Duality results on gradient estimates and Wasserstein controls

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§1 Framework and main results

X: Polish space

- $(P_x)_{x \in X} \subset \mathcal{P}(X)$: Markov kernel

$$Pf(x) := \int_X f \, dP_x,$$

$$P^* \mu(A) := \int_X P_x(A) \mu(dx)$$

Assume $P(C_b(X)) \subset C_b(X)$
(e.g. $P_x(dy) = P_t(x, dy)$: heat semigroup)

- d, $\tilde{d}$: lower semi-conti. pseudo-distance on X
(e.g. d: distance on X, $\tilde{d} = \mathrm{e}^{-kt} d$)

$$\Pi(\mu,\nu) := \left\{\pi \;\middle|\; \pi\circ p_1^{-1} = \mu,\ \pi\circ p_2^{-1} = \nu\right\}$$

(couplings of $\mu, \nu \in \mathcal{P}(X)$)

L^p-Wasserstein distance

For $p \in [1, \infty]$,

$$d_p^W(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \|d\|_{L^p(\pi)} \in [0,\infty]$$

L^p-Wasserstein control

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu,\nu) \qquad (C_p)$$

Gradient

$$|\nabla_d f|(x) := \lim_{r \downarrow 0} \sup_{y; d(x,y) \leq r} \left| \frac{f(x) - f(y)}{d(x, y)} \right|,$$

$$\|\nabla_d f\|_\infty := \sup_{x \in X} |\nabla_d f|(x)$$

Subgradient

$$|\nabla_d^- f|(x) := \lim_{r \downarrow 0} \sup_{y; d(x,y) \leq r} \left[\frac{f(x) - f(y)}{d(x, y)} \right]_+,$$

$$\|\nabla_d^- f\|_\infty := \sup_{x \in X} |\nabla_d^- f|(x)$$

L^q-gradient estimate

$$|\nabla_{\tilde{d}} P f|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

for $\forall f \in C_b^{\mathrm{Lip}(d)}$ when $q \in [1, \infty)$,

$$\|\nabla_{\tilde{d}} P f\|_\infty \le \|\nabla_d f\|_\infty \qquad (G_\infty)$$

when $q = \infty$

$$\Big([\nabla \rightsquigarrow \nabla^-] \Rightarrow [(G_q) \rightsquigarrow (G_q^-)]\Big)$$

The first duality result

Theorem A (K. '10 JFA)

For $p, q \in [1, \infty]$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q)$

(ii) Under Assumptions A1-A3, $(G_q) \Rightarrow (C_p)$

The second duality result

Theorem B (K. '10)

For $p, q \in [1, \infty]$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q^-)$

(ii) Under Assumptions B1-B3, $(G_q^-) \Rightarrow (C_p)$

Remarks (without Assumptions A/B)

- For $p' > p$,
  $(C_{p'}) \Rightarrow (C_p)$ and $(G_{q'}) \Rightarrow (G_q)$

- $(C_1) \Leftrightarrow (G_\infty)$ is well known
  (via Kantorovich-Rubinstein formula)

- $(C_\infty) \Rightarrow (G_1)$ is essentially well known
  (Coupling method)

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Interesting part: $(G_q) \Rightarrow (C_p)$ for $p \in (1, \infty]$

§2 Sketch of the proof of $(G_q) \Rightarrow (C_p)$

Recall:

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu,\nu) \qquad (C_p)$$

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

Recall:

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- (C_p) for $\forall p < \infty \Rightarrow (C_\infty)$

$\rightsquigarrow$ We may assume $p \in (1, \infty)$

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- (C_p) for $\forall p < \infty \Rightarrow (C_\infty)$

  $\rightsquigarrow$ We may assume $p \in (1,\infty)$

- (C_p) for $\mu = \delta_x, \nu = \delta_y \Rightarrow (C_p)$

  $\rightsquigarrow$ We show $\dfrac{d_p^W(P_x, P_y)^p}{p} \le \dfrac{\tilde{d}(x,y)^p}{p}$

Kantorovich duality

$$\frac{d_p^W(\mu,\nu)^p}{p} = \sup_{f\in C_b^{\mathrm{Lip}_d}} \left[\int_X f^* \, d\mu - \int_X f \, d\nu\right]$$

$$f^*(x) := \inf_{y\in X} \left[f(y) + \frac{1}{p} d(x,y)^p \right]$$

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$$\left(\left[\begin{array}{l} \forall x, \forall y,\ g(x) - f(y) \le \frac{1}{p} d(x,y)^p \\ \Rightarrow \frac{1}{p} \|d\|_{L^p(\pi)}^p \ge \int_X g \, d\mu - \int_X f \, d\nu \end{array} \right] \Rightarrow \text{“}\ge\text{”} \right)$$

Hamilton-Jacobi semigroup

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + t \cdot \frac{1}{p} \left(\frac{d(x,y)}{t} \right)^p \right]$$

$$\Rightarrow f^* = Q_1 f$$

We expect (under our assumptions):

