Two duality results on gradient estimates and Wasserstein controls

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§1 Introduction

Equivalent conditions for a lower Ricci curvature bound

(von Renesse & Sturm '05, etc...)

X: complete Riemannian manifold

P_t: heat semigroup associated with Δ

(i) $\mathrm{Ric} \geq k$

(ii) $d_p^W(P_t^* \mu, P_t^* \nu) \leq e^{-kt} d_p^W(\mu, \nu)$
for some $p \in [1, \infty]$

(iii) $|\nabla P_t f|(x) \leq e^{-kt} P_t(|\nabla f|^q)(x)^{1/q}$
for some $q \in [1, \infty]$

Our goal:

Generalization of (ii) $\Leftrightarrow$ (iii)

for p, q with $\dfrac{1}{p} + \dfrac{1}{q} = 1.$

(ii) L^p-Wasserstein control

$$d_p^W(P_t^* \mu, P_t^* \nu) \le \mathrm{e}^{-kt} d_p^W(\mu, \nu)$$

(iii) L^q-gradient estimate

$$|\nabla P_t f|(x) \le \mathrm{e}^{-kt} P_t(|\nabla f|^q)(x)^{1/q}$$

§2 Framework and main result

X: Polish space

- $(P_x)_{x \in X} \subset \mathcal{P}(X)$: Markov kernel

$$Pf(x) := \int_X f \, dP_x,$$

$$P^* \mu(A) := \int_X P_x(A) \mu(dx)$$

  Assume $P(C_b(X)) \subset C_b(X)$.
  (e.g. $P = P_t$: heat semigroup)

- d, $\tilde{d}$: lower semi-conti. pseudo-distance on X
  (e.g. $\tilde{d} = \mathrm{e}^{-kt} d$, d: Riemannian distance)

$$\Pi(\mu,\nu) := \left\{ \pi \;\middle|\; \pi \circ p_1^{-1} = \mu,\ \pi \circ p_2^{-1} = \nu \right\}$$

(couplings of $\mu, \nu \in \mathcal{P}(X)$)

L^p-Wasserstein distance

For $p \in [1, \infty]$,

$$d_p^W(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \|d\|_{L^p(\pi)} \in [0, \infty]$$

L^p-Wasserstein control

$$d_p^W(P^*\mu, P^*\nu) \leq \tilde{d}_p^W(\mu,\nu) \qquad (C_p)$$

Gradient

$$|\nabla_d f|(x) := \lim_{r \downarrow 0} \sup_{y \in B_r(x)} \left| \frac{f(x) - f(y)}{d(x, y)} \right|,$$

$$\|\nabla_d f\|_\infty := \sup_{x \in X} |\nabla_d f|(x)$$

Subgradient

$$|\nabla_d^- f|(x) := \lim_{r \downarrow 0} \sup_{y \in B_r(x)} \left[\frac{f(x) - f(y)}{d(x, y)} \right]_+,$$

$$\|\nabla_d^- f\|_\infty := \sup_{x \in X} |\nabla_d^- f|(x)$$

L^q-gradient estimate

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

for $q \in [1, \infty)$ and $f \in C_b^{\mathrm{Lip}}(X)$,

$$\|\nabla_{\tilde{d}} Pf\|_\infty \le \|\nabla_d f\|_\infty \qquad (G_\infty)$$

for $q = \infty$

$$\left(\begin{array}{c} \text{We define } (G_q^-) \text{ and } (G_\infty^-) \text{ similarly} \\ \text{by using } \nabla^- \text{ instead of } \nabla \end{array} \right)$$

The first duality result

Theorem A (K. '10 JFA)

For $p, q \in [1, \infty]$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q)$

(ii) Under Assumptions A1-A4 below,

$(G_q) \Rightarrow (C_p)$

The second duality result

Theorem B (K. '10)

For $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q^-)$

(ii) Under Assumptions B1-B4 below,

$(G_q^-) \Rightarrow (C_p)$

Examples satisfying A1-A4

- Cpl. Riem. mfds with $\mathrm{Ric} \geq k_0$
- Carnot groups
- Alexandrov spaces

($\dim X < \infty$, $P_x \ll v$ for a base measure v)

