Optimal transport and coupled diffusion by reflection

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[joint work with K.-Th. Sturm (Bonn)]

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1. Introduction

M: complete Riemannian manifold, $\dim M \geq 2$

$X^x(t)$: Brownian motion on M with $X(0) = x$

$$\mathrm{Ric} \geq K$$

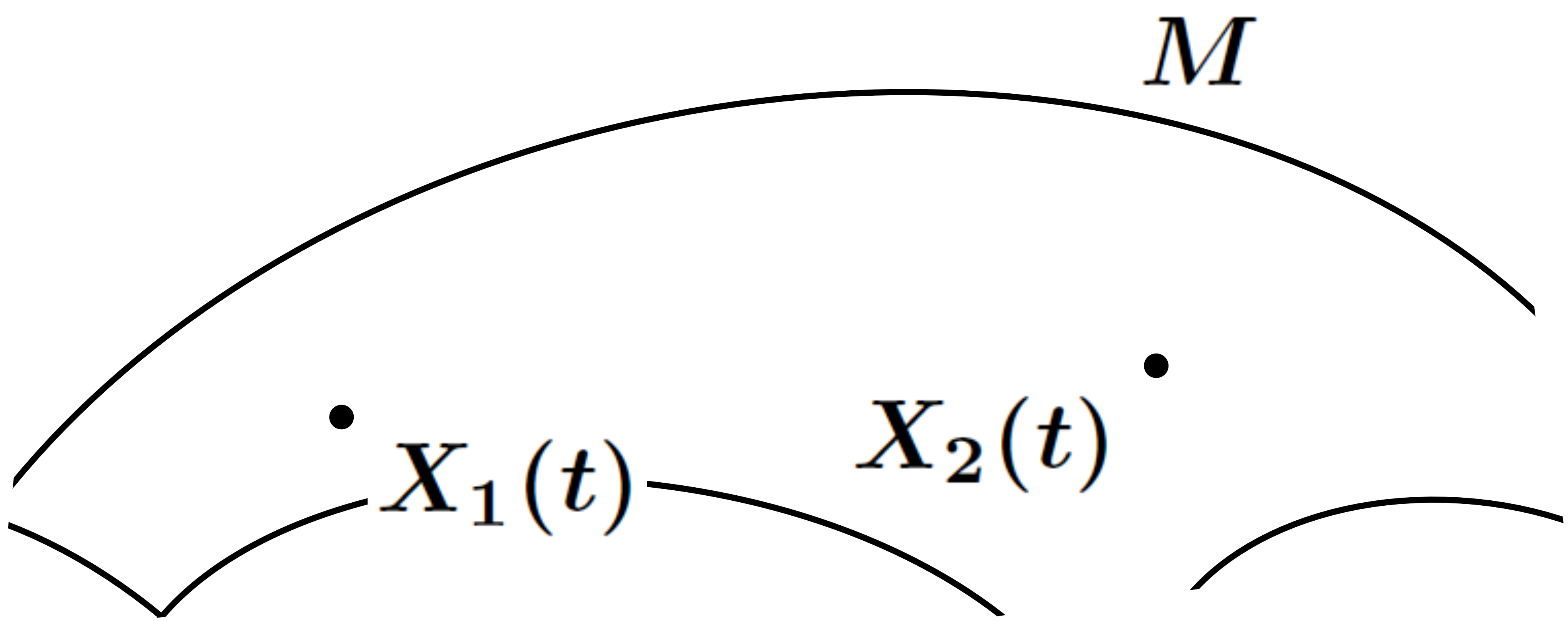
$$\Downarrow$$

Good control of couplings $(X_1(t), X_2(t))$ of two BMs $X^{x_1}(t)$ & $X^{x_2}(t)$

$(X_1(t), X_2(t))$: a coupling of $X^{x_1}(t)$ & $X^{x_2}(t)$

$\overset{\text{def}}{\Leftrightarrow} (X_i(t))_{t \geq 0} \overset{d}{=} (X^{x_i}(t))_{t \geq 0}$ $(i = 1, 2)$

$(X_1(t), X_2(t))$: coupling by parallel transport [F.-Y.Wang '97, von Renesse '04, Arnaudon & Coulibaly & Thalmaier '09, K., . . .]



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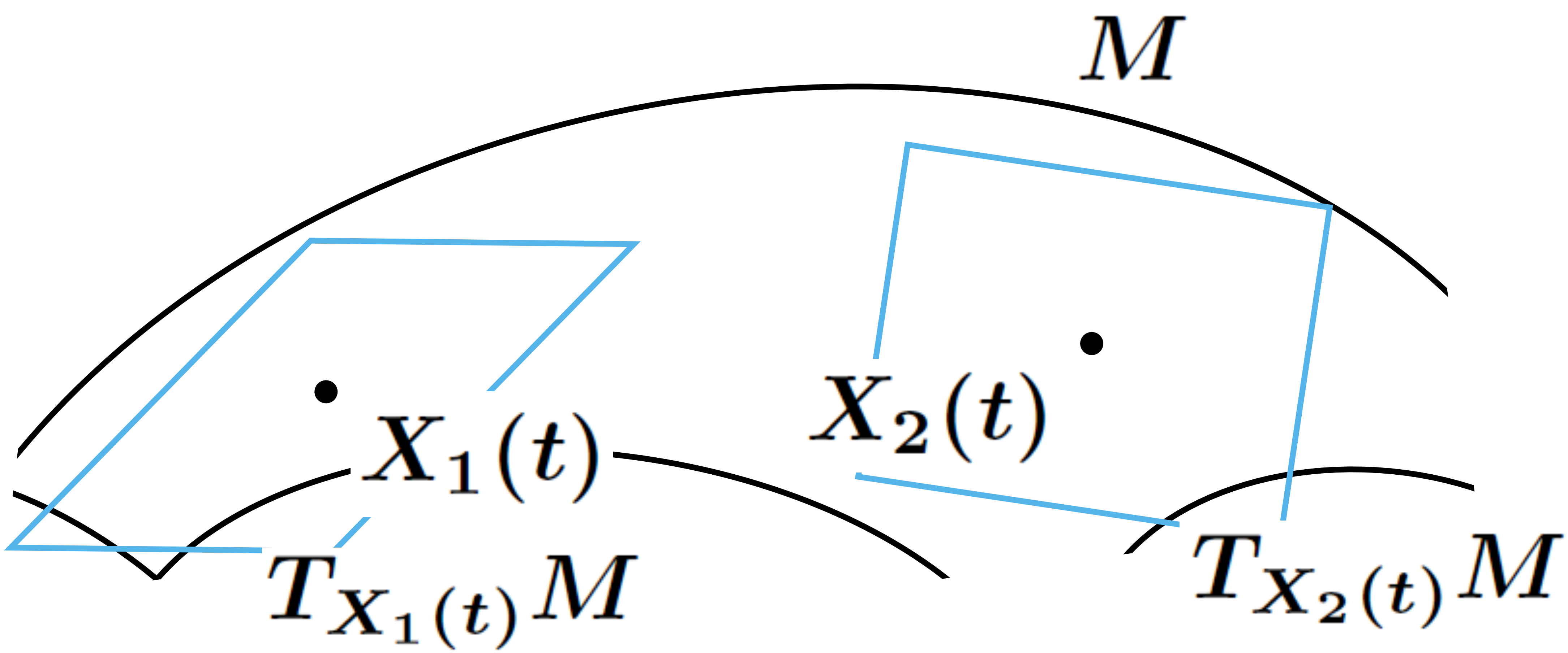

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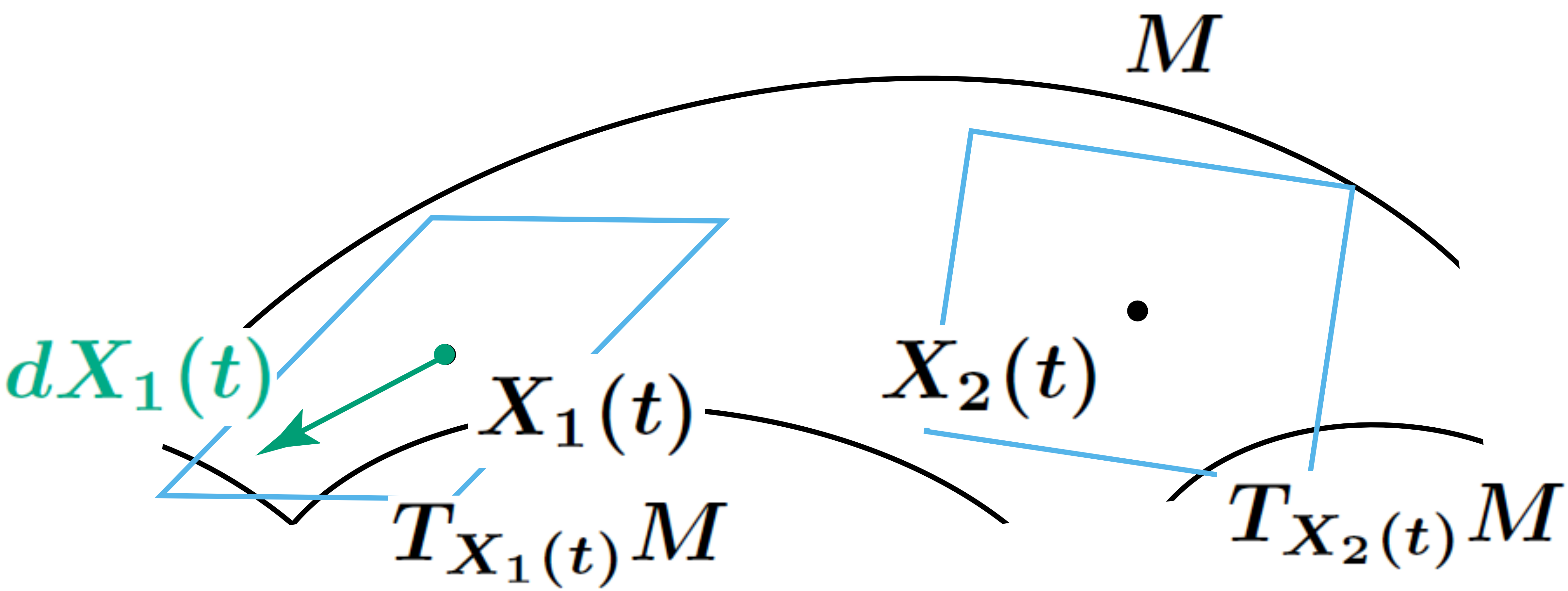

