Coupling methods under a backward Ricci flow

Kazumasa Kuwada

(Ochanomizu University
Universität Bonn)

§1 Framework

M: m-dim. manifold, $0 \leq T_1 < T_2 \leq \infty$

$(g(t))_{t \in [T_1, T_2]}$: smooth complete Riemannian metrics on M

———— ∘ ———— ∘ ————

$(X(t))_{t \in [T_1, T_2]}$: $g(t)$-Brownian motion, i.e.

$$f(t, X(t)) - f(T_1, X(T_1)) - \int_{T_1}^{t} \left(\frac{\partial}{\partial s} + \Delta_{g(s)} \right) f(s, X(s)) ds$$

is a local martingale for $\forall f$: smooth

(Construction: [Coulibaly '09] via SDE on $\mathcal{F}(M)$)

§2 Coupling by reflection

Theorem 1 [K. '10]

Suppose $\exists K \in \mathbb{R}$ s.t.

$$\partial_t g(t) \leq 2\,\mathrm{Ric}_{g(t)} - 2K \qquad (\star)$$

Theorem 1 [K. '10]

Suppose $\exists K \in \mathbb{R}$ s.t.

$$\partial_t g(t) \leq 2\,\mathrm{Ric}_{g(t)} - 2K \qquad (\star)$$

$\Rightarrow \exists (X_1(t), X_2(t))$: coupling of $g(t)$-BMs s.t.

$$\mathbb{P}[X_1(s) \neq X_2(s) \text{ for } T_1 \leq s \leq t]$$

$$\leq \mathbb{P}\left[\inf_{T_1 \leq s \leq t} U_{d_{g(T_1)}(X_1(0),X_2(0))}(s) > 0\right]$$

where $U_a(t)$ solves $U_a(T_1) = a$ and

$$dU_a(t) = 2\sqrt{2}dW(t) - KU_a(t)dt$$

Remarks

- Heuristically,
  "$d_{g(t)}(X_1(t), X_2(t)) \leq U(t)$" $\Rightarrow$ Thm
- Thm implies the following gradient estimate:

$$|dP_{T_1,t}f|_{g(T_1)} \leq \sqrt{\frac{K}{2\pi(\mathrm{e}^{K(t-T_1)}-1)}}\|f\|_\infty$$

$$\left(\begin{array}{c}\text{backward Ricci flow } (K = 0 \text{ \& ``='' in } (\star)) \\ \Rightarrow \text{[Coulibaly '09] via stoch. diff. geom.}\end{array}\right)$$