Examples satisfying B1-B4

- Cpl. Riem. mfds (no curvature assumption)
- Wiener space, d: Cameron-Martin norm

($P_x \ll v$ is not necessary)

Remarks

- For $p' > p$,
$$\begin{cases} (G_p) \Rightarrow (G_{p'}) \,/\, (G_p^-) \Rightarrow (G_{p'}^-) \\ (C_{p'}) \Rightarrow (C_p) \end{cases}$$
(without Assumptions A1-A4/B1-B4)

- $(G_\infty) \Leftrightarrow (C_1)$ is well known.
$$\left(\begin{array}{l} \text{via Kantorovich-Rubinstein formula;} \\ \text{without Assumptions A1-A4/B1-B4} \end{array} \right)$$

- $(C_\infty) \Rightarrow (G_1)$ is essentially well known.

§3 Applications

(0) Review: How did we obtain (C_p)?

- Coupling by parallel transport of B.m.'s:

$$\boxed{\mathrm{Ric} \geq k \Rightarrow (C_\infty)}$$

- Gradient flow formulation of the heat flow μ_t:
  $\partial_t \mu_t = -\nabla E(\mu_t)$
  - $\mathrm{Ric} \geq k \Leftrightarrow$ "$\mathrm{Hess}\, E \geq k$"
  - $\boxed{\mathrm{Hess}\, E \geq k \Rightarrow (C_2)}$ for $\mu_t \ (= P_t^* \mu)$

$\rightsquigarrow$ Extensions to a backward Ricci flow or singular spaces (e.g. Alex. sp.).

Remark

"(a lower Ricci bound) $\Rightarrow (C_p)$" in the literature.
No direct way "$(G_q) \Rightarrow (C_p)$" was known.

E.g. in von Renesse & Sturm '05,

$$\mathrm{Ric} \geq k$$

$\Downarrow$ coupling method

$$(C_\infty) \Rightarrow (C_p) \Rightarrow (C_1)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$(G_1) \Rightarrow (G_q) \Rightarrow (G_\infty) \Rightarrow \mathrm{Ric} \geq k$$

Bochner

(1) Manifolds with a time-dependent metric

$(M, g(t))$: cpl. Riem. mfds, $t \in [0, T]$

B_t: $g(t)$-B.m. ($\leftrightarrow$ generator $\partial_t + \frac{1}{2}\Delta_{g(\cdot)}$)

$P = P_{s \to t}$: heat semigroup associated with B_t.

$d = d_{g(t)}$, $\tilde{d} = \mathrm{e}^{-k(t-s)/2} d_{g(s)}$

($\tilde{d}$ is essentially different from d!)

$$\partial_t g(t) \leq \mathrm{Ric}_{g(t)} - k g(t) \qquad (\star)$$

- McCann & Topping '08: X: cpt, $k = 0$

$$(\star) \Leftrightarrow [(C_2)\ \forall s < t] \Leftrightarrow [(G_\infty)\ \forall s < t]$$

- Arnaudon, Coulibaly & Thalmaier '09:

$$(\star) \Rightarrow [(C_\infty)\ \forall s < t]$$

(cf. K. & Philipowski '09: B_t is non-explosive)

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- Theorem B $\Rightarrow$ $\boxed{\forall s < t, [(C_p) \Leftrightarrow (G_q^-)]}$

(When B_t is non-explosive)

(2) Hörmander-type operators on a Lie group

X: Lie group

$\{X_i\}_{i=1}^n$: left-invariant, lin. indep. vector fields satisfying the Hörmander condition

$$P_t := e^{tA}, \quad A := \sum_{i=1}^n X_i^2,$$

$$|\Gamma f| := \frac{1}{2}\left(A(f^2) - 2fAf\right) = \sum_{i=1}^n |X_i f|^2$$

<u>L^q-gradient estimate</u>

$$|\Gamma P_t f|(x) \le K_q(t) P_t(|\Gamma f|^{q/2})(x)^{2/q} \quad (G_q^*)$$

Known results

- 3-dim. Heisenberg group, $K_q(t) \equiv K_q > 1$
  - $q > 1$: Driver & Melcher '05
  - $q = 1$: H.-Q. Li '06 / Bakry, Baudoin, Bonnefont & Chafaï '08
- X: general, $q > 1$: Melcher '08 ($K_q(t) \equiv K_q$ if X: nilpotent)
- X: group of type H, $q = 1$, $K_q(t) \equiv K_q$: Eldredge '10
- $X = SU(2)$, $q > 1$, $K_q(t) = K_q \mathrm{e}^{-t}$: Baudoin & Bonnefont '09