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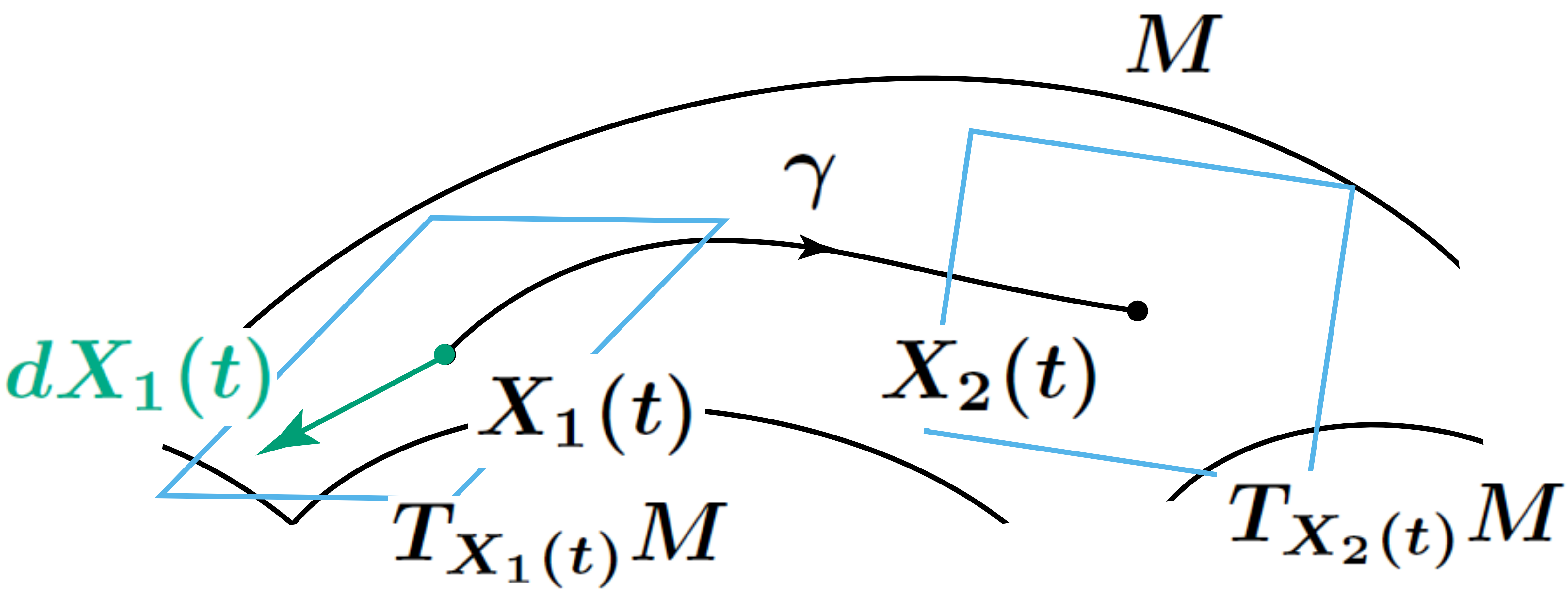

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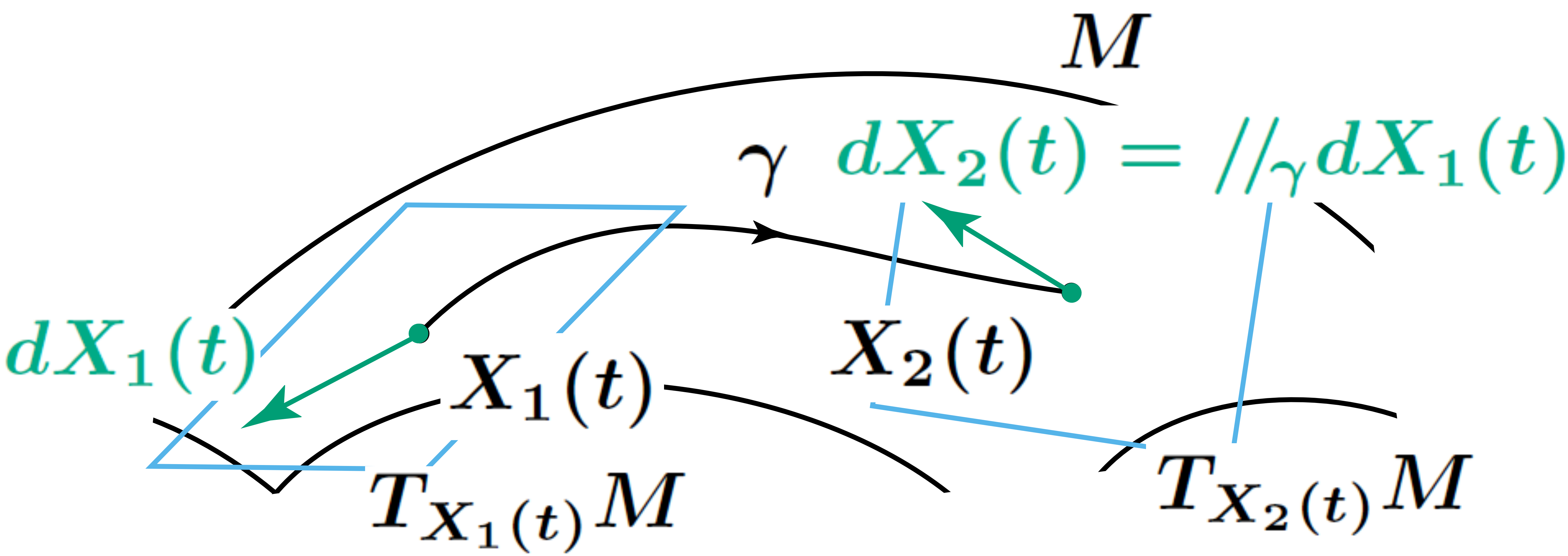
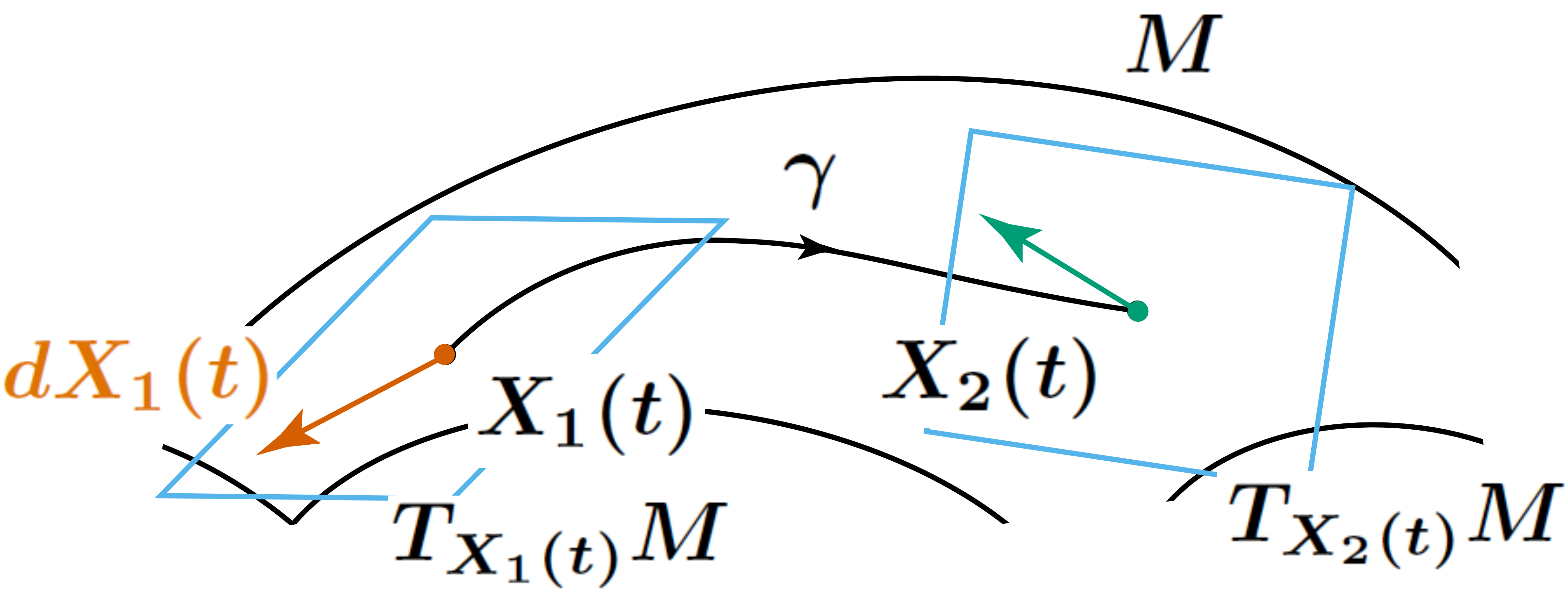


$(X_1(t), X_2(t))$: coupling by reflection
[Kendall '86, Cranston '91, F.-Y.Wang '97, '05, von Renesse '04, K., ...]



