}{t} \right)^p \right]$$

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$$\left(\begin{array}{l} \left[\begin{array}{l} \forall x, \forall y,\ g(x) - f(y) \leq \dfrac{1}{p} d(x,y)^p \\ \Rightarrow \dfrac{1}{p} \|d\|_{L^p(\pi)}^p \geq \displaystyle\int_X g \, d\mu - \int_X f \, d\nu \end{array} \right] \\ \Rightarrow \text{“}\geq\text{”} \end{array} \right)$$

$Q_t f$: Hamilton-Jacobi semigroup

Under Assumptions 1 & 2,

- $Q_\cdot f(x)$: Lipschitz, $Q_t f(\cdot)$: d-Lipschitz
- Hamilton-Jacobi equation

$$\partial_t Q_t f = -\frac{1}{q} |\nabla_d Q_t f|^q \quad v\text{-a.e.}$$

([Lott & Villani '07]

[Balogh, Engoulatov, Hunziker & Maasalo])

Assumption 3:

$$\begin{cases} \tilde{\gamma} : [0,1] \to X \quad \tilde{d}\text{-minimal geodesic}, \\ \tilde{\gamma}_0 = y,\ \tilde{\gamma}_1 = x, \\ \tilde{d}(\tilde{\gamma}_s, \tilde{\gamma}_t) = |t-s|\tilde{d}(x,y) \end{cases}$$

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$$\frac{d_p^W(P^*\delta_x, P^*\delta_y)^p}{p} = \sup_f \left[PQ_1 f(x) - Pf(y)\right]$$

interpolation $$\boxed{=} \sup_f \left[\int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t)) dt\right]$$

$$\partial_t(PQ_t f(\tilde{\gamma}_t))$$

$$\left(\text{“=”}\ P(\partial_t Q_t f)(\tilde{\gamma}_t) + \langle \nabla PQ_t f(\tilde{\gamma}_t), \dot{\tilde{\gamma}}_t \rangle\right)$$

HJ eq.
Ass. 4

$$\leq -\frac{1}{q} P(|\nabla_d Q_t f|^q)(\tilde{\gamma}_t)$$

$$+ \tilde{d}(x,y) \left|\nabla_{\tilde{d}} PQ_t f\right| (\tilde{\gamma}_t)$$

(G_q)

$$\leq \tilde{d}(x,y)\sigma - \frac{1}{q}\sigma^q \leq \frac{\tilde{d}(x,y)^p}{p}$$

$$\left(\sigma := P(|\nabla_d Q_t f|^q)(\tilde{\gamma}_t)^{1/q}\right) \qquad \blacksquare$$

4. Application: Hörmander-type operators on a Lie group

3-dim. Heisenberg group

$X := \mathbb{R}^3$, v: Lebesgue

$$(x, y, z) \cdot (x', y', z')$$
$$:= \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')\right)$$

$$X_1 := \partial_x - \frac{y}{2}\partial_z \,, \quad X_2 := \partial_y + \frac{x}{2}\partial_z$$

$$A := \frac{1}{2}\left(X_1^2 + X_2^2\right),$$

$P := T_t = \mathrm{e}^{tA}$ (t: fixed)

$|\Gamma f| := |X_1 f|^2 + |X_2 f|^2$: carré du champ

<u>L^q-gradient estimate</u>

$\exists K_q > 1$,

$$|\Gamma T_t f|(x) \le K_q T_t(|\Gamma f|^{q/2})(x)^{2/q} \qquad (G_q^*)$$

- $q > 1$: [Driver & Melcher '05]
- $q = 1$: [H.-Q. Li '06],
  [Bakry, Baudoin, Bonnefont & Chafaï '08]

Carnot-Caratheodory distance

For $V \in T_x X$,

$$|V| := \begin{cases} \sqrt{a_1^2 + a_2^2} & \text{if } V = a_1 X_1 + a_2 X_2, \\ \infty & \text{otherwise.} \end{cases}$$

$$d(x, y) := \inf \left\{ \int_0^1 |\dot{\gamma}_s| ds \;\middle|\; \begin{array}{l} \gamma_0 = x, \\ \gamma_1 = y \end{array} \right\}$$