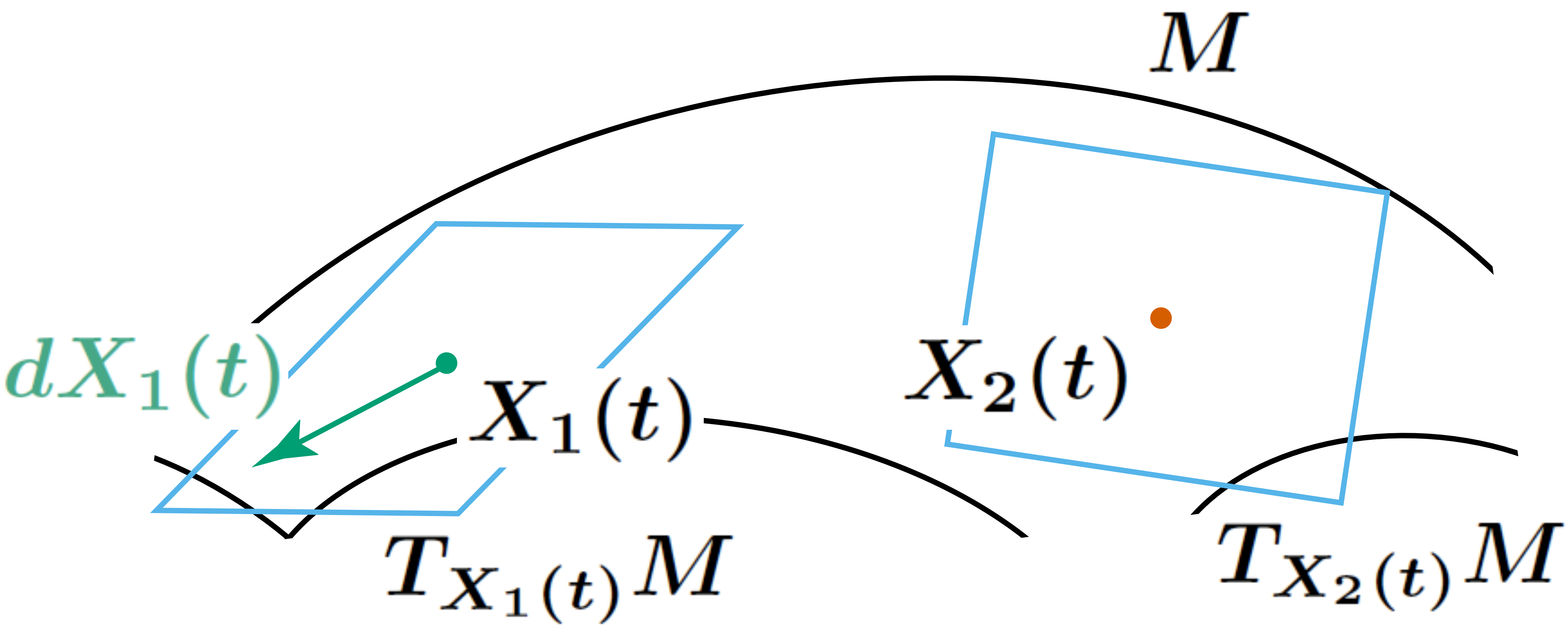
§3 Idea of the proof

Construct $(\boldsymbol{X}_1(t), \boldsymbol{X}_2(t))$

where $d\boldsymbol{X}_2(t) =$ "(local) reflection" of $d\boldsymbol{X}_1(t)$