- $Q_{.} f(x)$: Lipschitz, $Q_t f(\cdot)$: d-Lipschitz
- Hamilton-Jacobi equation

$$\partial_t Q_t f = -\frac{1}{q} |\nabla_d Q_t f|^q$$

$$\begin{cases} \tilde{\gamma} \ : \ [0,1] \longrightarrow X \quad \tilde{d}\text{-minimal geodesic}, \\ \tilde{\gamma}_0 = y, \ \tilde{\gamma}_1 = x, \\ \tilde{d}(\tilde{\gamma}_s, \tilde{\gamma}_t) = |t-s|\tilde{d}(x,y) \end{cases}$$

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$$\frac{d_p^W(P_x, P_y)^p}{p} = \sup_f \left[PQ_1 f(x) - Pf(y) \right]$$

$$\text{interpolation} \boxed{=} \sup_f \left[\int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t)) dt \right]$$

$$\partial_t(PQ_tf(\tilde{\gamma}_t))$$

$$\left(\text{“}=\text{”}\ P(\partial_t Q_t f)(\tilde{\gamma}_t) + \langle \nabla PQ_t f(\tilde{\gamma}_t), \dot{\tilde{\gamma}}_t \rangle\right)$$

$$\underset{\text{upp. grad.}}{\overset{\text{HJ eq.}}{\boxed{\leq}}} -\frac{1}{q}P(|\nabla_d Q_t f|^q)(\tilde{\gamma}_t) + \tilde{d}(x,y)\left|\nabla_{\tilde{d}} PQ_t f\right|(\tilde{\gamma}_t)$$

$$\overset{(G_q)}{\boxed{\leq}} \tilde{d}(x,y)\sigma - \frac{1}{q}\sigma^q \leq \frac{\tilde{d}(x,y)^p}{p}$$

$$\left(\sigma := P(|\nabla_d Q_t f|^q)(\tilde{\gamma}_t)^{1/q}\right) \qquad \blacksquare$$

§3 Assumptions

(1) Assumptions A1-A3

v: Radon measure on X with $\mathrm{supp}(v) = X$

Assumption A1

- d: compatible with the topology on X
- d: length metric
- $\forall r > 0$, $\forall x$, $\{y \mid d(x, y) \leq r\}$: compact
- local (uniform) volume doubling condition
- $(1, \rho)$-local Poincaré inequality ($\exists \rho \geq 1$)

Assumptions A1 $\Rightarrow$ HJ eq. for $Q_t f$ υ-a.e.

(Lott & Villani '07 JMPA

Balogh, Engoulatov, Hunziker & Maasalo '09)

Assumption A2

$\tilde{d}$: conti. length metric

Assumption A3

$$P_x \ll \upsilon,\ x \mapsto \frac{dP_x}{d\upsilon}(y)\text{: conti}$$

(2) Assumptions B1-B3

“Assumption B1”

- d: length pseudo-metric
- $Q_t f$ is measurable for $\forall f \in C_b(X)$
- $d(x, \cdot)^2$ is locally semiconcave
- d-geodesic is locally uniformly extendable

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Assumption B1 $\Rightarrow$ HJ eq. for $Q_t f$ w.r.t. ∇^-

& local uniform bound of $\frac{1}{s}(Q_{t+s}f - Q_t f)$

(Extension of an argument in Villani’s book)

Assumption B2

$\tilde{d}$: length pseudo-metric

Assumption B3

$\forall f \in C_b(X)$, $PQ_t f$: $\tilde{d}$-upper semi-conti.

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When $[\tilde{d}(x_n, x) \to 0] \Rightarrow [x_n \to x$ in $X]$,

- P: strong Feller, or
- $\forall f \in C_b(X)$, $Q_t f$: u.s.c.
  (true if X: vect sp., $\tilde{d}$: seminorm)

$\Rightarrow$ Assumption B3

Examples

Examples satisfying A1-A2

- Complete Riemannian manifold with $\mathrm{Ric} \geq k_0$
- Carnot groups (see below)

($\dim X < \infty$, $P_x \ll v$ for a base measure v)

Examples satisfying B1-B2

- Cpl. Riem. mfds (no curvature assumption)
- Vecter space, d: Hilbertian seminorm

($P_x \ll v$ is not necessary)

§4 Related results and Applications

(1) Connection with geometry, curvature bound

Equivalent conditions for a lower Ricci curvature bound

von Renesse & Sturm '05 CPAM, etc...

