Coupling methods for diffusion processes under time-dependent metric

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§1 Framework and introduction

M: m-dim. manifold, $0 \le T_1 < T_2 \le \infty$,

$(g(t))_{t \in [T_1, T_2]}$: smooth complete Riemannian metrics on M.

Examples

- $\partial_t g(t) = 2 \, \mathrm{Ric}_{g(t)}$ (backward Ricci flow).
- $\partial_t g(t) \le 2 \, \mathrm{Ric}_{g(t)} - 2K g(t)$ (time-dependent extension of "$\mathrm{Ric} \ge K$").

$\mathcal{A}_t := \Delta_{g(t)} + Z_t$ time-dependent generator.
$(X_t)_{t \in [T_1, T_2]}$: $\mathcal{A}_t$-diffusion, i.e.

$$f(t, X_t) - f(T_1, X_{T_1}) - \int_{T_1}^{t} \left(\frac{\partial}{\partial s} + \mathcal{A}_s \right) f(s, X_s) ds$$

is a local martingale for $\forall f$: smooth

SDE for U_t on $\mathcal{F}(M)$

[Arnaudon, Coulibaly & Thalmaier '08/Coulibaly '09]

$$dU_t = \sum_i H_i(t, U_t) \circ dW_t^i + Z_t^H(U_t)dt - \frac{1}{2}\sum_{\alpha,\beta} \frac{\partial g}{\partial t}(t, U_t e_\alpha, U_t e_\beta) V_{\alpha\beta}(U_t)dt.$$

$H_i(t,u)$: $g(t)$-horizontal lift of ue_i,

$(V_{\alpha,\beta})_{\alpha,\beta=1}^d$: vertical vector fields,

Z_t^H: $g(t)$-horizontal lift of Z_t.

- ⋆ $U_0 \in \mathcal{O}_{g(0)}(M) \Rightarrow U_t \in \mathcal{O}_{g(t)}(M)$
- ⋆ $\pi(U_t)$ is an $\mathcal{A}_t$-diffusion process

Coupling: a pair of particles (X_t, Y_t)

each of which moves as an $\mathcal{A}_t$-diffusion.

What we consider:

(i) Coupling by parallel transport

(ii) Coupling by reflection

(iii) Coupling by spacetime parallel transport

Purpose

A nice control of "distance" between X_t and Y_t .

$\Rightarrow$ (Sharper) monotonicity of transportation costs

$\Rightarrow$ Functional inequalities

Transportation cost

$c : M \times M \to \mathbb{R}$: cost function

($c(x, y)$: cost of bringing a unit mass from x to y)

For $\mu, \nu \in \mathcal{P}(M)$,

$$\Pi(\mu, \nu) := \left\{ \pi \;\middle|\; \begin{array}{l} \pi(E \times M) = \mu(E), \\ \pi(M \times E) = \nu(E) \end{array} \right\}$$

(set of all couplings between μ and ν),

$$\mathcal{T}_c(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{M \times M} c(x, y)\pi(dxdy)$$

(Minimal total transportation cost from μ to ν)

Known results related to Coupling by parallel transport

Known results related to Coupling by parallel transport

[McCann & Topping '10]

Suppose M: cpt. & $Z_t \equiv 0$.

$\Rightarrow$ The following are equivalent:

(1) $\partial_t g(t) \le 2\,\mathrm{Ric}_{g(t)}$

(2) $\mathcal{T}_{d^2}(\mu_t, \nu_t) \searrow$ for $\forall \mu_t, \nu_t$: heat distributions

(3) $\mathrm{Lip}_{g(s)}(T_{s,t} f) \le \mathrm{Lip}_{g(t)}(f)$

[Arnaudon & Coulibaly & Thalmaier '09]

Suppose

$$4(\nabla Z_t)^\flat + \partial_t g(t) \leq 2\,\mathrm{Ric}_{g(t)} - 2K g(t).$$

$\Rightarrow$ For $\forall x, y \in M$,

$\exists\ (X_t, Y_t)$: coupled $\mathcal{A}_t$-diffusions from (x, y)

s.t.

$$\mathrm{e}^{Kt} d_{g(t)}(X_t, Y_t) \searrow \quad \text{a.s.}$$

$(\Rightarrow \forall \varphi \nearrow,\ \mathcal{T}_{\varphi(d)}(\mu_t, \nu_t) \searrow)$

(cf. [K. & Philipowski '09] for non-explosion)