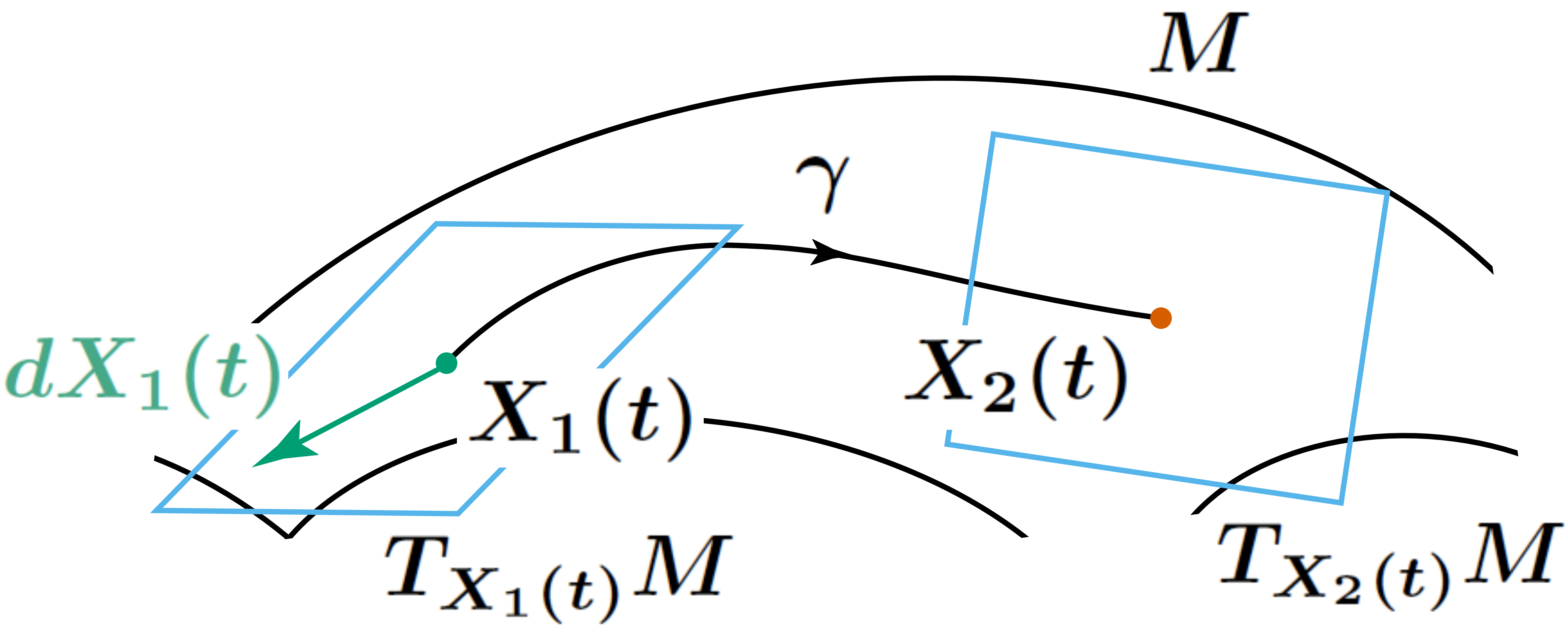
Construct $(X_1(t), X_2(t))$

where $dX_2(t) =$ "(local) reflection" of $dX_1(t)$



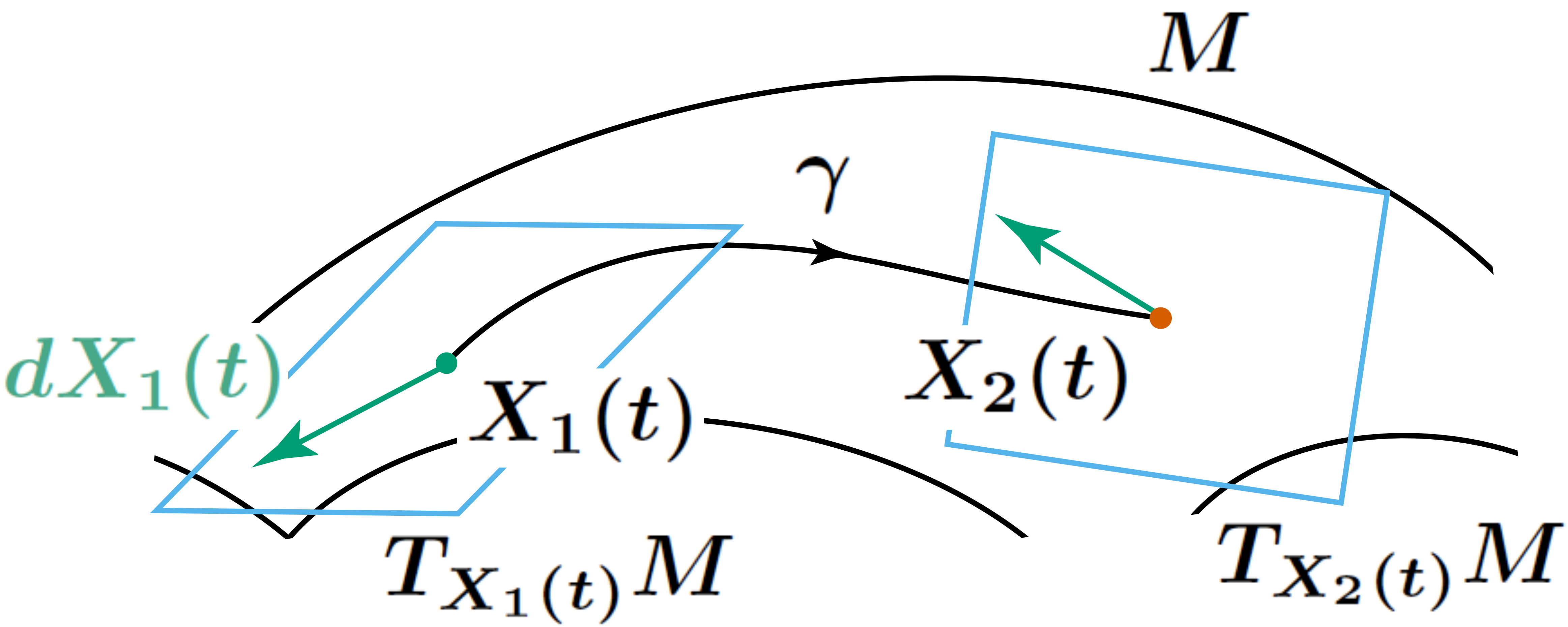
