Duality on gradient estimates and Wasserstein controls

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§1 Motivation

Equivalent conditions for a lower Ricci curvature bound

(von Renesse & Sturm '05, etc...)

X: complete Riemannian manifold

P_t: heat semigroup associated with Δ

(i) $\mathrm{Ric} \geq k$,
(ii) $d_p^W(P_t^*\mu, P_t^*\nu) \leq \mathrm{e}^{-kt} d_p^W(\mu, \nu)$ for some $p \in [1, \infty]$,
(iii) $\lvert\nabla P_t f\rvert(x) \leq \mathrm{e}^{-kt} P_t(\lvert\nabla f\rvert^q)(x)^{1/q}$ for some $q \in [1, \infty]$.

Our goal:

Generalization of (ii) $\Leftrightarrow$ (iii), to obtain a (ii)/(iii)-type estimate from the other one.

§2 Framework and main result

(X, d): Polish metric space.

- $(P_x)_{x \in X} \subset \mathcal{P}(X)$: Markov kernel.
  $P : \mathcal{B}_b(X) \to \mathcal{B}_b(X)$

$$Pf(x) := \int_X f \, dP_x,$$

$$P^* \mu(A) := \int_X P_x(A) \mu(dx).$$

  (e.g. $P = P_t$: heat semigroup)

- $\tilde{d}$: continuous distance function on X.
  (e.g. $\tilde{d} = \mathrm{e}^{-kt} d$)

L^p-Wasserstein distance

For $p \in [1, \infty]$,

$$d_p^W(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \|d\|_{L^p(\pi)} \in [0, \infty].$$

($\Pi(\mu, \nu)$: couplings of μ and ν)

Gradient

$$|\nabla_d f|(x) := \lim_{r \downarrow 0} \sup_{y \in B_r(x)} \left| \frac{f(y) - f(x)}{d(y, x)} \right|,$$

$$\|\nabla_d f\|_\infty := \sup_{x \in X} |\nabla_d f|(x).$$

L^p-Wasserstein control

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu,\nu) \qquad (C_p)$$

for $p \in [1,\infty]$ and $\mu,\nu \in \mathcal{P}(X)$.

L^q-gradient estimate

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

for $q \in [1,\infty)$ and $f \in C_b^{\mathrm{Lip}}(X)$,

$$\|\nabla_{\tilde{d}} Pf\|_\infty \le \|\nabla_d f\|_\infty \qquad (G_\infty)$$

for $q = \infty$.

v: Radon measure on X with $\operatorname{supp}(v) = X$.

Assumption 1 (X, d): proper length space.

Assumption 2 (X, d, v) supports

- local (uniform) volume doubling condition,
- $(1, \rho)$-local Poincaré inequality ($\exists \rho \geq 1$).

Assumption 3 $\tilde{d}$: geodesic distance.

Assumption 4 $P_x \ll v$, $x \mapsto \dfrac{dP_x}{dv}(y)$: continuous.

Theorem (K.)

For $p, q \in [1, \infty]$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q)$.

(ii) Under Assumption 1-4, $(G_q) \Rightarrow (C_p)$.

Remarks

- For $p' > p$, $\begin{cases} (G_p) \Rightarrow (G_{p'}), \\ (C_{p'}) \Rightarrow (C_p). \end{cases}$
  (without Assumption 1-4)

- $(G_\infty) \Leftrightarrow (C_1)$ is well known.
  $\left(\begin{array}{l} \text{via Kantorovich-Rubinstein formula;} \\ \text{without Assumption 1-4} \end{array} \right)$

- $(C_\infty) \Rightarrow (G_1)$ is essentially well known.

Remark

To obtain (C_p), we have used some notion of lower curvature bound which is different from (G_q).

E.g. in von Renesse & Sturm '05,

$$\mathrm{Ric} \geq k$$

$\Downarrow$ coupling method

$$(C_\infty) \Rightarrow (C_p) \Rightarrow (C_1)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$(G_1) \Rightarrow (G_q) \Rightarrow (G_\infty) \Rightarrow \mathrm{Ric} \geq k.$$

Bochner

§3 Hörmander-type operators on a Lie group

X: Lie group with a right-Haar measure v.

$\{X_i\}_{i=1}^n$: left-invariant vector fields satisfying the Hörmander condition.

$$P_t := \mathrm{e}^{tA}, \quad A := \sum_{i=1}^n X_i^2.$$

$$|\nabla f|^2 := \frac{1}{2}\left(A(f^2) - 2fAf\right) = \sum_{i=1}^n |X_i f|^2.$$

<u>L^q-Gradient estimate</u>

$$|\nabla P_t f|(x) \le K_q(t) P_t(|\nabla f|^q)(x)^{1/q}. \quad (G_q^*)$$

Known results

- 3-dim. Heisenberg group, $K_q(t) \equiv K_q > 1$
  - $q > 1$: Driver & Melcher '05.
  - $q = 1$: H.-Q. Li '06 / Bakry & Baudoin & Bonnefont & Chafaï '08.
- X: general, $q > 1$: Melcher '08 ($K_q(t) \equiv K_q$ if X: nilpotent).
- X: group of type H, $q = 1$, $K_q(t) \equiv K_q$: Eldredge '10.
- $X = SU(2)$, $q > 1$, $K_q(t) = K_q e^{-t}$: Baudoin & Bonnefont '09.