§2 Coupling by reflection

Thm 1 [K. '10]

Suppose $\exists K \in \mathbb{R}$ s.t.

$$4(\nabla Z_t)^\flat + \partial_t g(t) \le 2 \operatorname{Ric}_{g(t)} - 2K g(t)$$

$\Rightarrow \forall x, y \in M$,

$\exists (X_t, Y_t)$: coupled $\mathcal{A}_t$-diff. from (x, y) s.t.

$$\mathbb{P}\Big[\inf_{T_1 \le s \le t} d_{g(s)}(X_s, Y_s) > 0\Big] \le \mathbb{P}\Big[\inf_{T_1 \le s \le t} \rho_s > 0\Big]$$

where ρ_t solves $\rho_{T_1} = d_{g(T_1)}(x, y)$ and

$$d\rho_t = 2\sqrt{2} dB_t - K \rho_t dt$$

Rem

- Heuristically,
  "$d_{g(t)}(X_t, Y_t) \leq \rho_t$" $\Rightarrow$ Thm 1
- (RHS) $= \varphi_{t-T_1}(d_{g(T_1)}(x, y))$,
  where

$$\varphi_s(a) := \sqrt{\frac{2}{\pi}} \int_0^{\frac{a}{2\sqrt{2}\sqrt{\beta(s)}}} e^{-x^2/2} dx,$$

$$\beta(s) := \frac{e^{Ks} - 1}{K}$$

Cor 1 [K. & Sturm]

Let $T_1 < T \leq T_2$. $\forall \mu_t, \nu_t$: heat distributions,

$$\mathcal{T}_{\varphi_{T-t}}(\mu_t, \nu_t) \searrow$$

Cor 2 [K.'10]

$$\left\| |\nabla P_{T_1,t} f|_{g(T_1)} \right\|_\infty \leq \frac{1}{2\sqrt{\pi\beta(t - T_1)}} \operatorname{osc} f$$

(cf. [Coulibaly '09] via stoch. diff. geom.)

§3 Sketch of the proof

(Suppose $T_1 = 0$ here, for simplicity)

Construct $(\boldsymbol{X}_t, \boldsymbol{Y}_t)$

where $d\boldsymbol{Y}_t =$ "(local) reflection" of $d\boldsymbol{Y}_t$

$\Rightarrow$ For $\sigma_t := d_{g(t)}(X_t, Y_t)$,

$$\boxed{d\sigma_t = 2\sqrt{2}\,dB_t + dV_t - dL_t}$$

- γ: unit-speed min. geod. from X_t to Y_t
- $\{e_i\}_{i=1}^m$: ONB of $T_{X_t}M$ with $e_1 = \dot{\gamma}$

$$\Downarrow$$

Mart. part:

$$\sqrt{2}\langle \nabla d_{g(t)}(X_t, Y_t), dX_t \otimes dY_t \rangle_{g(t)}$$

$$= \sqrt{2}\left(\langle \dot{\gamma}, dX_t \rangle + \langle \dot{\gamma}, dY_t \rangle \right)$$

$$= 2\sqrt{2}\langle \dot{\gamma}, dX_t \rangle$$

$\Rightarrow$ For $\sigma_t := d_{g(t)}(X_t, Y_t)$,

$$d\sigma_t = 2\sqrt{2}dB_t + dV_t - dL_t$$

- γ: unit-speed min. geod. from X_t to Y_t
- $\{e_i\}_{i=1}^m$: ONB of $T_{X_t}M$ with $e_1 = \dot{\gamma}$

$$\Downarrow$$

Bdd. var. part

(Ass.) $\Rightarrow dV_t \leq -2K\sigma_t dt$

$L_t \geq 0$: "local time" at $g(t)$-cut locus $\mathrm{Cut}_{g(t)}$

$\Rightarrow$ For $\sigma_t := d_{g(t)}(X_t, Y_t)$,

$$\boxed{d\sigma_t = 2\sqrt{2}dB_t + dV_t - dL_t}$$

- γ: unit-speed min. geod. from X_t to Y_t
- $\{e_i\}_{i=1}^m$: ONB of $T_{X_t}M$ with $e_1 = \dot{\gamma}$

$$\Downarrow$$

$$\boxed{\sigma_t \leq \rho_t}$$

Difficulty:

How to extract L_t?

and how small $\{t \mid (X_t, Y_t) \in \mathrm{Cut}_{g(t)}\}$ is?