Carnot-Caratheodory distance

For $V \in T_x X$,

$$|V| = \begin{cases} \left(\sum_{i=1}^{n} {a_i}^2 \right)^{1/2} & \text{if } V = \sum_{i=1}^{n} a_i X_i(x), \\ \infty & \text{otherwise.} \end{cases}$$

$$d(x, y) := \inf \left\{ \int_0^1 |\dot{\gamma}_s| ds \;\middle|\; \begin{array}{l} \gamma_0 = x, \\ \gamma_1 = y \end{array} \right\}$$

v: right-Haar measure, $P = P_t$.

Proposition

(i) $(X, d, v; P)$ satisfies Assumption A1-A4

(ii) $(G_q^*) \Rightarrow (G_q)$

Corollary

$(G_q^*) \Rightarrow (C_p)$ for $q \in [1, \infty]$.

Examples

3-dim. Heisenberg group

$X = \mathbb{R}^3$, v: Lebesgue

$$(x, y, z) \cdot (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')\right)$$

$$X_1 = \partial_x - \frac{y}{2}\partial_z \,, \quad X_2 = \partial_y + \frac{x}{2}\partial_z$$

Associated diffusion (B_t^1, B_t^2, B_t^3) from (x, y, z):

$$B_t^1 = W_t^1, \qquad B_t^2 = W_t^2,$$

$$B_t^3 = z + \frac{1}{2}\int_0^t W_t^1 dW_t^2 - W_t^2 dW_t^1,$$

where (W_t^1, W_t^2): 2-dim. BM from (x, y)

(C_∞): For each $t > 0$,

$\exists$ a coupling $(\boldsymbol{B}_t, \tilde{\boldsymbol{B}}_t)$ of $(B_{\boldsymbol{t}}^{\mathbf{1}}, B_{\boldsymbol{t}}^{\mathbf{2}}, B_{\boldsymbol{t}}^{\mathbf{3}})$ s.t.

$$d(\boldsymbol{B}_t, \tilde{\boldsymbol{B}}_t) \leq K_1 d(\boldsymbol{B}_0, \boldsymbol{B}_0) \quad \mathbb{P}\text{-a.s.}$$

Definition

X: a group of type H iff, for $\mathcal{X}$: Lie alg. associated with X with a scalar product $\langle\cdot,\cdot\rangle$,

- $\mathcal{X} = \mathcal{V} \oplus \mathcal{Z}$ with $[\mathcal{V},\mathcal{V}] = \mathcal{Z}$, $[\mathcal{V},\mathcal{Z}] = [\mathcal{Z},\mathcal{Z}] = 0$.
- $J : \mathcal{Z} \to \mathrm{End}\,\mathcal{V}$ given by

$$\langle J(Z)V_1, V_2\rangle := \langle Z, [V_1, V_2]\rangle$$

satisfies $J(Z)^2 = -\|Z\|\mathrm{Id}$.

$\{X_i\}_{i=1}^n$ will be an ONB of $\mathcal{V}$.

Remarks

- (Heisenberg) $\subset$ (type H)

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- (Heisenberg) $\subset$ (type H)
- (type H) $\subset$ (stratified, step 2 nilpotent)
- (type H) $\cap$ (free step 2 nilpotent)
  $= \{$3-dim. Heisenberg$\}$
- For type H,
  possible dimension is completely determined

§4 Sketch of the proof of $(G_q) \Rightarrow (C_p)$

Recall:

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu,\nu) \qquad (C_p)$$

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

Recall:

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu,\nu) \qquad (C_p)$$

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

- $p = 1$ $(q = \infty)$: well-known
- (C_p) for $\forall p < \infty \Rightarrow (C_\infty)$