Proposition

(i) $(X, d, v; P)$ satisfies Assumptions 1-4

(ii) $(G_q^*) \Rightarrow (G_q)$

Corollary

$(G_q^*) \Rightarrow (C_p)$ for $p \in [1, \infty]$

(C_∞): For each $t > 0$ and $(\boldsymbol{B}_0, \tilde{\boldsymbol{B}}_0)$,
$\exists$ a coupling $(\boldsymbol{B}_t, \tilde{\boldsymbol{B}}_t)$ of (B_t^1, B_t^2, B_t^3) s.t.

$$d(\boldsymbol{B}_t, \tilde{\boldsymbol{B}}_t) \le K_1 d(\boldsymbol{B}_0, \tilde{\boldsymbol{B}}_0) \quad \mathbb{P}\text{-a.s.}$$

Remark

$\exists C_1, C_2 > 0$ s.t.

$$C_1 \|\boldsymbol{b}^{-1}\boldsymbol{a}\| \le d(\boldsymbol{a}, \boldsymbol{b}) \le C_2 \|\boldsymbol{b}^{-1}\boldsymbol{a}\|,$$

where $\|(x, y, z)\| = \left((x^2 + y^2)^2 + z^2\right)^{1/4}$

Extension of (G_q^*)

- [Melcher '08]:
  X: general, $q > 1$ ($K_q(t) \equiv K_q$ if X: nilp.)
- [Eldredge '10]:
  X: group of type H, $q = 1$, $K_q(t) \equiv K_q$
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  $X = SU(2)$, $q > 1$, $K_q(t) = K_q \mathrm{e}^{-t}$

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Proposition (and Corollary) is still valid

$\Rightarrow$ Our thm also implies (C_p) in these cases

5. Application: heat semigroup on Alexandrov spaces

Alexandrov sp.: metric space

with “sectional curvature $\geq k$”

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Heat flow on cpt. Alex. sp., as a gradient flow of

- Dirichlet energy in L^2 ($\Rightarrow \exists p_t(x,y)$: conti.) [Kuwae, Machigashira & Shioya '01]

- relative entropy on $(\mathcal{P}_2(X), d_2{}^W)$ ($\Rightarrow (C_2)$) [Savaré '07, Ohta '09]

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$\Rightarrow (G_2) \Rightarrow p_t(x, \cdot)$: Lipschitz

6. Duality under different assumptions (Work in progress)

Subgradient

$$|\nabla_d^- f|(x) := \lim_{r\downarrow 0} \sup_{y\in B_r(x)} \left[\frac{f(x)-f(y)}{d(x,y)}\right]_+$$

$$|\nabla_{\tilde{d}}^- Pf|(x) \le P(|\nabla_d^- f|^q)(x)^{1/q} \qquad (G_q^-)$$

Theorem [K.]

For $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q^-)$

(ii) Under Assumptions 5-7, $(G_q^-) \Rightarrow (C_p)$

Assumption 5 d, $\tilde{d}$: geodesic distance

Assumption 6

$\exists D > 0$ s.t. for $\forall \gamma$: d-min. geod., $\forall \boldsymbol{\lambda} \in [0, 1]$,

$$
\begin{aligned}
d(x, \gamma(\boldsymbol{\lambda}))^2 \\
&\geq (1 - \boldsymbol{\lambda}) d(x, \gamma(0))^2 + \boldsymbol{\lambda} d(x, \gamma(1))^2 \\
&\quad - D \boldsymbol{\lambda} (1 - \boldsymbol{\lambda}) d(\gamma(0), \gamma(1))^2
\end{aligned}
$$

Assumption 7

$\exists r > 0$ s.t. if $d(x, y) < r$ then

$\exists \gamma$: d-min. geod. from x with $|\gamma| = r$ and $y \in \gamma$

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Examples

- X: any cpl. Riem. mfd., d: Riem. distance
  - Ass. 6 $\Leftarrow$ local lower sect. curv. bound
  - Ass. 7 $\Leftarrow$ local positivity of inj. radius

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For Ass. 6 & 7, it is sufficient to hold locally.

Examples

- X: any cpl. Riem. mfd., d: Riem. distance
  - Ass. 6 $\Leftarrow$ local lower sect. curv. bound
  - Ass. 7 $\Leftarrow$ local positivity of inj. radius
- X: Wiener space, d: Cameron-Martin norm
  (No absolute continuity is necessary for P!)