$\left(\begin{array}{l}\text{When } \partial_t g(t) \equiv 0, \\ \text{[Kendall '86, Cranston '91, F.-Y. Wang '94/'05]}\end{array}\right)$

Our method:

Show "$\sigma_t \leq \rho_t$" in a weak sense

without extracting L_t explicitly

by using Random walk approximation

(When $\partial_t g(t) \equiv 0$, [von Renesse '04, K.'10])

Approximation by coupled RWs $(X_1^{\varepsilon}(t), X_2^{\varepsilon}(t))$

Thm 2 [K.'10] (invariance principle)

$$X^{\varepsilon} \xrightarrow{d} X \text{ as } \varepsilon \to 0.$$

Why does it works?

Discrete Itô formula:

$$\boldsymbol{\sigma}^{\varepsilon}_{\varepsilon^2 n} = \boldsymbol{\sigma}^{\varepsilon}_{\varepsilon^2 (n-1)} + \varepsilon \boldsymbol{\lambda}^{\varepsilon}_{n} + \varepsilon^2 \boldsymbol{\Lambda}^{\varepsilon}_{n} + o(\varepsilon^2).$$

When $(X^{\varepsilon}_{\varepsilon^2(n-1)}, Y^{\varepsilon}_{\varepsilon^2(n-1)}) \in \mathrm{Cut}_{g(\varepsilon^2(n-1))}$,

Dividing a min. geod. into two pieces:

Disadvantage: Discreteness of the Itô formula.

- (invariance principle)

  $$\varepsilon \sum_n \lambda_n^\varepsilon \to 2\sqrt{2}B.\ \text{in law}$$

- (Law of large numbers)

  $$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon^2 \sum_n \Lambda_n^\varepsilon \le \overline{\lim_{\varepsilon\to 0}} \left(-K\varepsilon^2 \sum_n \sigma_{\varepsilon^2 n}^\varepsilon \right)$$

(!) Λ_n^ε depends on λ_n^ε

(!) $\boxed{\Lambda_n^\varepsilon \le -K\sigma_{\varepsilon^2 n}^\varepsilon \text{ a.s.}}$ is NOT true

Thm 3 [K.'10] (unif. nonexplosion/localization)

$$\sup_{\varepsilon} \mathbb{P}_x \left[\sup_{0 \le s \le t} d_{g(s)}(o, X^{\varepsilon}(s)) > R \right] \to 0$$

as $R \to \infty$

$\Rightarrow$ Uniform control of $o(\varepsilon^2)$

$\Rightarrow$ $\varepsilon^2 \sum_n \Lambda_n^{\varepsilon} \approx \varepsilon^2 \sum_n \mathbb{E}[\Lambda_n^{\varepsilon} \mid \mathcal{F}_{n-1}]$

with arbitrary high probability (as $\varepsilon \to 0$).

$\star\ \mathbb{E}[\Lambda_n^{\varepsilon} \mid \mathcal{F}_{n-1}] \le -K\sigma^{\varepsilon}_{\varepsilon^2(n-1)}$

$\Rightarrow \forall \delta > 0,$

$$\sigma^{\varepsilon}_{\varepsilon^2 n} \leq \sigma^{\varepsilon}_0 + \varepsilon \sum_{m \leq n} \lambda^{\varepsilon}_m - K\varepsilon^2 \sum_{m \leq n} \sigma^{\varepsilon}_{\varepsilon^2 m} + \delta$$

with arbitrary high probability (as $\varepsilon \to 0$).

$\Downarrow$

$$\mathbb{P}\left[\inf_{n \leq \varepsilon^{-2} T} \sigma^{\varepsilon}_{\varepsilon^2 n} > \delta\right] \leq \mathbb{P}\left[\inf_{n \leq \varepsilon^{-2} T} \rho^{\varepsilon}_{\varepsilon^2 n} > \delta'\right] + \text{(error)}$$

$\Rightarrow$ Thm 1 ($\varepsilon \to 0$, $\delta \to 0$)

§4 Coupling by $\mathcal{L}$-parallel transport

(joint work with R. Philipowski)

Perelman's $\mathcal{L}$-distance

$$\gamma \ : \ [\tau_1, \tau_2] \longrightarrow M, \ [\tau_1, \tau_2] \subset [T_1, T_2]$$