Construct $(X_1(t), X_2(t))$

where $dX_2(t) =$ "(local) reflection" of $dX_1(t)$



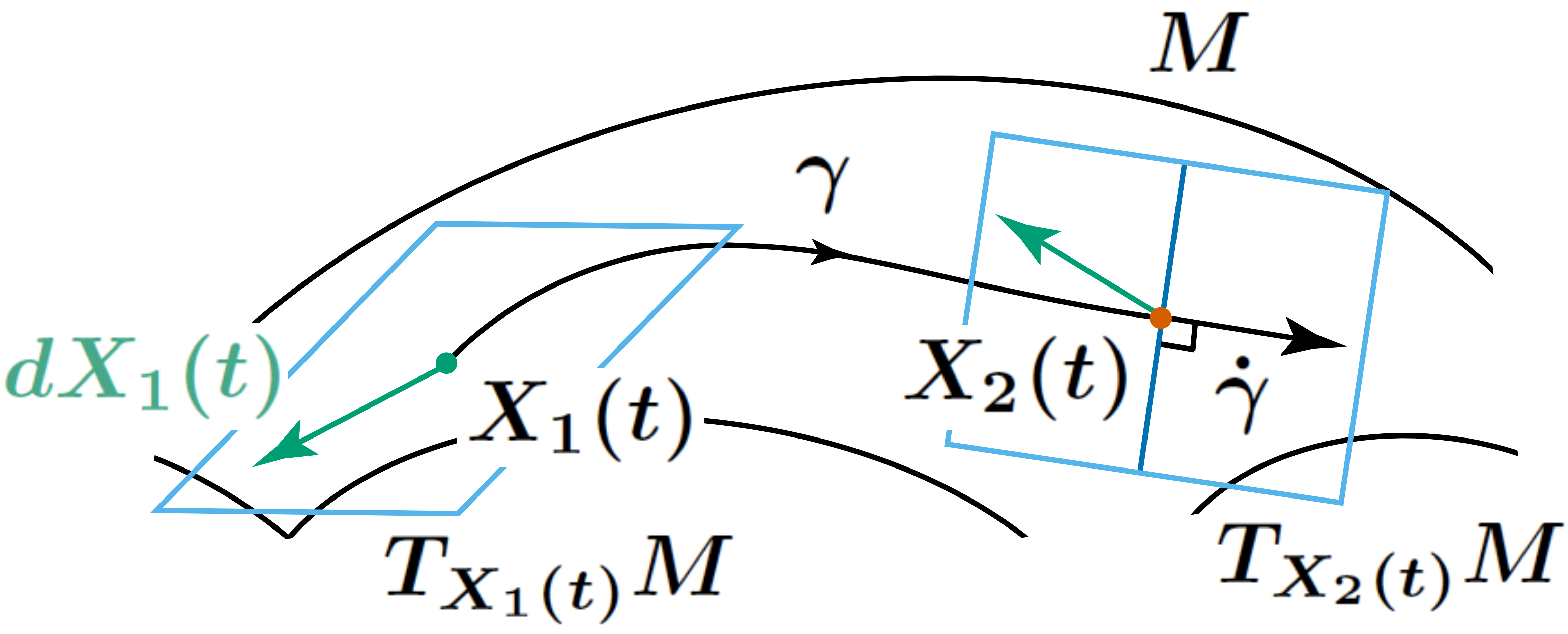
Construct $(\boldsymbol{X}_1(t), \boldsymbol{X}_2(t))$