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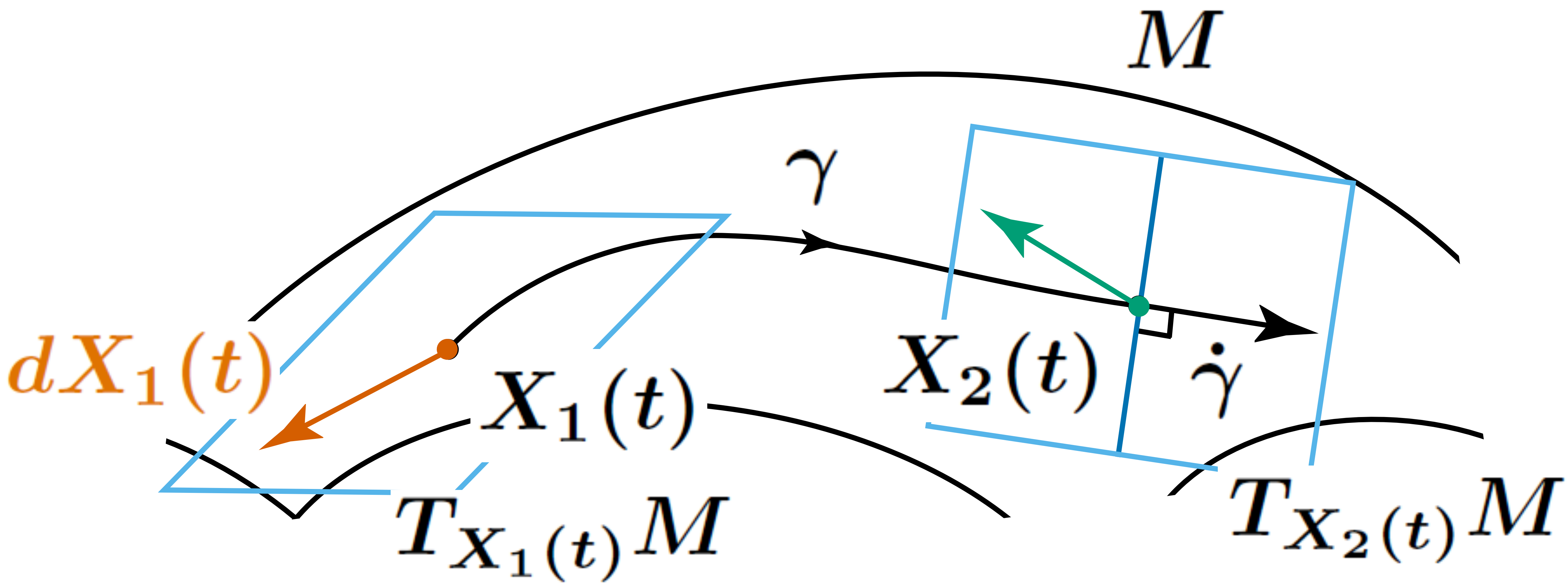

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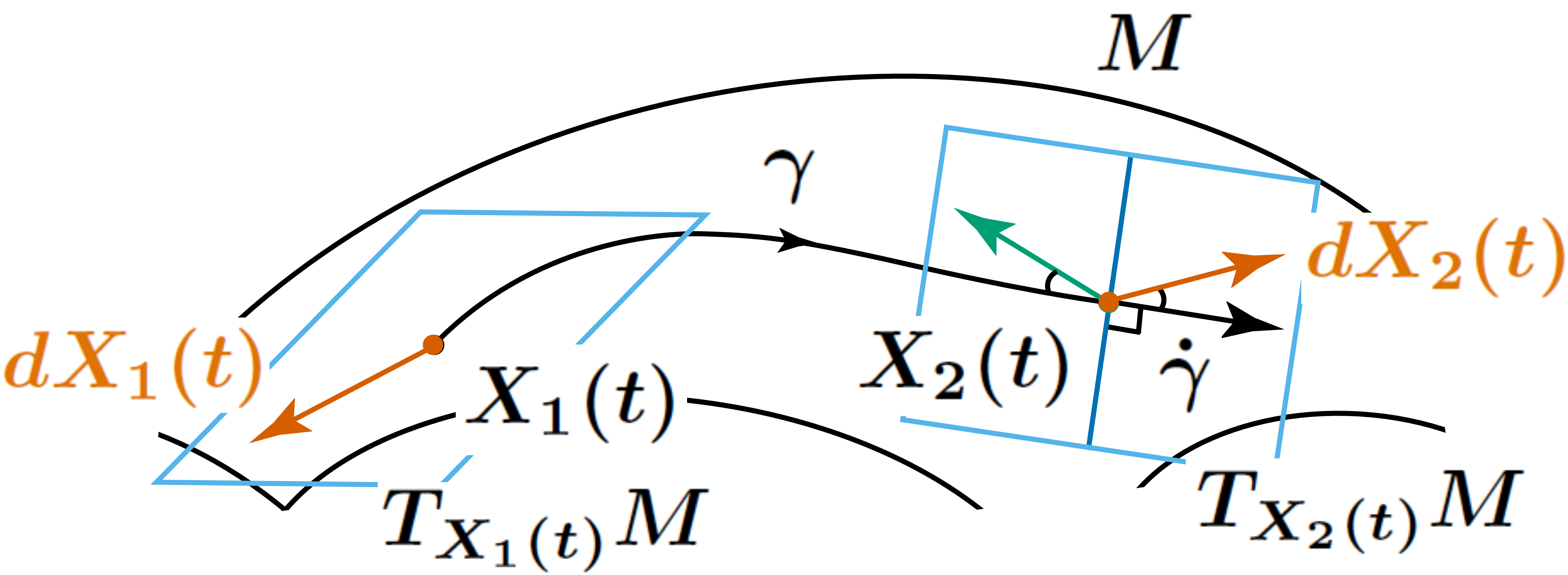
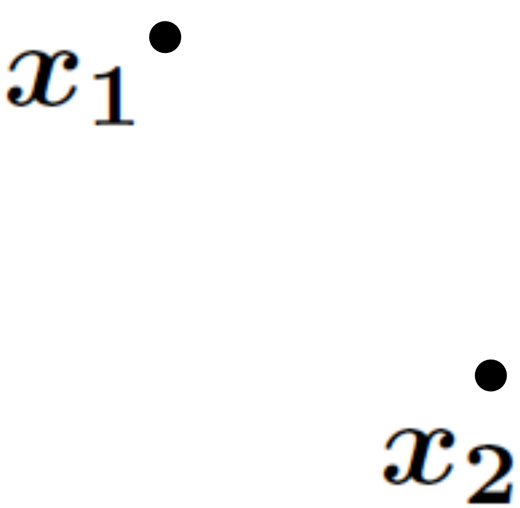


Example ($M = \mathbb{R}^n$)

parallel transport



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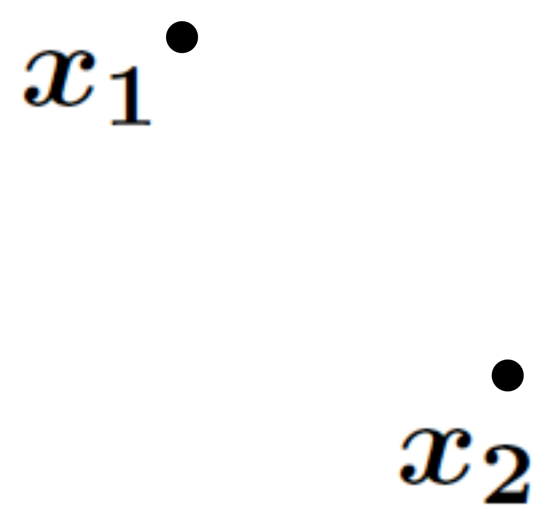

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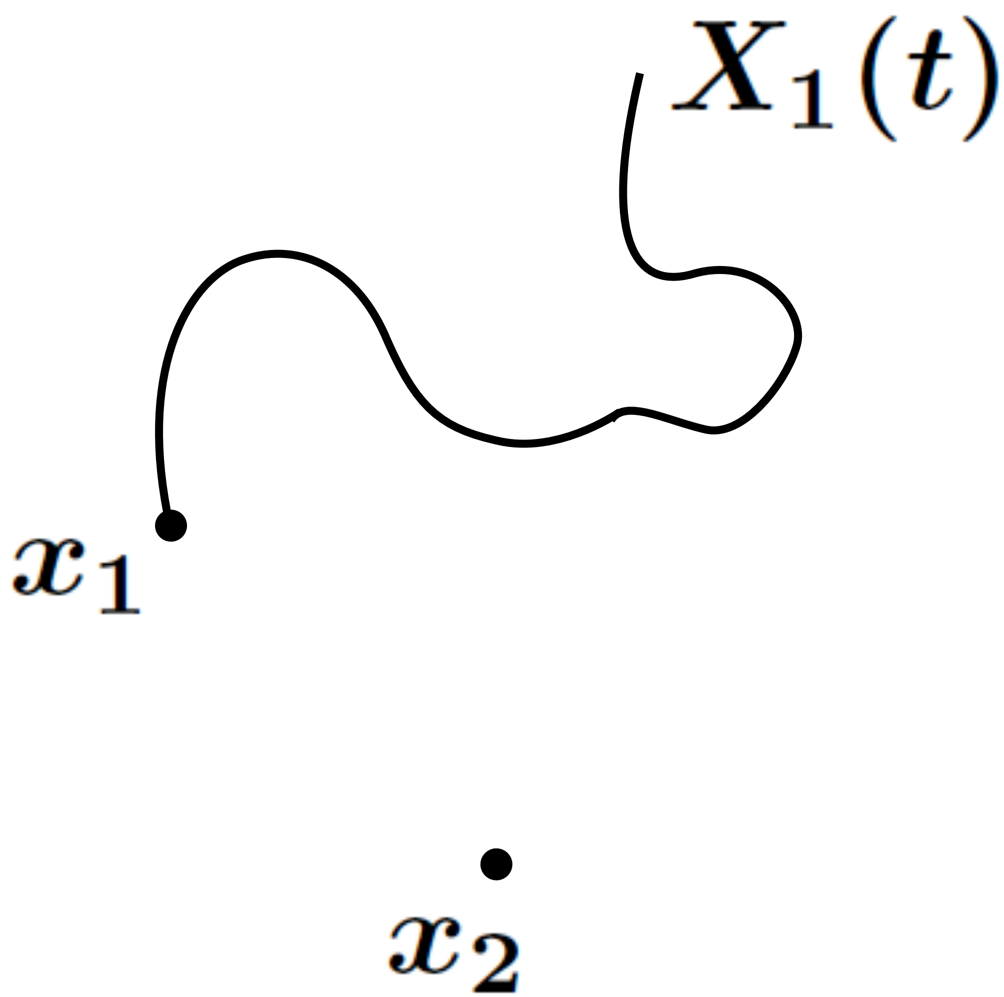