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left(|\dot{\gamma}(\tau)|^2_{g(\tau)} + R_{g(\tau)}(\gamma(\tau)) \right) d\tau$$

$$L(\tau_1, x; \tau_2, y) := \inf \left\{ \mathcal{L}(\gamma) \ \middle| \ \begin{array}{l} \gamma(\tau_1) = x, \\ \gamma(\tau_2) = y \end{array} \right\}$$

Normalization

Given $T_1 \leq \bar{\tau}_1 < \bar{\tau}_2 \leq T_2$,

$$\Theta_t(x, y) := 2(\sqrt{\bar{\tau}_2 t} - \sqrt{\bar{\tau}_1 t}) L(\bar{\tau}_1 t, x; \bar{\tau}_2 t, y) - 2m(\sqrt{\bar{\tau}_2 t} - \sqrt{\bar{\tau}_1 t})^2$$

Thm 4 [K. & Philipowski '10]

Suppose $\partial_t g(t) = 2\,\mathrm{Ric}_{g(t)}$, $Z_t \equiv 0$,

$$\inf_{x\in M,\, t\in[T_1,T_2]} \mathrm{Ric}_{g(t)}(x) > -\infty$$

$\Rightarrow$ $\exists (X_1(\tau), X_2(\tau))$: coupling of $g(\tau)$-BMs

s.t. $(\Theta_t(X_1(\bar\tau_1 t), X_2(\bar\tau_2 t)))_{t\in[1,T_2/\bar\tau_2]}$

is a (local) supermartingale

- If $\sup_{x\in M,\tau\in[T_1,T_2]} \left|\mathrm{Rm}_{g(\tau)}\right|_{g(\tau)}(x) < \infty$,

  then $(\Theta_t(X_1(\bar\tau_1 t), X_2(\bar\tau_2 t)))$: supermart.

Cor 3 [K. & Philipowski '10]

$\forall \varphi$: $\nearrow$, $\sqcup\!\sqcup$ & $\forall \mu_t, \nu_t$: heat distributions,
$\mathcal{T}_{\varphi(\Theta_t)}(\mu_{\bar{\tau}_1 t}, \nu_{\bar{\tau}_2 t}) \searrow$

- [Topping '09]: $\mathcal{T}_{\Theta_t}(\mu_{\bar{\tau}_1 t}, \nu_{\bar{\tau}_2 t}) \searrow$ when M:cpt, via optimal transport techniques

Strategy of the Proof

- Properties of $\mathcal{L}$-distance
  being analogous to the Riem. distance
  - $\mathcal{L}$-geodesic, 1st & 2nd variation of $\mathcal{L}$-length, $\mathcal{L}$-index lemma, $\mathcal{L}$-cut locus
- Approximation by RWs
- Coupling of $dX_1^\varepsilon(\bar{\tau}_1 t)$ and $dX_2^\varepsilon(\bar{\tau}_2 t)$ by spacetime-parallel transport along $\mathcal{L}$-geodesic

Spacetime parallel transport

For $\gamma : [s,t] \longrightarrow M$ & V: vector field along γ,

V: spacetime parallel

$$\overset{\text{def}}{\Leftrightarrow} \nabla^{g(u)}_{\dot\gamma(u)} V(u) = -\frac{1}{2}\partial_u g(u)^{\#} V(u)$$

$\mathfrak{L}$-geodesic

$\gamma : [s,t] \longrightarrow M$: $\mathfrak{L}$-geodesic

$$\overset{\text{def}}{\Leftrightarrow} \nabla^{g(u)}_{\dot\gamma_u} \dot\gamma_u = \frac{1}{2}\nabla^{g(u)} R_{g(u)} - 2\,\mathrm{Ric}^{\#}_{g(u)}(\dot\gamma_u) - \frac{1}{2u}\dot\gamma_u$$

$\sqrt{u}\dot\gamma_u$ is NOT spacetime parallel to γ!

§5 Convergence of geodesic random walks

Proof of Thm 2 (Invariance principle)

Reduced to prove tightness

($\Leftarrow$! of the mart. pbm. for $\partial_t + \Delta_{g(\cdot)}$)

Thm 3 $\Rightarrow$ localize the problem. The rest is easy.

Proof of Thm 3 (unif. non-explosion estimate)

- Discrete Itô formula for $d_{g(t)}(o, X_t^\varepsilon)$
- local comparability of metrics
- Take some care on singularity at $X_t^\varepsilon = o$
- Comparison thm for difference eq.'s