Optimal transport and coupled diffusion by reflection

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This talk is based on a joint work with Karl-Theodor Sturm (Universität Bonn).

Given $-\infty < T_1 < T_2 \le \infty$, let M be an m-dimensional manifold with $\partial M = \emptyset$ and $m \ge 2$, and $(g(t))_{t\in[T_1,T_2]}$ a family of smooth complete Riemannian metrics on M. Let Δ_t, ∇^t, Ric_t and d_t be the Laplace-Beltrami operator, the covariant derivative, the Ricci curvature and the Riemannian distance with respect to $g(t)$ respectively. Take a family of C^1-vector fields $(Z(t))_{t\in[T_1,T_2]}$ on M which is continuous as a function of t. Let $(X(t))_{t\in[T_1,T_2]}$ be a diffusion process associated with a time-dependent generator $\mathscr{L}_t := \Delta_t/2 + Z(t)$. Let $(\nabla Z(t))^\flat$ be a $(0,2)$-tensor given by

$$(\nabla Z(t))^\flat(V_1,V_2) := \frac{1}{2}\left(\langle \nabla^t_{V_1} Z(t), V_2\rangle + \langle \nabla^t_{V_2} Z(t), V_1\rangle\right).$$

Assumption 1 Given $K \in \mathbb{R}$ and $N \in [m,\infty]$, the following holds:

(i) When $N = \infty$, $2(\nabla Z(t))^\flat + \partial_t g(t) \le \mathrm{Ric}_t - Kg(t)$.

(ii) When $N < \infty$, $g(t)$ is independent of t and

$$\frac{4}{N-m} Z(t)\otimes Z(t) + 2(\nabla Z(t))^\flat \le \mathrm{Ric}_t - Kg(t).$$

Note that the condition (ii) means that the N-dimensional Bakry-Émery Ricci tensor associated with $\mathscr{L}_t$ is bounded from below by K.

The first goal of this talk is to show that the so-called coupling by reflection or the Kendall-Cranston coupling (see [1, 3] and references therein) yields the monotonicity of an optimal transportation cost. To begin with, we introduce some notations in order to describe the cost function. Let $K \in \mathbb{R}$ and $N \in [m,\infty]$. Set $\bar{R} \in (0,\infty]$ by

$$\bar{R} := \begin{cases} \sqrt{\dfrac{N-1}{K}}\pi & \text{if } K > 0, \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\mathrm{diam}(M) \le \bar{R}$ under Assumption 1. Moreover, $\mathrm{diam}(M) = \bar{R} < \infty$ holds if and only if $Z(t) \equiv 0$ and M is isometric to the sphere of constant sectional curvature $K/(m-1)$ (see [2]). We define s_K and c_K as a usual comparison function as follows:

$$s_K(\theta) := \begin{cases} \dfrac{1}{\sqrt{K}}\sin(\sqrt{K}\theta) & K > 0, \\ \theta & K = 0, \\ \dfrac{1}{\sqrt{-K}}\sinh(\sqrt{-K}\theta) & K < 0, \end{cases} \qquad c_K(\theta) := \begin{cases} \cos(\sqrt{K}\theta) & K > 0, \\ 1 & K = 0, \\ \cosh(\sqrt{-K}\theta) & K < 0 \end{cases}$$

and $t_K := s_K/c_K$. Let $\Psi = \Psi_{K,N} : (-\bar{R},\bar{R}) \to \mathbb{R}$ be given by

$$\Psi_{K,N}(u) := \begin{cases} -Kt_{K/(N-1)}\left(\dfrac{u}{2}\right) & \text{if } N < \infty, \\ -\dfrac{K}{2}u & \text{otherwise.} \end{cases}$$

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Let us define a diffusion process $\rho(t)$, $t \in [T_1, T_2]$ on $(-\bar{R}, \bar{R})$ with an initial condition $\rho(T_1) = a \in [0, \bar{R})$ by

$$d\rho(t) = 2d\beta(t) + \Psi(\rho(t))dt.$$

For $t \in [T_1, T_2]$, let us define $\varphi_t : [0, \infty) \to [0, 1]$ by

$$\varphi_t(a) := \mathbb{P}[\rho(s) > 0 \text{ for } s \in [T_1, t + T_1]].$$

By using it, we state our main theorems as follows:

Theorem 1 Under Assumption 1, for any $x_1, x_2 \in M$, let $\mathbf{X}(t) = (X_1(t), X_2(t))_{t \geq 0}$ be a coupling by reflection of two $\mathscr{L}_t$-diffusion processes starting from (x_1, x_2), constructed in [1]. Then, for any $t \in [T_1, T_2]$, $\mathbb{E}\left[\varphi_{t-s}(d_s(\mathbf{X}(s)))\right]$ is a nonincreasing function of $s \in [T_1, t]$.

Theorem 2 For a probability measure $\mu^{(i)}$ on M, let $\mu_t^{(i)}$ be a distribution of $X(t)$ under $\mathbb{P}_{\mu^{(i)}}$ $(i = 1, 2)$. For $t, s \in [T_1, T_2]$ with $s \leq t$, let $\mathcal{T}_{\varphi_{t-s}(d_s)}(\mu_s^{(1)}, \mu_s^{(2)})$ be the optimal transportation cost between $\mu_s^{(1)}$ and $\mu_s^{(2)}$ associated with the cost function $\varphi_{t-s}(d_s)$, that is,

$$\mathcal{T}_{\varphi_{t-s}(d_s)}(\mu_s^{(1)}, \mu_s^{(2)}) := \inf\left\{ \int_{M \times M} \varphi_{t-s}(d_s(x, y))\pi(dxdy) \;\middle|\; \pi \text{ is a coupling of } \mu_s^{(1)} \text{ and } \mu_s^{(2)} \right\}.$$

Then, for any $t \in [T_1, T_2]$, $\mathcal{T}_{\varphi_{t-s}(d_s)}(\mu_s^{(1)}, \mu_s^{(2)})$ is an nonincreasing function of $s \in [T_1, t]$.

In what follows, we will state some additional results related to Theorem 1.

- In the case $N < \infty$, $t = s$ and $\mu^{(i)} = \delta_{x_i}$, Theorem 2 means that the total variation between $\mu_t^{(1)}$ and $\mu_t^{(2)}$ is bounded from above by that between two heat distributions in the space form of dimension N and constant sectional curvature $K/(N-1)$ with initial distribution δ_{y_1} and δ_{y_2}, where y_1 and y_2 will be chosen to satisfy $d(x_1, x_2) = d(y_1, y_2)$ (when $N = \infty$, distributions of an Ornstein-Uhlenbeck process appears instead)
- We can give an explicit expression of φ_t. In any case, $\varphi_t(\cdot)$ is concave.
- Theorem 2 implies a gradient bound of the following type for the diffusion semigroup P_t:

  $$\|\nabla P_t f\|_\infty \leq C(t)(\sup f - \inf f).$$

  It immediately yields the strong Feller property. Moreover, when $K \geq 0$, $\lim_{t \to \infty} C(t) = 0$ holds. This fact implies the Liouville property when $g(t)$ is independent of t.
- The monotonicity in Theorem 2 is preserved under the Gromov-Hausdorff convergence with a uniform upper dimension bound and a uniform lower Ricci curvature bound (when the corresponding heat flow also converges; it certainly holds when $Z \equiv 0$ and M is compact). Thus the conclusion of the last item is still valid for the Gromov-Hausdorff limit.

References

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