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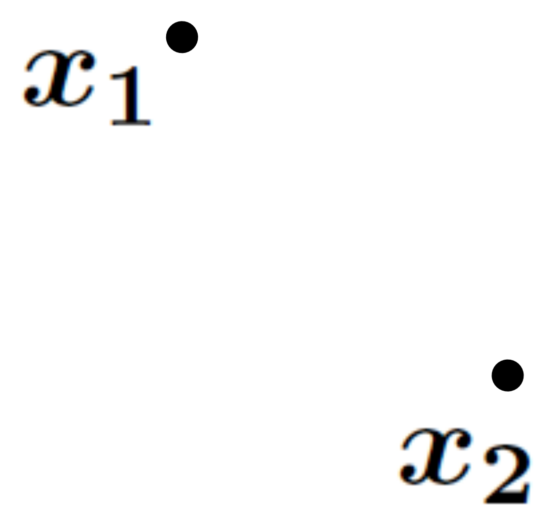

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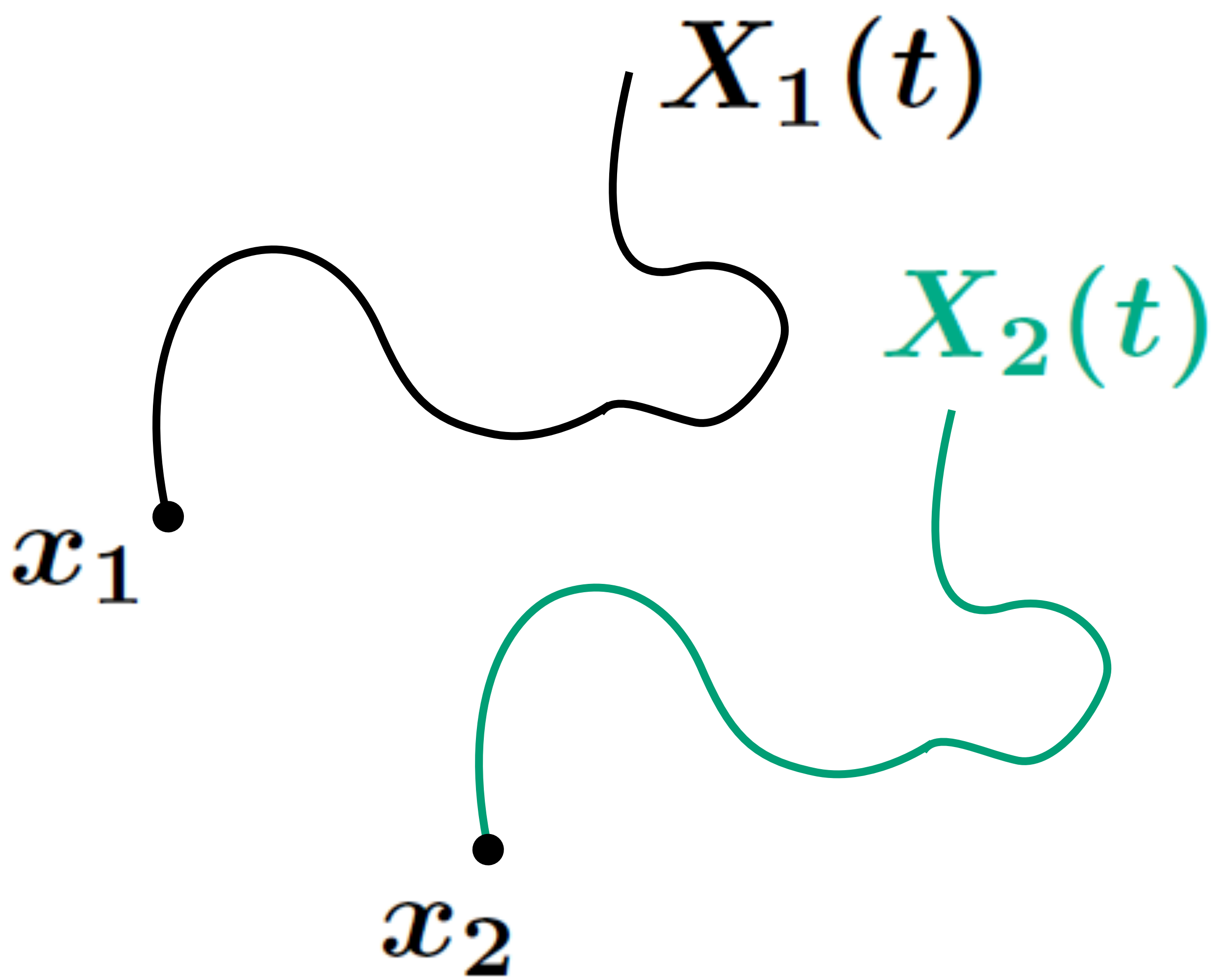

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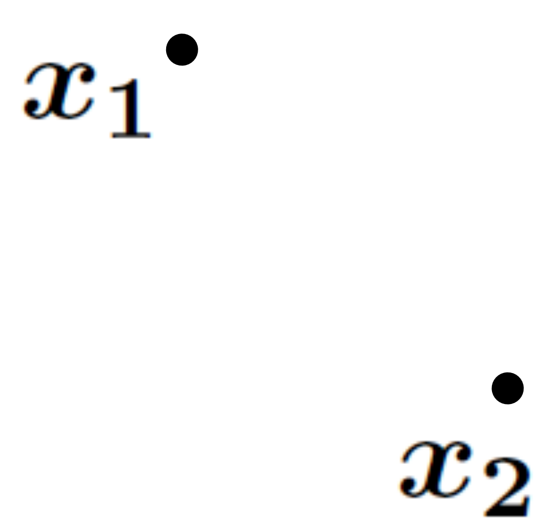

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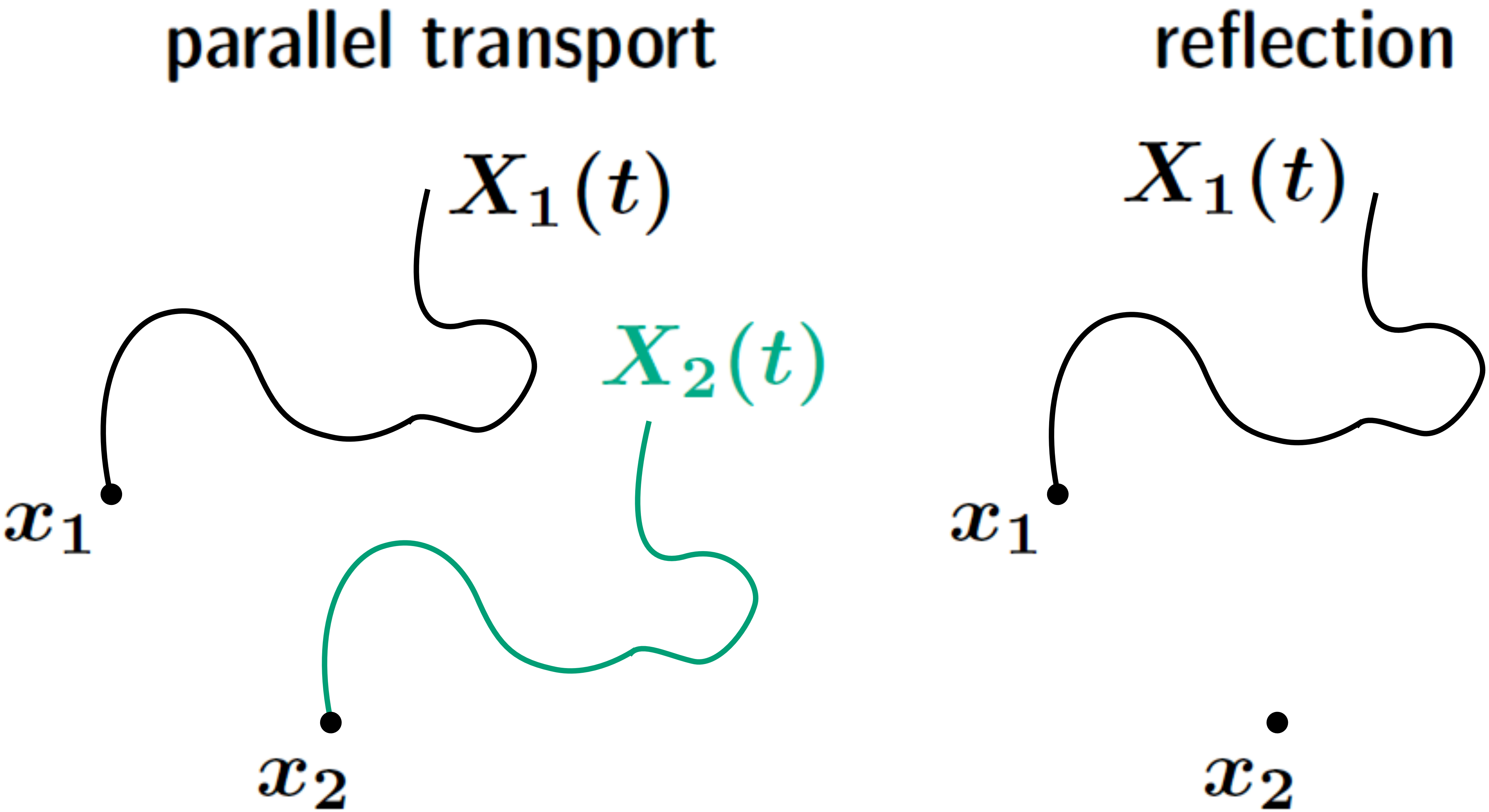