where $d\boldsymbol{X}_2(t) =$ "(local) reflection" of $d\boldsymbol{X}_1(t)$



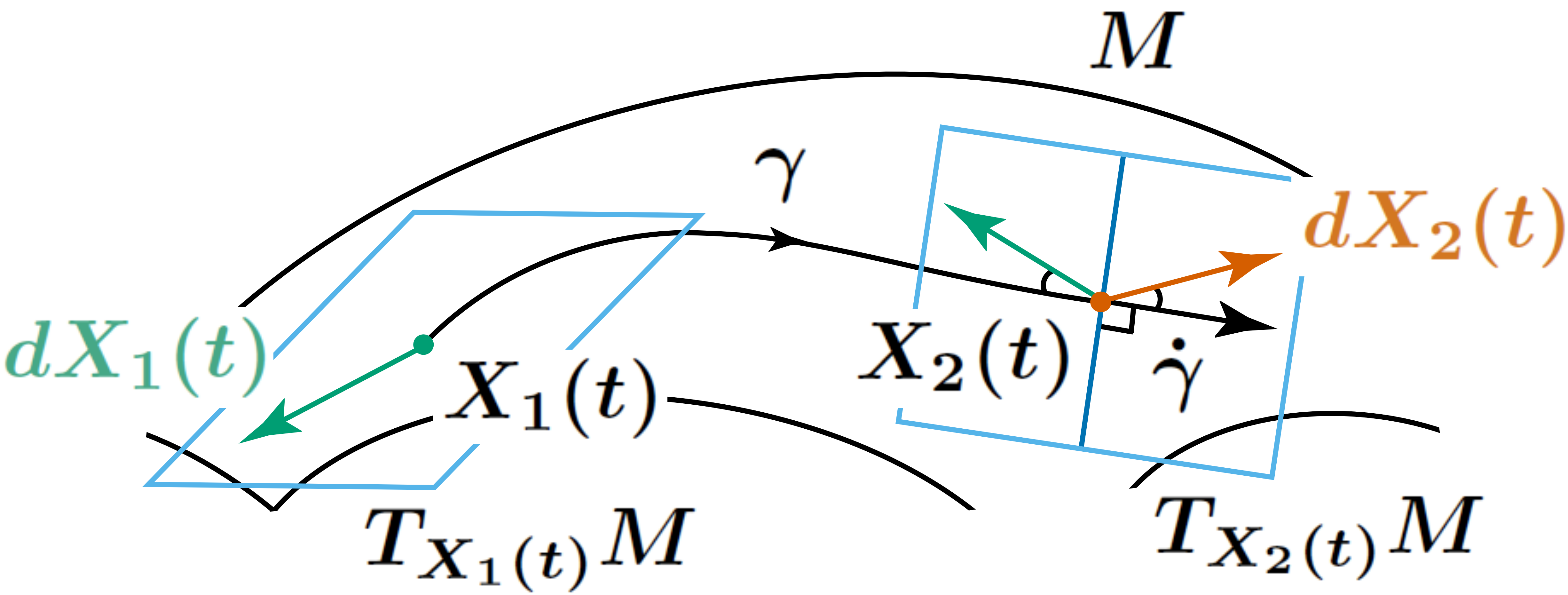
Construct $(X_1(t), X_2(t))$

where $dX_2(t) =$ "(local) reflection" of $dX_1(t)$



Construct $(X_1(t), X_2(t))$

where $dX_2(t) =$ "(local) reflection" of $dX_1(t)$

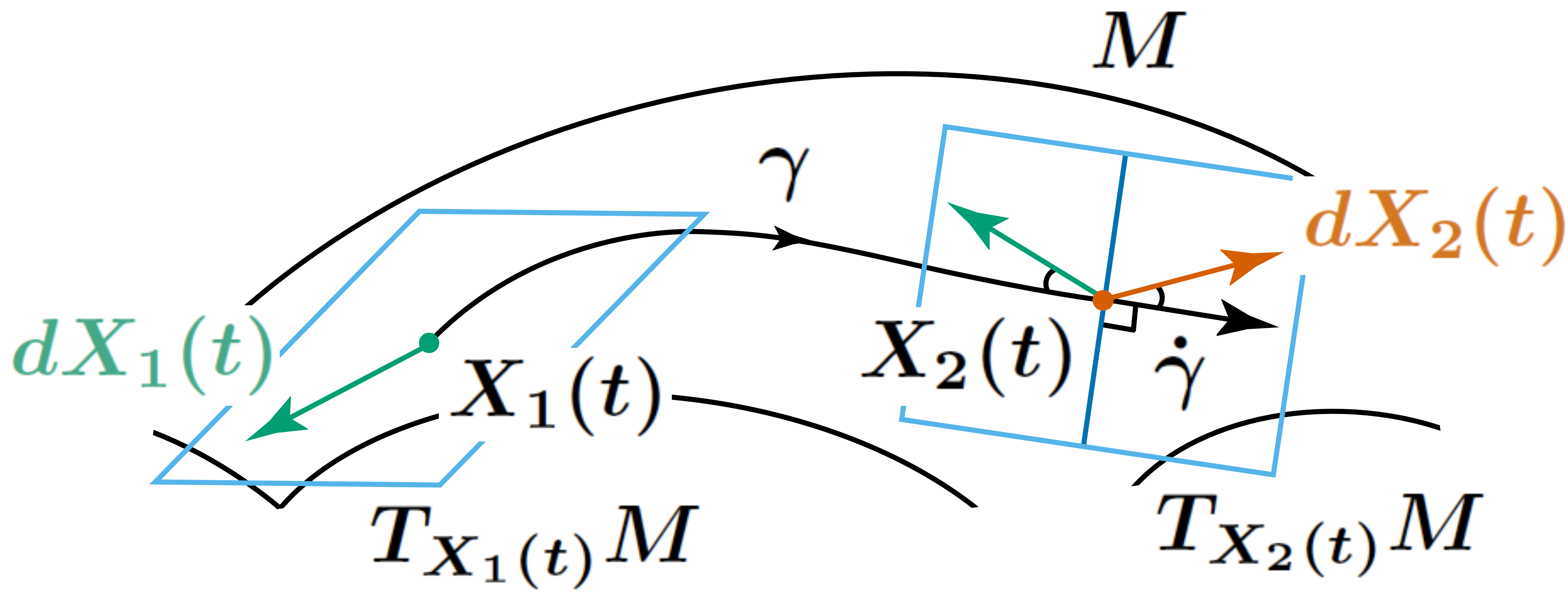


Construct $(X_1(t), X_2(t))$

where $dX_2(t) =$ "(local) reflection" of $dX_1(t)$

$\Rightarrow$ For $\rho(t) := d_{g(t)}(X_1(t), X_2(t))$,

$$d\rho(t) = 2\sqrt{2}dW(t) + dA_t - dL_t$$



$$d\rho(t) = 2\sqrt{2}dW(t) + dA_t - dL_t$$

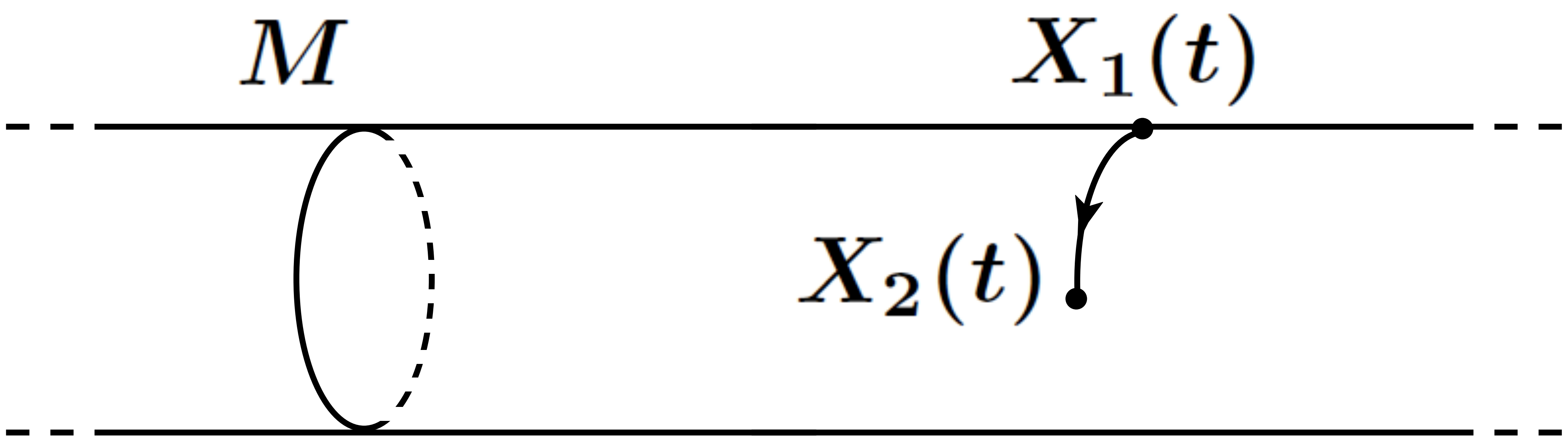
- (reflection) $\rightsquigarrow 2\sqrt{2}dW(t)$