Carnot-Caratheodory distance

For $V \in T_x X$,

$$|V| = \begin{cases} \left(\sum_{i=1}^{n} {a_i}^2\right)^{1/2} & \text{if } V = \sum_{i=1}^{n} a_i X_i(x), \\ \infty & \text{otherwise.} \end{cases}$$

$$d(x, y) := \inf \left\{ \int_0^1 |\dot{\gamma}_s| ds \;\middle|\; \begin{array}{l} \gamma_0 = x, \\ \gamma_1 = y \end{array} \right\}.$$

Proposition

(X, d, v), $P = P_t$: as above.

(i) $(X, d, v; P)$ satisfies Assumption 1-4

(ii) $(G_q^*) \Rightarrow (G_q)$ with $\tilde{d} = K_q(t)d$.

Corollary

$(G_q^*) \Rightarrow (C_p)$ for $q \in [1, \infty]$.

§4 Sketch of the proof of $(G_q) \Rightarrow (C_p)$

Recall:

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu, \nu), \qquad (C_p)$$

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q}. \qquad (G_q)$$

- The case $p = 1$ ($q = \infty$) is well-known.
- $d_p^W(\mu, \nu) \xrightarrow{p\to\infty} d_\infty^W(\mu, \nu) \in [0, \infty]$.
  $\Rightarrow$ We may assume $p < \infty$.
- For (C_p), it suffices to show

$$d_p^W(P_x, P_y) \le \tilde{d}(x, y).$$

General theory of the Hamilton-Jacobi semigroup

(Lott & Villani '07 / Balogh & Engoulatov & Hunziker & Maasalo '09)

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + t \cdot \frac{1}{p} \left(\frac{d(x,y)}{t} \right)^p \right].$$

- Under Assumption 1,
  $Q_\cdot f \in C_b^{\mathrm{Lip}}([0,\infty) \times X)$ if $f \in C_b^{\mathrm{Lip}}(X)$.
- Under Assumption 1-2, for $\forall t > 0$, $\mathfrak{v}$-a.e.

$$\partial_t Q_t f = -\frac{1}{q} |\nabla_d Q_t f|^q .$$

(Note: $q^{-1} u^q = \sup_{s \geq 0} \left(us - p^{-1} s^p \right)$)

Kantorovich duality

$$d_p^W(\mu,\nu)^p = \sup_{f\in C_b^{\mathrm{Lip}}} \left[\int_X f^*\,d\mu - \int_X f\,d\nu\right],$$

$$f^*(x) := \inf_{y\in X} [\, f(y) + d(x,y)^p \,]$$

$$= p\,Q_1(p^{-1}f)(x).$$

$$\Downarrow$$

$$\frac{d_p^W(P_x,P_y)^p}{p} = \sup_f \left[PQ_1 f(x) - Pf(y)\right].$$

$\exists \gamma : [0,1] \to X$: $\tilde{d}$-min. geod. of const. speed, $\gamma_0 = y$, $\gamma_1 = x$. (Assumption 3)

$$\Downarrow$$

$$\frac{d_p^W(P_x, P_y)^p}{p} = \sup_f [PQ_1 f(x) - Pf(y)]$$

$$\text{“=”} \sup_f \left[\int_0^1 \partial_t (PQ_t f(\gamma_t)) dt \right].$$

interpolation

$$\partial_t(PQ_tf(\gamma_t))$$

$$\text{“}=\text{”} \langle \nabla PQ_tf(\gamma_t), \dot{\gamma}_t\rangle + P(\partial_t Q_t f)(\gamma_t)$$

up. grad. HJ eq. $\boxed{\le} \tilde{d}(x,y) \left|\nabla_{\tilde{d}} PQ_t f\right| (\gamma_t)$

$$- \frac{1}{q} P(|\nabla_d Q_t f|^q)(\gamma_t)$$

$(G_q)\boxed{\le} \tilde{d}(x,y)\sigma - \frac{1}{q}\sigma^q \le \frac{\tilde{d}(x,y)^p}{p}.$

$$\left(\sigma := P(|\nabla_d Q_t f|^q)(\gamma_t)^{1/q}\right)$$

Hence

$$\frac{d_p^W(P_x, P_y)^p}{p} = \sup_f \left[\int_0^1 \partial_t (PQ_t f(\gamma_t))dt\right]$$
$$\leq \sup_f \int_0^1 \frac{\tilde{d}(x, y)^p}{p} dt$$
$$= \frac{\tilde{d}(x, y)^p}{p}.$$

■

§5 Questions

(i) When does $(C_p) \Rightarrow (C_{p'})$ / $(G_{p'}) \Rightarrow (G_p)$ occur for $p' > p$?
(OK if X: Riem., $P = P_t$)

(ii) When does $(C_\infty) \Rightarrow$ "pathwise control" occur?
(in the case $P = P_t$)

(iii) Relation between Bakry-Émery's Γ_2-criterion and (G_q) (in the case $P = P_t$, $\tilde{d} = \mathrm{e}^{-kt} d$)
(When does $|\nabla_d f| = \Gamma(f, f)^{1/2}$ hold?).

(iv) Relation with other "lower curvature bounds"...