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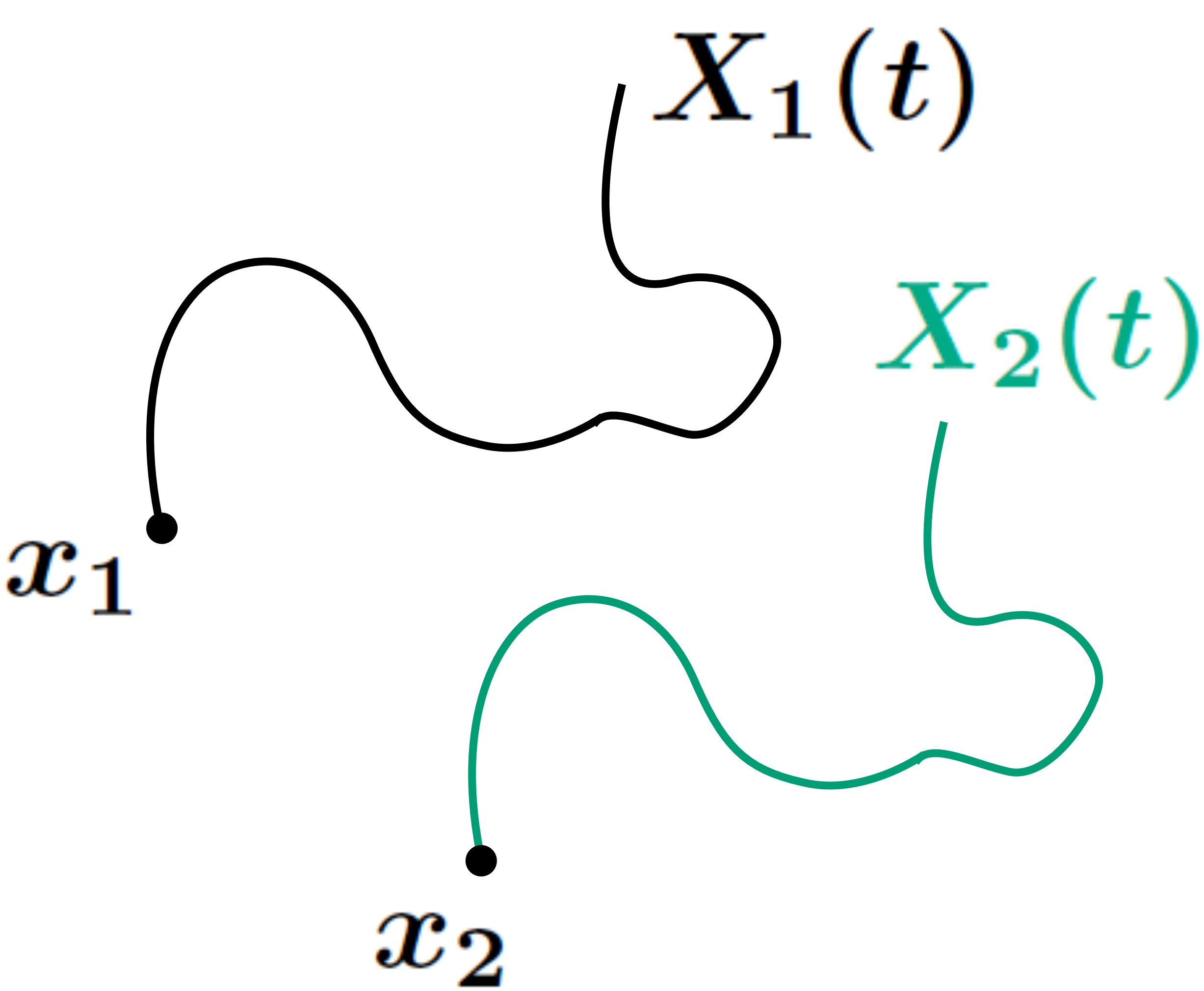

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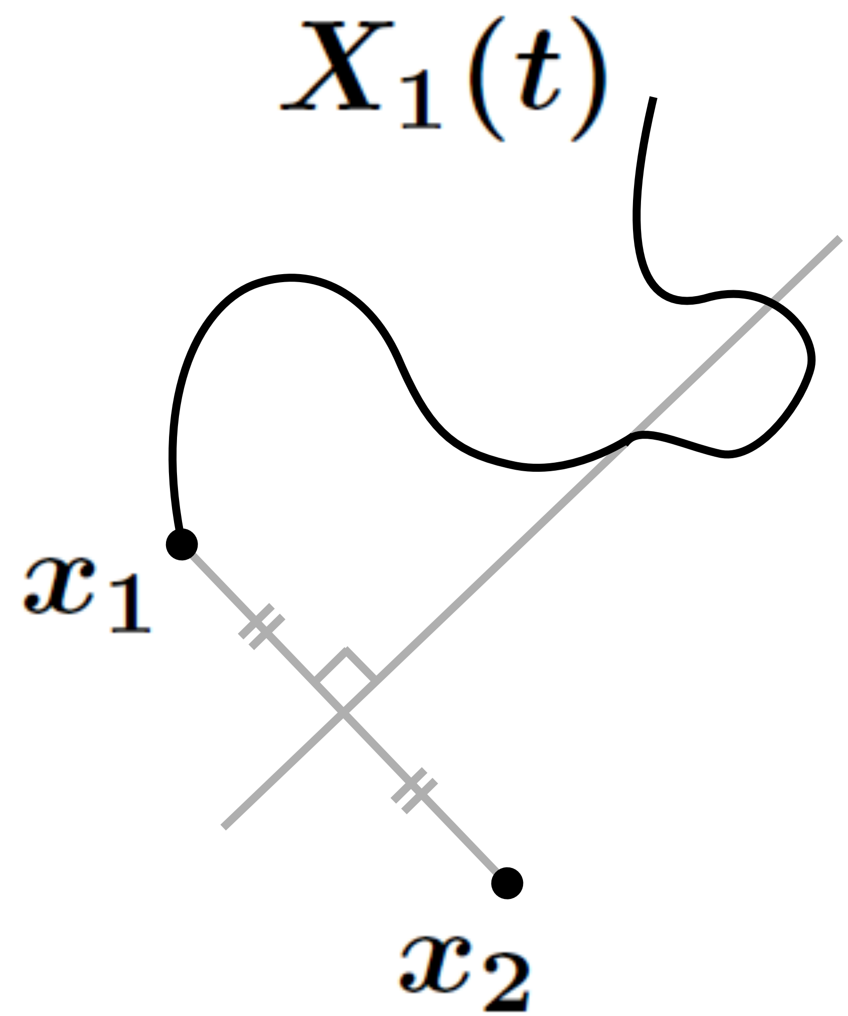

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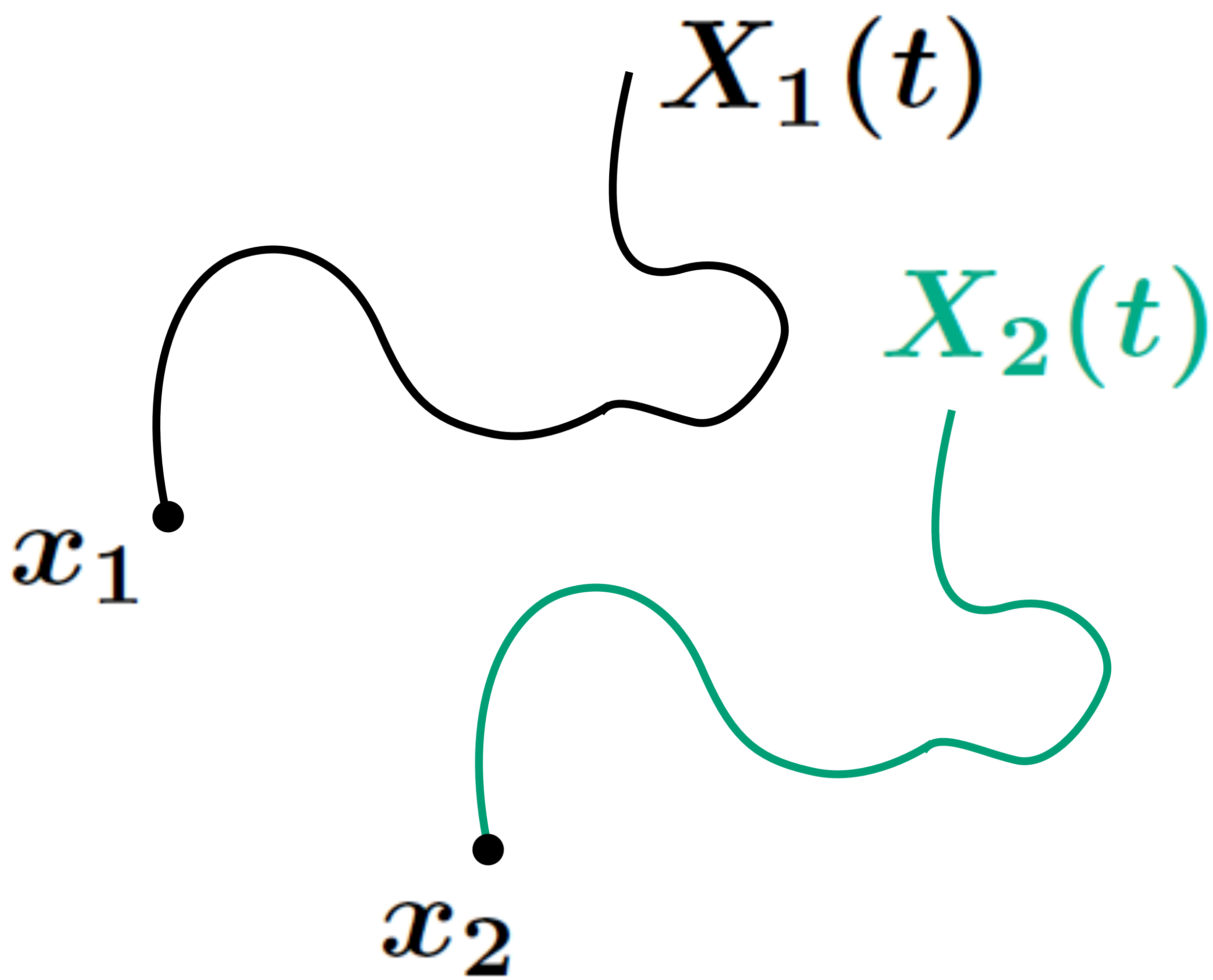