- $(\star) \Rightarrow dA_t \leq -K\rho(t)dt$
- $L_t \geq 0$: "local time" at singular points of $d_{g(t)}$

$\Rightarrow \rho(t) \leq U(t)$

$$d\rho(t) = 2\sqrt{2}dW(t) + dA_t - dL_t$$

- (reflection) $\rightsquigarrow 2\sqrt{2}dW(t)$
- $(\star) \Rightarrow dA_t \leq -K\rho(t)dt$
- $L_t \geq 0$: "local time" at singular points of $d_{g(t)}$

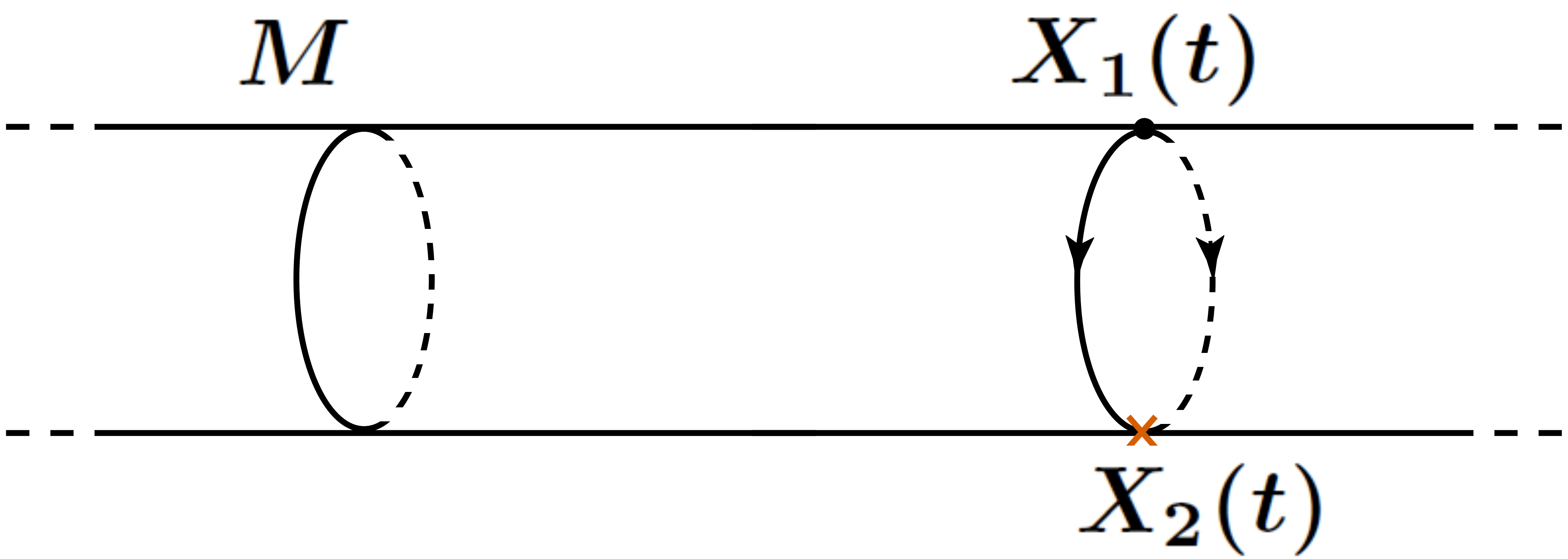
$\Rightarrow \rho(t) \leq U(t)$



$$d\rho(t) = 2\sqrt{2}dW(t) + dA_t - dL_t$$

- (reflection) $\rightsquigarrow 2\sqrt{2}dW(t)$
- $(\star) \Rightarrow dA_t \leq -K\rho(t)dt$
- $L_t \geq 0$: "local time" at singular points of $d_{g(t)}$

