Duality on gradient estimates and Wasserstein controls

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§1 Framework and the main result

(X, d): Polish metric space

- $(P(x,\cdot))_{x\in X} \subset \mathcal{P}(X)$: Markov kernel

$$Pf(x) := \int_X f(y)\, dP(x, dy),$$

$$P\mu(A) := \int_X P(x, A)\mu(dx)$$

Assume $P(C_b(X)) \subset C_b(X)$
(e.g. $P(x, dy) = P_t(x, dy)$: heat semigroup)

- $\tilde{d}$: another distance on X
(e.g. $\tilde{d} = \mathrm{e}^{-kt} d$)

$\Pi(\mu,\nu)$: set of couplings between $\mu,\nu \in \mathcal{P}(X)$

<u>L^p-Wasserstein distance</u>

For $p \in [1,\infty]$,

$$d_p^W(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \|d\|_{L^p(\pi)} \in [0,\infty]$$

<u>L^p-Wasserstein control</u>

$$\boxed{d_p^W(P\mu, P\nu) \leq \tilde{d}_p^W(\mu,\nu)} \qquad (C_p)$$

Gradient

$$|\nabla_d f|(x) := \lim_{r\downarrow 0} \sup_{y;d(x,y)\le r} \left| \frac{f(x) - f(y)}{d(x,y)} \right|$$

L^q-gradient estimate

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

$$\|\nabla_{\tilde{d}} Pf\|_\infty \le \|\nabla_d f\|_\infty \qquad (G_\infty)$$

for $\forall f \in C_b^{\mathrm{Lip}(d)}$

Theorem [K. '10]

For $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q)$

(ii) Under Assumptions below, $(G_q) \Rightarrow (C_p)$

Assumptions

(i) $d, \tilde{d}$: geodesic distance, (X, d): loc. cpt.

(ii) $\exists v$: pos. Radon meas. on X, $\mathrm{supp}[v] = X$ supporting

- (local uniform) volume doubling condition
- (local uniform) (1,ρ)-Poincaré inequality

(iii) $P(x, \cdot) \ll v$, $x \mapsto \dfrac{dP(x, \cdot)}{dv}(y)$: conti.

Remarks (without Assumptions)

- $p' > p$: $(C_{p'}) \Rightarrow (C_p)$ & $(G_{q'}) \Rightarrow (G_q)$
- $(C_1) \Leftrightarrow (G_\infty)$ is well known
- [von Renesse & Sturm '05]
  X: cpl. Riem. mfd., $P = P_t$, $\tilde{d} = \mathrm{e}^{-Kt} d$

$$\Rightarrow \boxed{(C_p) \Leftrightarrow (G_q) \Leftrightarrow \mathrm{Ric} \geq K}$$

(No direct proof of $(G_q) \Rightarrow (C_p)$)

§2 Sketch of the proof of $(G_q) \Rightarrow (C_p)$

Recall:

$$d_p^W(P\mu, P\nu) \leq \tilde{d}_p^W(\mu, \nu) \qquad (C_p)$$

$$|\nabla_{\tilde{d}} Pf|(x) \leq P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

Recall:

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What we prove: For $p \in (1, \infty)$, (G_q) implies

$$\frac{d_p^W(P\delta_x, P\delta_y)^p}{p} \le \frac{\tilde{d}(x,y)^p}{p}$$

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Def(Hamilton-Jacobi semigroup):

$$Q_t f(x) := \inf_{y \in X} \left(f(y) + \frac{t}{p} \left(\frac{d(x,y)}{t} \right)^p \right)$$

By the Kantorovich duality,

$$\frac{d_p^W(P\delta_x, P\delta_y)^p}{p} = \sup_{f \in C_b^{\mathrm{Lip}(d)}} [PQ_1 f(x) - Pf(y)]$$

$$= \sup_f \left[\int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t)) dt \right]$$

($\tilde{\gamma}$: $\tilde{d}$-minimal geodesic from y to x)

$$\leq \cdots \leq \frac{\tilde{d}(x,y)^p}{p}$$

(HJ eq. "$\partial_t Q_t f = -q^{-1} |\nabla_d Q_t f|^q$", (G_q), ...)

§3 Applications

(1) Hörmander-type operators on a Lie group

Diffusion process on the Heisenberg group

$\boldsymbol{X} := \mathbb{R}^3$, v: Lebesgue

$$dB_t^1 = dW_t^1, \qquad dB_t^2 = dW_t^2,$$

$$dB_t^3 = \frac{1}{2} W_t^1 dW_t^2 - \frac{1}{2} W_t^2 dW_t^1,$$

where (W_t^1, W_t^2): 2-dim. BM

$\rightsquigarrow$ $\exists$ the associated sub-Riemannian structure

Gradient estimates:

$$|\nabla P_t f|(x) \le K P_t(|\nabla f|^q)(x)^{1/q}$$

(K must be > 1)

- $q > 1$: [Driver & Melcher '05]
- $q = 1$: [H.-Q. Li '06],
  [Bakry, Baudoin, Bonnefont & Chafaï '08]

Our Theorem $\Rightarrow$ (C_∞)

i.e., $\forall t > 0$, $x, y \in M$, $\exists \pi \in \Pi(P_t \delta_x, P_t \delta_y)$
s.t. $d(z,w) \le K d(x,y)$ π-a.e.(z,w)

Extension of the gradient estimate

X: Lie group, $P = P_t = \mathrm{e}^{tL}$

L: Hörmander op. assoc. with left-inv. vector fields

- X: general, $q > 1$: [Melcher '08] ($K_q(t) \equiv K_q$ if X: nilpotent)
- X: group of type H, $q = 1$, $K_q(t) \equiv K_q$: [Eldredge '10]

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Our theorem is also applicable to them!

(2) Heat semigroup on Alexandrov spaces

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with "sectional curvature $\geq k$"

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Heat flow on cpt. Alex. sp., as a gradient flow of

- Dirichlet energy in L^2 ($\Rightarrow \exists p_t(x, y)$: conti.) [Kuwae, Machigashira & Shioya '01]

- relative entropy on Wasserstein space ($\Rightarrow (C_2)$) [Savaré '07, Ohta '09]

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  [Kuwae, Machigashira & Shioya ’01]

- relative entropy on Wasserstein space ($\Rightarrow$ (C_2))
  [Savaré ’07, Ohta ’09]

These two notions coincide [Gigli, K. & Ohta ’10]

$\Rightarrow$ (G_2), $p_t(x, \cdot)$: Lipschitz

Analytic characterizations of "$\mathrm{Ric} \geq K_0$":

[G.K.O. op.cit.] (M:cpl. mfd $\Leftarrow$ [vR. & S. op.cit.])

For $K_0 \in \mathbf{R}$, the following are equivalent:

- $d_2^W(P_t\mu, P_t\nu) \leq \mathrm{e}^{-K_0 t} d_2^W(\mu, \nu)$,
  $\forall \mu, \nu \in \mathcal{P}(X), \forall t > 0$

- $|\nabla P_t f|(x) \leq \mathrm{e}^{-K_0 t} P_t(|\nabla f|^2)(x)^{1/2}$
  a.e.x, $\forall f \in W^{1,2}(X), \forall t > 0$

- "Weak form" of Γ_2-condtion
  $$\Delta(|\nabla f|^2) - 2\langle \nabla f, \nabla \Delta f \rangle \geq 2K_0 \langle \nabla f, \nabla f \rangle$$