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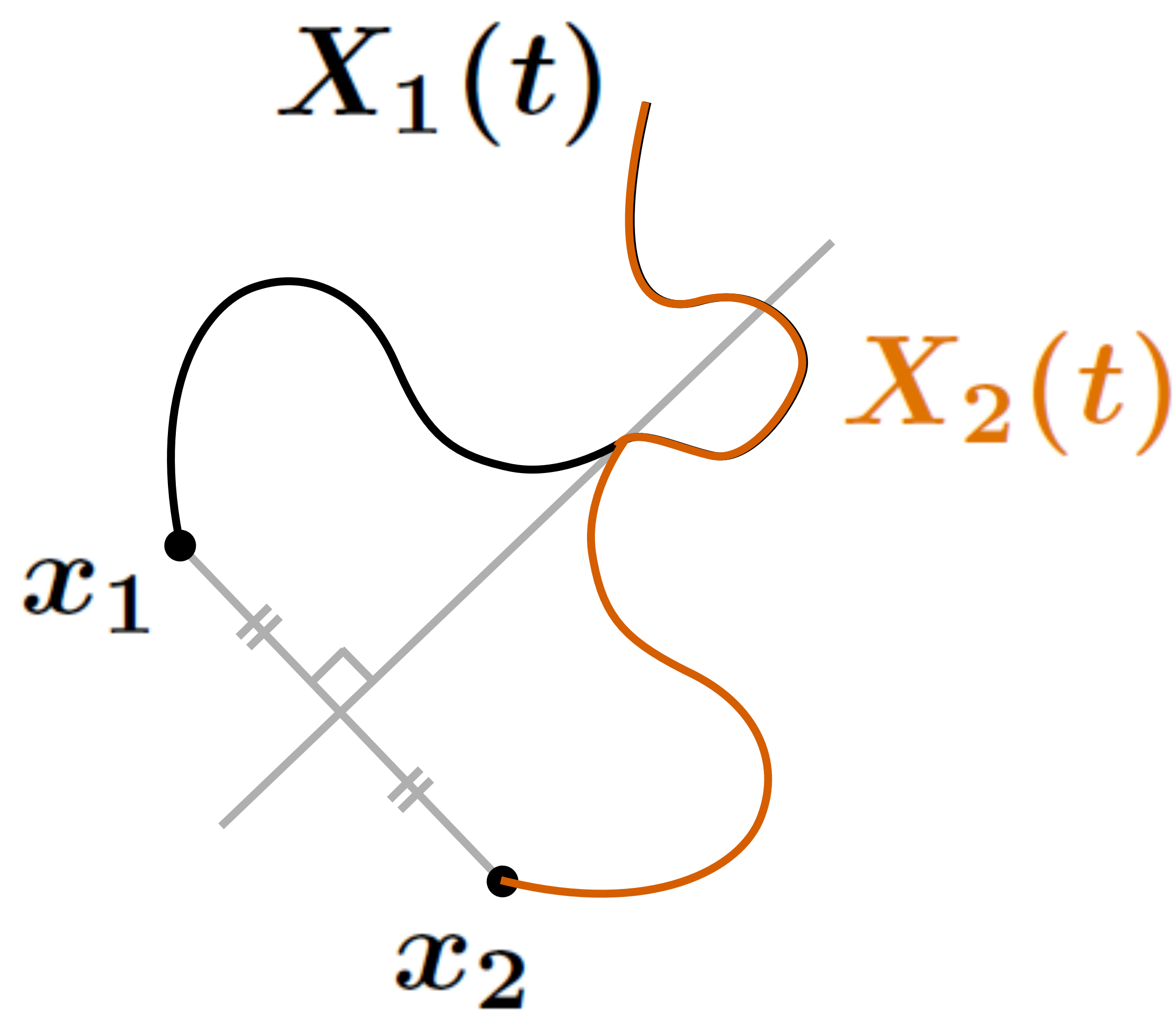

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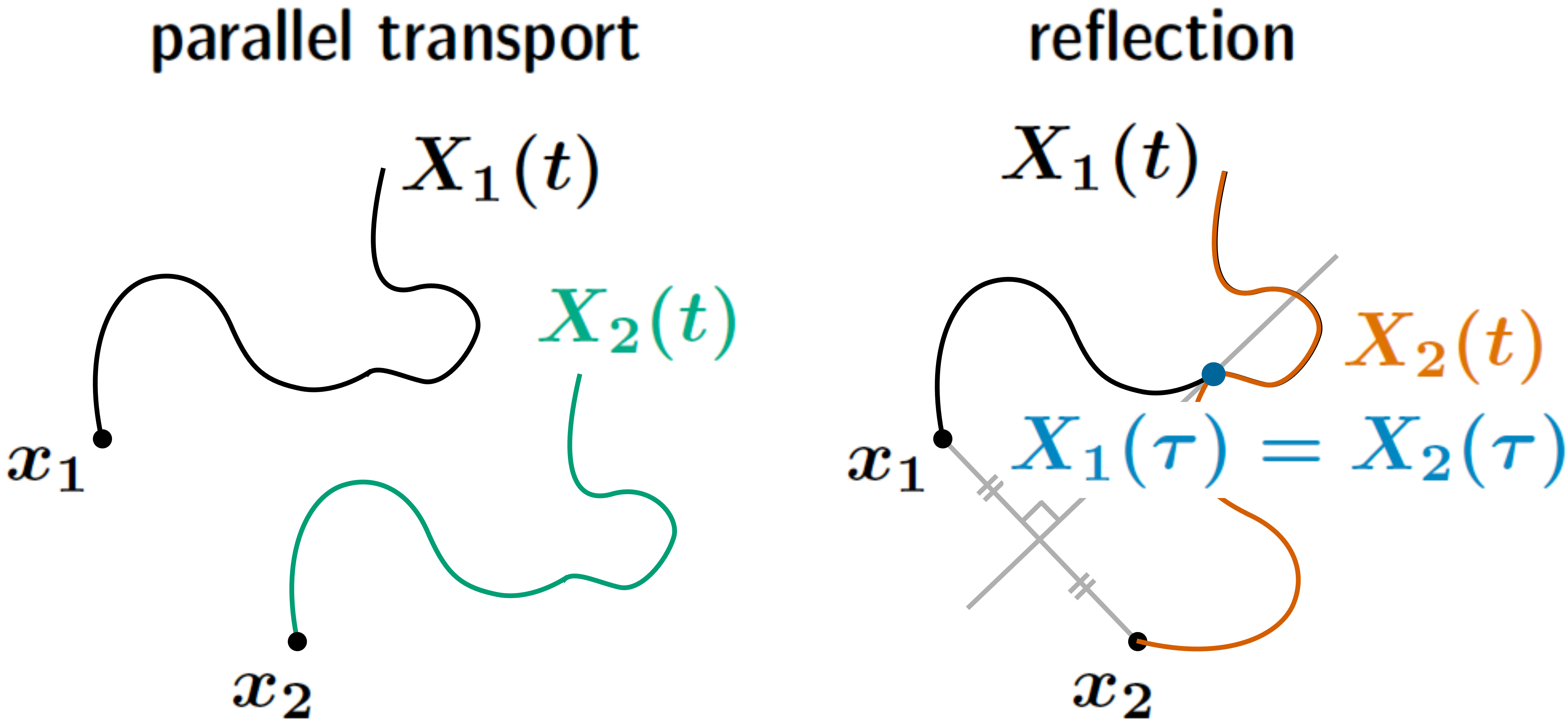

$$\tau := \inf\{t \geq 0 \mid X_1(s) = X_2(s)\, \forall s \geq t\}$$

What is obtained?

- Coupling by parallel transport

- Coupling by reflection

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$\Rightarrow$ Pathwise contraction:
$\mathrm{e}^{Kt} d(X_1(t), X_2(t)) \searrow$ $\mathbb{P}$-a.s.

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What is obtained? (P_t: heat semigroup)

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  $\Rightarrow$ Pathwise contraction:
  $\mathrm{e}^{Kt} d(X_1(t), X_2(t)) \searrow$ $\mathbb{P}$-a.s.