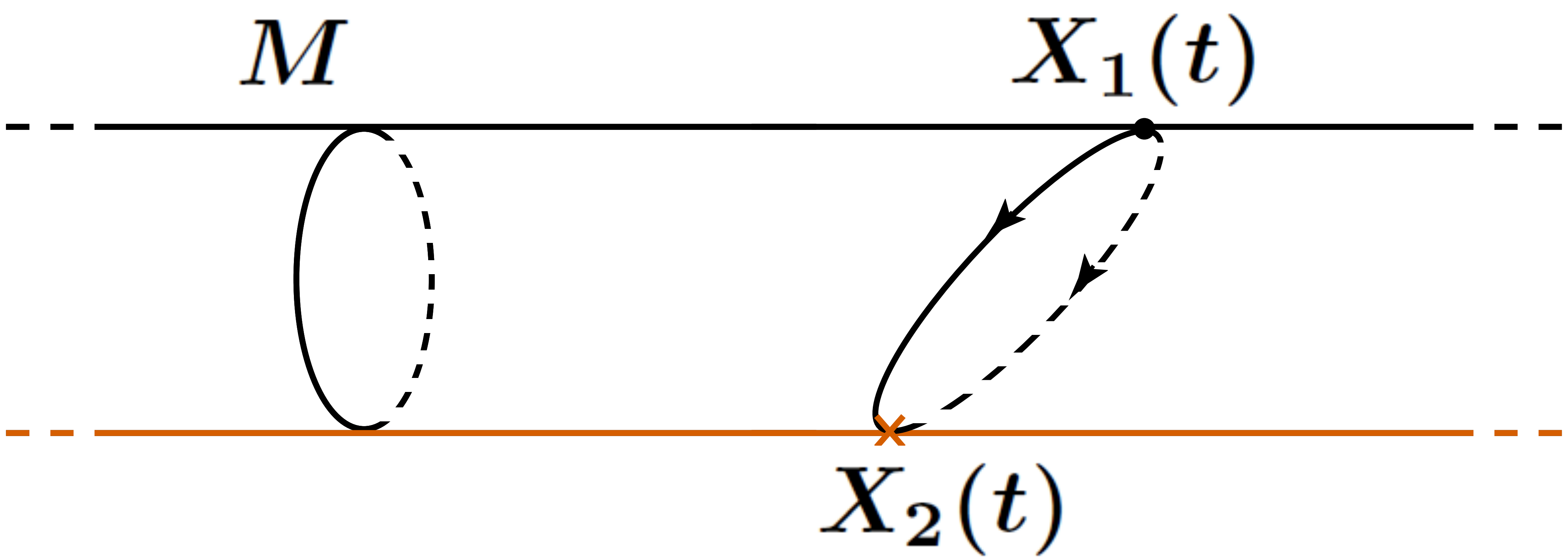
$\Rightarrow \rho(t) \leq U(t)$



$$d\rho(t) = 2\sqrt{2}dW(t) + dA_t - dL_t$$

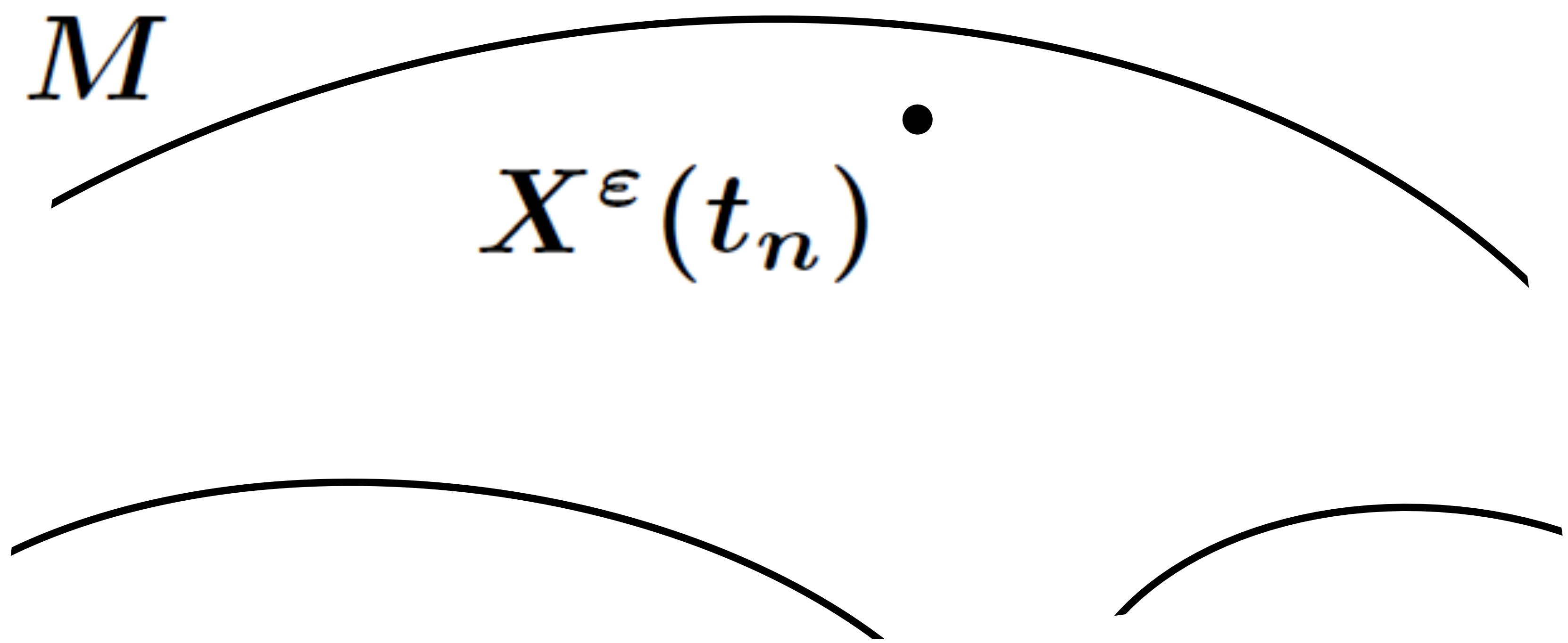
- (reflection) $\rightsquigarrow 2\sqrt{2}dW(t)$
- $(\star) \Rightarrow dA_t \leq -K\rho(t)dt$
- $L_t \geq 0$: "local time" at singular points of $d_{g(t)}$