  $\rightsquigarrow$ We may assume $p \in (1, \infty)$

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- $p = 1$ $(q = \infty)$: well-known
- (C_p) for $\forall p < \infty \Rightarrow (C_\infty)$

  $\rightsquigarrow$ We may assume $p \in (1, \infty)$

∘ (C_p) for $\mu = \delta_x, \nu = \delta_y \Rightarrow (C_p)$

$\rightsquigarrow$ We show $\dfrac{d_p^W(P_x, P_y)^p}{p} \le \dfrac{\tilde{d}(x,y)^p}{p}$

Hamilton-Jacobi semigroup

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + t \cdot \frac{1}{p} \left(\frac{d(x,y)}{t} \right)^p \right]$$

We expect (under our assumptions):

- $Q_{\cdot} f(x)$: Lipschitz, $Q_t f(\cdot)$: d-Lipschitz
- Hamilton-Jacobi equation

$$\boxed{\partial_t Q_t f = -\frac{1}{q} |\nabla_d Q_t f|^q}$$

(Note: $q^{-1} u^q = \sup_{s \geq 0} \left(us - p^{-1} s^p \right)$)

Kantorovich duality

$$d_p^W(\mu,\nu)^p = \sup_{f\in C_b^{\mathrm{Lip}}} \left[\int_X f^* \, d\mu - \int_X f \, d\nu\right],$$

$$\begin{aligned} f^*(x) &:= \inf_{y\in X} \left[\, f(y) + d(x,y)^p \,\right] \\ &= p\, Q_1(p^{-1} f)(x) \end{aligned}$$

$$\Downarrow$$

$$\frac{d_p^W(\mu,\nu)^p}{p} = \sup_f \left[\int_X Q_1 f \, d\mu - \int_X f \, d\nu\right]$$

$$\begin{cases} \tilde{\gamma} \ : \ [0,1] \to X \quad \tilde{d}\text{-minimal geodesic}, \\ \tilde{\gamma}_0 = y, \ \tilde{\gamma}_1 = x, \\ \tilde{d}(\tilde{\gamma}_s, \tilde{\gamma}_t) = |t-s|\tilde{d}(x,y) \end{cases}$$

———— ∘ ———— ∘ ————

$$\frac{d_p^W(P_x, P_y)^p}{p} = \sup_f \left[PQ_1 f(x) - Pf(y) \right]$$

$$\text{interpolation} \boxed{=} \sup_f \left[\int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t)) dt \right]$$

$$
\begin{aligned}
&\partial_t(PQ_tf(\tilde{\gamma}_t)) \\
&\quad \left(\text{“}{=}\text{”}\ P(\partial_t Q_t f)(\tilde{\gamma}_t) + \langle \nabla PQ_t f(\tilde{\gamma}_t), \dot{\tilde{\gamma}}_t \rangle \right) \\
\text{HJ eq.}\quad &\boxed{\leq} - \frac{1}{q} P(|\nabla_d Q_t f|^q)(\tilde{\gamma}_t) \\
\text{up. grad.}\quad &\qquad + \tilde{d}(x,y) \left| \nabla_{\tilde{d}} PQ_t f \right| (\tilde{\gamma}_t) \\
(G_q) &\boxed{\leq} \tilde{d}(x,y)\sigma - \frac{1}{q}\sigma^q \leq \frac{\tilde{d}(x,y)^p}{p}.
\end{aligned}
$$

$$\left(\sigma := P(|\nabla_d Q_t f|^q)(\tilde{\gamma}_t)^{1/q} \right) \qquad \blacksquare$$

§5 Assumptions

(1) Assumptions A1-A4

v: Radon measure on X with $\mathrm{supp}(v) = X$.

Assumption A1 (X, d): proper length space, compatible with the original topology

Assumption A2 (X, d, v) supports

- local (uniform) volume doubling condition
- $(1, \rho)$-local Poincaré inequality $(\exists \rho \geq 1)$

Assumption A3 $\tilde{d}$: continuous geodesic distance

Assumption A4 $P_x \ll v$, $x \mapsto \dfrac{dP_x}{dv}(y)$: conti.