  $\Rightarrow$ Bakry-Émery type gradient estimates:
  $|\nabla P_t f|^q \le \mathrm{e}^{-tqK} P_t(|\nabla f|^q)$

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$\Rightarrow$ Estimate to $\mathbb{P}[\tau > t]$

$\Rightarrow$ $\|\nabla P_t f\|_\infty \le C_{K,N}(t) \operatorname{osc}(f)$

Optimal transportation cost

For $c : M \times M \to \mathbb{R}$, $\mu_1, \mu_2 \in \mathcal{P}(M)$,

$$\mathcal{T}_c(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_2)} \int_{M \times M} c \, d\pi$$

$$\begin{aligned} \pi \in \Pi(\mu_1, \mu_2) \overset{\text{def}}{\Leftrightarrow} \; & \pi \in \mathcal{P}(M \times M), \\ & \pi(A \times M) = \mu_1(A), \\ & \pi(M \times A) = \mu_2(A) \end{aligned}$$

⋆ $(X_1(t), X_2(t))$: coupling of $X^{x_1}(t)$ & $X^{x_2}(t)$
$\Rightarrow \mathbb{P}^{(X_1(t), X_2(t))} \in \Pi(\mathbb{P}^{X^{x_1}(t)}, \mathbb{P}^{X^{x_2}(t)})$

Formulation via optimal transportation

- Coupling by parallel transport

- Coupling by reflection

Formulation via optimal transportation

- Coupling by parallel transport

$\Rightarrow \mathcal{T}_{(\mathrm{e}^{Kt}d)^p}(P_t^*\mu_1, P_t^*\mu_2) \searrow$ in t.

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$$\Rightarrow \mathcal{T}_{(\mathrm{e}^{Kt}d)^p}(P_t^*\mu_1, P_t^*\mu_2) \searrow \text{ in } t.$$

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- Coupling by reflection

$\Rightarrow$ (monotonicity of a transportation cost)

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Formulation via optimal transportation

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$\Rightarrow \mathcal{T}_{(\mathrm{e}^{Kt} d)^p}(P_t^* \mu_1, P_t^* \mu_2) \searrow$ in t.

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$\Rightarrow$ (monotonicity of a transportation cost)

$\Leftrightarrow$ a gradient estimate for $P_t f$

$\Rightarrow \|\nabla P_t f\|_\infty \le C_{N,K}(t) \operatorname{osc}(f)$

Motivation: Stochastic analysis on singular spaces

- Construction of coupling by reflection
  $\Leftarrow$ differentiable structure on M
  (How do we formulate it on singular sp.'s?)

- "Monotonicity of transportation cost" is robust
  $\Rightarrow$ Stable under Gromov-Hausdorff conv.

- Potential connection with gradient flow theory

$$\left(\text{e.g. } \text{"}\mathrm{Hess\,Ent} \geq K\text{"} \Rightarrow \mathcal{T}_{(\mathrm{e}^{Kt}d)^2}(P_t^*\mu_1, P_t^*\mu_2) \searrow \right)$$

2. Framework and main results

General framework

Z: C^1-vector field

$X^x(t)$: diffusion process associated with $\frac{1}{2}\Delta + Z$

($X(t)$: BM $\Leftrightarrow Z = 0$)

Bakry-Émery Ricci tensor:

$n := \dim M$. For $N \in [n, \infty]$,

$$\mathrm{Ric}^{Z,N} := \mathrm{Ric} - 2(\nabla Z)^{\mathrm{sym}} - \frac{4}{N-n} Z \otimes Z$$

Assumption

Let $K \in \mathbb{R}$. Either (i) or (ii) holds:

(i) $\mathrm{Ric}^{Z,N} \geq K$

(ii) $N = \infty$, g depends on t,

$$\mathrm{Ric}^{Z,\infty}_{g(t)} \geq \partial_t g(t) + K$$

Remark

- (i) $\Leftrightarrow$ Bakry-Émery's curv.-dim.cond.

When $Z = 0$,

- (i) $\Leftrightarrow$ $n \leq N$ and $\mathrm{Ric} \geq K$
- $K = 0$ & "$=$" in (ii) $\Rightarrow$ backward Ricci flow

Set $\bar{R} := \sqrt{\frac{N-1}{K \vee 0}}\pi$

<u>Remark</u> [K. '11 preprint]

- [Bonnet-Myers] $\mathrm{diam}(M) \leq \bar{R}$
- [Max. diam.] When $K > 0$ & $N < \infty$,

$$\mathrm{diam}(M) = \bar{R} \Leftrightarrow N = n, Z = 0, \quad M \overset{\text{isom}}{\simeq} \mathbb{S}^n_K$$

(Z can be of non-gradient type)

Theorem 1 [K. & Sturm]

$(X_1(t), X_2(t))$: a coupling by refl. of two BMs.

$\Rightarrow \exists \varphi = \varphi^{N,K} : [0,\infty) \times \overline{[0,\bar{R})} \to [0,1]$

s.t. for $t > 0$,

$$\mathbb{E}[\varphi_{t-s}(d_s(X_1(s), X_2(s)))] \searrow$$

in $s \in [0,t]$

Theorem 2 [ibid.]

For $t > 0$, $\mu_1, \mu_2 \in \mathcal{P}(M)$,

$$\mathcal{T}_{\varphi_{t-s}(d_s)}(P_s^* \mu_1, P_s^* \mu_2) \searrow \text{ in } s \in [0,t]$$

Definition of $\varphi_t^{K,N}(a)$ (for $N \in \mathbb{N}$)

$$\varphi_t^{K,N}(a) := \frac{1}{2} \left\| \tilde{P}_t^* \delta_{\tilde{x}} - \tilde{P}_t^* \delta_{\tilde{y}} \right\|_{\mathrm{TV}}$$

- $\tilde{P}_t$: heat semigr. on the spaceform $\mathbb{M}_{K,N}$ ($\mathbb{M}_{N,K}$: sphere, Euclidean sp. or hyperbolic sp.)
- $d(\tilde{x}, \tilde{y}) = a$

Comparison functions / processes

$$s_K(u) := \frac{1}{\sqrt{K}} \sin(\sqrt{K} u),$$

$$c_K(u) := \cos(\sqrt{K} u)$$

$$\Psi_{K,N}(u) := -K \frac{s_{K/(N-1)}(u/2)}{c_{K/(N-1)}(u/2)}$$

- $\rho^a(t)$: $\overline{(-\bar{R}, \bar{R})}$-valued process, $\rho^a(0) = a$,

$$d\rho^a(t) = 2d\beta(t) + \Psi(\rho(t))dt,$$

($\beta(t)$: BM on $\mathbb{R}$)

Definition of $\varphi_t^{K,N}(a)$ (general case)

P_t^ρ: transition semigroup of $(\frac{1}{2}\rho_t^a)_a$

$$\varphi_t^{K,N}(a) := \frac{1}{2}\left\|(P_t^\rho)^*\delta_{a/2} - (P_t^\rho)^*\delta_{-a/2}\right\|_{\mathrm{TV}}$$

Example of $\rho^a(t)$

- $K = 0$

  $\Rightarrow \rho^a(t) = a + 2\beta(t)$

  (1-dim. BM, independent of N)

- $N = \infty$

  $\Rightarrow d\rho^a(t) = 2d\beta(t) - \frac{K}{2}\rho^a(t)dt$

  (Ornstein-Uhlenbeck processes)

Properties of φ_t

- $\varphi_t \nearrow$, concave, $\varphi_t(0) = 0$ ($\Rightarrow \varphi_t(d)$: dist.)
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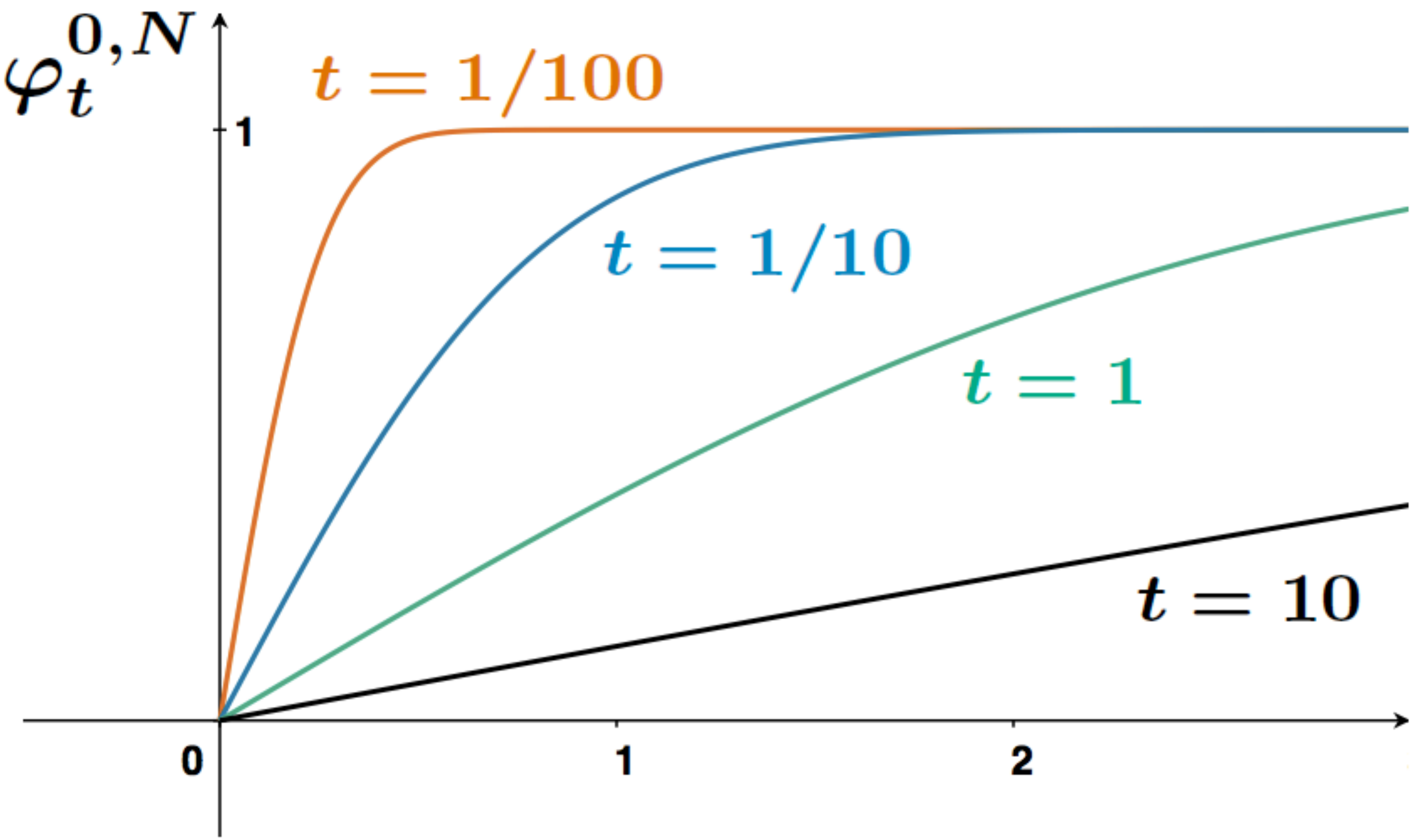