X: cpl. Riem. mfd

P_t: heat semigroup associated with Δ

(i) $\mathrm{Ric} \ge k$

(ii) $d_p^W(P_t^* \mu, P_t^* \nu) \le \mathrm{e}^{-kt} d_p^W(\mu, \nu)$
for some $p \in [1, \infty]$, $\forall t > 0$

(iii) $|\nabla P_t f|(x) \le \mathrm{e}^{-kt} P_t(|\nabla f|^q)(x)^{1/q}$
for some $q \in [1, \infty]$, $\forall t > 0$

Remark

"(a lower Ricci bound) $\Rightarrow$ (C_p)" in the literature

No direct way "$(G_q) \Rightarrow (C_p)$" was known

E.g. in von Renesse & Sturm '05,

$$
\begin{array}{ccccccc}
\mathrm{Ric} \geq k & & & & & & \\
\Downarrow \ \text{coupling method} & & & & & & \\
(C_\infty) & \Rightarrow & (C_p) & \Rightarrow & (C_1) & & \\
\Downarrow & & & & \Downarrow & & \\
(G_1) & \Rightarrow & (G_q) & \Rightarrow & (G_\infty) & \Rightarrow & \mathrm{Ric} \geq k \\
 & & & & & \text{Bochner} &
\end{array}
$$

(2) Hörmander-type operators on a Lie group

3-dim. Heisenberg group

$X := \mathbb{R}^3$, v: Lebesgue

$$(x, y, z) \cdot (x', y', z') := (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx'))$$

$$X_1 := \partial_x - \frac{y}{2}\partial_z , \quad X_2 := \partial_y + \frac{x}{2}\partial_z$$

$$A := \frac{1}{2}\left(X_1^2 + X_2^2\right), \quad P = P_t := \mathrm{e}^{tA}$$

Diffusion $\boldsymbol{B}_t = (B_t^1, B_t^2, B_t^3)$ associated with A starting from (x, y, z), i.e.,

$$
\begin{aligned}
d\boldsymbol{B}_t &= \boldsymbol{X}_1(\boldsymbol{B}_t)dW_t^1 + \boldsymbol{X}_2(\boldsymbol{B}_t)dW_t^2, \\
\boldsymbol{B}_0 &= (x, y, z)
\end{aligned}
$$

More explicitly,

$$
B_t^1 = W_t^1, \qquad B_t^2 = W_t^2,
$$

$$
B_t^3 = z + \frac{1}{2}\int_0^t W_t^1 dW_t^2 - W_t^2 dW_t^1,
$$

where (W_t^1, W_t^2): 2-dim. BM starting from (x, y)

$|\Gamma f| := |X_1 f|^2 + |X_2 f|^2$: carré du champ

L^q-gradient estimate

$\exists K_q > 1,$

$$|\Gamma P_t f|(x) \leq K_q P_t(|\Gamma f|^{q/2})(x)^{2/q} \qquad (G_q^*)$$

- $q > 1$: Driver & Melcher '05 JFA
- $q = 1$: H.-Q. Li '06 JFA/ Bakry, Baudoin, Bonnefont & Chafaï '08 JFA

Carnot-Caratheodory distance

For $V \in T_x X$,

$$|V| := \begin{cases} \sqrt{a_1^2 + a_2^2} & \text{if } V = a_1 X_1 + a_2 X_2, \\ \infty & \text{otherwise.} \end{cases}$$

$$d(x, y) := \inf \left\{ \int_0^1 |\dot{\gamma}_s| ds \;\middle|\; \begin{array}{l} \gamma_0 = x, \\ \gamma_1 = y \end{array} \right\}$$

Proposition

(i) $(X, d, v; P)$ satisfies Assumption A1-A3

(ii) $(G_q^*) \Rightarrow (G_q)$

Corollary

$(G_q^*) \Rightarrow (C_p)$ for $p \in [1, \infty]$

(C_∞): For each $t > 0$,

$\exists$ a coupling $(\boldsymbol{B}_t, \tilde{\boldsymbol{B}}_t)$ of (B_t^1, B_t^2, B_t^3) s.t.

$$d(\boldsymbol{B}_t, \tilde{\boldsymbol{B}}_t) \le K_1 d(\boldsymbol{B}_0, \boldsymbol{B}_0) \quad \mathbb{P}\text{-a.s.}$$

Remark $\exists C_1, C_2 > 0$ s.t.

$$C_1 \|\boldsymbol{b}^{-1}\boldsymbol{a}\| \le d(\boldsymbol{a}, \boldsymbol{b}) \le C_2 \|\boldsymbol{b}^{-1}\boldsymbol{a}\|,$$

where $\|(x, y, z)\| = \left((x^2 + y^2)^2 + z^2\right)^{1/4}$

Extension of (G_q^*)

- X: general, $q > 1$: Melcher '08 SPA ($K_q(t) \equiv K_q$ if X: nilpotent)
- X: group of type H, $q = 1$, $K_q(t) \equiv K_q$: Eldredge '10 JFA
- $X = SU(2)$, $q > 1$, $K_q(t) = K_q e^{-t}$: Baudoin & Bonnefont '09 Math. Z.

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Proposition (and Corollary) is still valid
$\Rightarrow$ Theorem A implies (C_p)