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Assumption A2 (X, d, v) supports

- local (uniform) volume doubling condition
- $(1, \rho)$-local Poincaré inequality $(\exists \rho \geq 1)$

Assumptions A1-A2 $\Rightarrow$ HJ eq. for $Q_t f$ v-a.e.

(Lott & Villani '07

Balogh, Engoulatov, Hunziker & Maasalo '09)

Local volume doubling condition

$\exists D > 0$, $\exists R_1 > 0$ s.t. $\forall x \in X$, $\forall r < R_1$

$$v(B_{2r}(x)) \leq D v(B_r(x)).$$

$(1, \rho)$-local Poincaré inequality

$\forall R > 0$, $\exists \lambda \geq 1$, $\exists C_P > 0$ s.t. $\forall r < R$,

$$⨍_{B_r(x)} |f - f_{x,r}| dv \leq C_P r \left(⨍_{B_{\lambda r}(x)} g^\rho \, dv \right)^{1/\rho}$$

for $\forall f$ and $\forall g$: upper gradient of f, where

$$f_{x,r} := \frac{1}{v(B_r(x))} \int_{B_r(x)} f \, dv =: ⨍_{B_r(x)} f \, dv.$$

(2) Assumptions B1-B4

Assumption B1

For $\forall x, y \in \boldsymbol{X}$ with $d(x, y) < \infty$,

$\exists$ a minimal geodesic γ from x to y.

The same is also true for $\tilde{d}$.

Assumption B2

$Q_t f$ is measurable for $\forall f \in C_b(\boldsymbol{X})$

Assumption B3

$$\lim_{r \downarrow 0} \sup_{y;\ \tilde{d}(x,y) \le r} PQ_t f(y) \le PQ_t f(x)$$

“Assumption B4”

The following holds locally:

- $\exists D \geq 0$ s.t. for $\forall \gamma$: d-min. geod., $d(x, \gamma(\lambda))^2$ is D-semiconcave, i.e.:

$$d(x, \gamma(\lambda))^2 \geq (1-\lambda) d(x, \gamma(0))^2 + \lambda d(x, \gamma(1))^2 - D\lambda(1-\lambda) d(\gamma(0), \gamma(1))^2 \qquad (\dagger(D))$$

- $\forall y, x \in X$, with $d(x, y)$: small, y is on a min. geod. from x of given length

Assumption B4

For $\forall K \subset X$ cpt., $\exists D_K \geq 0, \exists \eta_K > 0$ s.t.

(i) for $\forall \gamma$: d-min. geod. with $\gamma(0) \in K$, $d(\gamma(0), \gamma(1)) < \eta_K$, $d(\gamma(0), x) < \eta_K$ and $\lambda \in (0,1)$, $(\dagger(D_K))$ holds.

(ii) For $\forall x \in K$, $\forall y \in X$ with $d(x,y) < \eta_K$, $\exists \gamma$: min. geod. with $d(x, \gamma(1)) = \eta_K$ and $\boxed{\gamma(\lambda) = y}$ for some $\lambda \in (0,1)$.

Example B

- X: cpl. Riem. mfd.
  - Ass. B4 (i) $\Leftarrow$ local lower sect. curv. bound
  - Ass. B4 (ii) $\Leftarrow$ local positivity of inj. radius
- X: Wiener space, d: Cameron-Martin norm
  - Ass. B4 (i) with $D_K \equiv 1$ ("$=$" holds!)
  - Ass. B4 (ii) is obvious

§6 Questions

(i) When (weak) $\Rightarrow$ (strong) ?
i.e. $(C_p) \Rightarrow (C_{p'})$ or $(G_{p'}) \Rightarrow (G_p)$
for $p' > p$
(OK if X: Riem., $P = P_t$)

(ii) When "$(C_p) \Rightarrow$ (pathwise control)"
in the case $P = P_t$?

(iii) "Bakry-Émery's Γ_2-criterion $\Leftrightarrow (G_q)$"
in the case $P = P_t$, $\tilde{d} = \mathrm{e}^{-kt} d$?
(When $|\nabla_d f| = |\Gamma f|^{1/2}$?).

(iv) Relation with other "lower curvature bounds"...