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- $N < N' \Rightarrow \varphi_t^{K,N}(a) \le \varphi_t^{K,N'}(a)$
- $\partial^+ \varphi_t(0) \le \frac{1}{\sqrt{2\pi}} \left(\frac{e^{Kt} - 1}{K} \right)^{-1/2}$

3. Applications

(Suppose $N \in \mathbb{N}$ & "g: indep. of t" for simplicity)

Theorem 2: $\mathcal{T}_{\varphi_{t-s}(d_s)}(P_s^* \mu_1, P_s^* \mu_2) \searrow$

$$\Downarrow$$

$$\mathcal{T}_{\varphi_0(d)}(P_t^* \delta_x, P_t^* \delta_y) \leq \mathcal{T}_{\varphi_t(d)}(\delta_x, \delta_y)$$

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$$\Downarrow$$

Corollary 1 (Comparison thm for total variations)

$$\left\| P_t^*\delta_x - P_t^*\delta_y \right\|_{\mathrm{TV}} \le \left\| \tilde{P}_t^*\delta_{\tilde{x}} - \tilde{P}_t^*\delta_{\tilde{y}} \right\|_{\mathrm{TV}}$$

Corollary 2 (Gradient estimate)

For any bounded measurable f on M,

$$\|\nabla P_t f\|_\infty \le \partial^+ \varphi_t(0)\, \mathrm{osc}(f)$$

Remark (duality; cf. [K. '10 JFA])

$$\mathcal{T}_{\varphi_0(d)}(P_t\delta_x, P_t\delta_y) \le \mathcal{T}_{\varphi_t(d)}(\delta_x, \delta_y)$$

$\Updownarrow$ Kantorovich-Rubinstein

$$\sup_{x \ne y} \left| \frac{P_t f(x) - P_t f(y)}{\varphi_t(d(x,y))} \right| \le \sup_{x \ne y} \left| \frac{f(x) - f(y)}{\varphi_0(d(x,y))} \right|$$

Stability under GH-convergence

(M_m, g_m): n-dim. cpt. Riem. mfds, $\mathrm{Ric}_{g_m} \geq K$

Suppose

$$(M_m, d_m, \mathrm{vol}_{g_m}) \xrightarrow{\text{mGH}} (M_\infty, d_\infty, v_\infty)$$

$\Downarrow$

For $\mu^{(m)} \in \mathcal{P}(M_m)$

with $\mu^{(m)} \to \mu^{(\infty)} \in \mathcal{P}(M_\infty)$,

$P_t \mu^{(m)} \to$ a "heat distribution" μ_t^∞ on M_∞

[Gigli '10, Ambrosio, Gigli & Savaré '11]

Theorem 3 [K. & S., op.sit.]

$(M_\infty, d_\infty, v_\infty)$: as above, $N \geq n$

$\mu_1(t), \mu_2(t)$: heat distributions on M_∞

$\Rightarrow$ For $t > 0$,

$$\mathcal{T}_{\varphi_{t-s}^{K,N}(d)}(\mu_1(t), \mu_2(t)) \searrow$$

in $s \in [0, t]$

4. Idea of the proof of Thm 1

(Under same simplification as before)

$(\tilde{X}_1(t), \tilde{X}_2(t))$: coupling by reflection on $\mathbb{M}_{K,N}$

s.t. $\tilde{d}(\tilde{X}_1(0), \tilde{X}_2(0)) = d(x_1, x_2)$

Lemma 1 (cf. [K.'07 J. Theoret. Probab.])

$$\mathbb{E}[\varphi_{t-s}(\tilde{d}(\tilde{X}_1(s), \tilde{X}_2(s)))]\text{: const. in } s$$

Proposition 1

"$d(X_1(s), X_2(s)) \le \tilde{d}(\tilde{X}_1(s), \tilde{X}_2(s))$"

Strategy of the proof of Proposition 1

- Itô formula for $d(X_1(s), X_2(s))$
- Index lemma and SDE comparison

Prop.1 & $\varphi_t \nearrow$

$$\Rightarrow \mathbb{E}[\varphi_{t-s}(d(X_1(s), X_2(s)))]$$

$$\leq \mathbb{E}[\varphi_{t-s}(\tilde{d}(\tilde{X}_1(s), \tilde{X}_2(s)))]$$

$$\overset{\text{Lem.1}}{=} \mathbb{E}\left[\varphi_t(\tilde{d}(\tilde{X}_1(0), \tilde{X}_2(0)))\right]$$

$$= \varphi_t(d(x_1, x_2))$$

$\Rightarrow$ Theorem 1 ($\because$ Markov property of (X_1, X_2)) $\square$

5. Cost function φ_t

Properties of φ_t

- $\varphi_t \nearrow$, concave, $\varphi_t(0) = 0$
- $\varphi_\cdot(a) \searrow$
- $\varphi_0 = 1_{(0,\infty)}$
- $N < N' \Rightarrow \varphi_t^{K,N}(a) \leq \varphi_t^{K,N'}(a)$
- $\partial^+ \varphi_t(0) \leq \frac{1}{\sqrt{2\pi}} \left(\frac{e^{Kt} - 1}{K} \right)^{-1/2}$

(A) $\varphi_t(\cdot) \nearrow$ & $\varphi_.(a) \searrow$ & $\varphi_0(\cdot) = 1_{(0,\infty)}$

These are a consequence of the following:

Lemma 2(cf. [K.'07 op.sit.])

$\tilde{\tau} := \inf\{t \geq 0 \mid \tilde{d}(\tilde{X}_1(t), \tilde{X}_2(t)) = 0\}$

Then

$$\mathbb{P}[\,\tilde{\tau} > t\,] = \varphi_t(\tilde{d}(\tilde{x}_1, \tilde{x}_2))$$

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(B) $N < N' \Rightarrow \varphi_t^{K,N}(a) \leq \varphi_t^{K,N'}(a)$

Prop.1

$$\tilde{d}^N(\tilde{X}_1^N(t), \tilde{X}_2^N(t)) \leq \tilde{d}^{N'}(\tilde{X}_1^{N'}(t), \tilde{X}_2^{N'}(t))$$

$$\Downarrow$$

$$\tilde{\tau}^N \leq \tilde{\tau}^{N'}$$

$$\Downarrow$$

$$\varphi_t^N(a) = \mathbb{P}[\tilde{\tau}^N > t] \leq \mathbb{P}[\tilde{\tau}^{N'} > t] = \varphi_t^{N'}(a)$$

(B) $N < N' \Rightarrow \varphi_t^{K,N}(a) \leq \varphi_t^{K,N'}(a)$

Prop.1

$$\tilde{d}^{N}(\tilde{X}_1^{N}(t), \tilde{X}_2^{N}(t)) \leq \tilde{d}^{N'}(\tilde{X}_1^{N'}(t), \tilde{X}_2^{N'}(t))$$

$$\Downarrow$$

$$\tilde{\tau}^{N} \leq \tilde{\tau}^{N'}$$

$$\Downarrow$$

$$\varphi_t^{N}(a) = \mathbb{P}[\tilde{\tau}^{N} > t] \leq \mathbb{P}[\tilde{\tau}^{N'} > t] = \varphi_t^{N'}(a)$$

$$\Downarrow$$

$$\partial^+ \varphi_t^{K,N}(0) \leq \partial^+ \varphi_t^{K,N'}(0) \leq \partial^+ \varphi_t^{K,\infty}(0)$$

Properties of φ_t

- $\varphi_t \nearrow$, concave, $\varphi_t(0) = 0$
- $\varphi_\cdot(a) \searrow$
- $\varphi_0 = 1_{(0,\infty)}$
- $N < N' \Rightarrow \varphi_t^{K,N}(a) \leq \varphi_t^{K,N'}(a)$
- $\partial^+ \varphi_t(0) \leq \frac{1}{\sqrt{2\pi}} \left(\frac{e^{Kt} - 1}{K} \right)^{-1/2}$

(C) φ_t: concave

Proposition 2

$\exists \xi_{t,K,N} \in \mathcal{P}([0,\infty))$ s.t.

$$\varphi_t(a) = \int_{[0,\infty)} \chi\left(\frac{a}{2\sqrt{u}}\right) \xi_{t,K,N}(du),$$

$$\chi(r) := \frac{1}{\sqrt{2\pi}} \int_{-r}^{r} \mathrm{e}^{-x^2/2} dx$$

$$\Rightarrow \partial^+ \varphi_t^{K,N}(0) = \frac{1}{\sqrt{2\pi}} \int_{[0,\infty)} \frac{\xi_{t,K,N}(du)}{\sqrt{u}}$$

Expression of $\xi_{t,K,N}$

(i) $\xi_{t,K,\infty} = \delta_{\gamma(t)}, \quad \gamma(t) := \dfrac{e^{Kt} - 1}{K}$

(ii) When $N < \infty$,

$$\xi_{t,K,N}(E) = \mathbb{P}\left[\int_0^t \frac{ds}{c_{K/(N-1)}(h_s)^2} \in E\right],$$

$$dh_t = d\beta_t + \hat{\Psi}(h_t)dt,$$

$$\hat{\Psi}(a) := \frac{N-2}{2}\frac{c_{K/(N-1)}(a)}{s_{K/(N-1)}(a)} + \frac{\Psi_{K,N}(a)}{2(N-1)}$$