$\Rightarrow \rho(t) \leq U(t)$

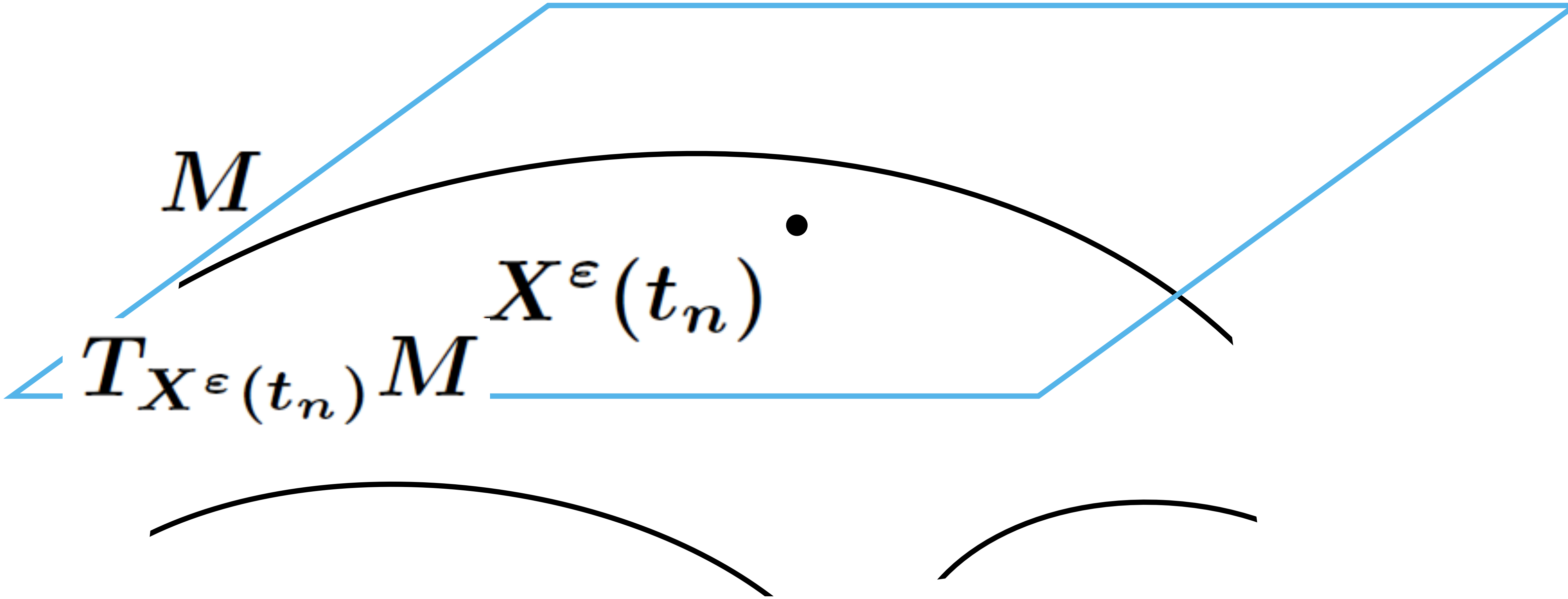


Approximation by coupled RWs $(X_1^\varepsilon(t), X_2^\varepsilon(t))$

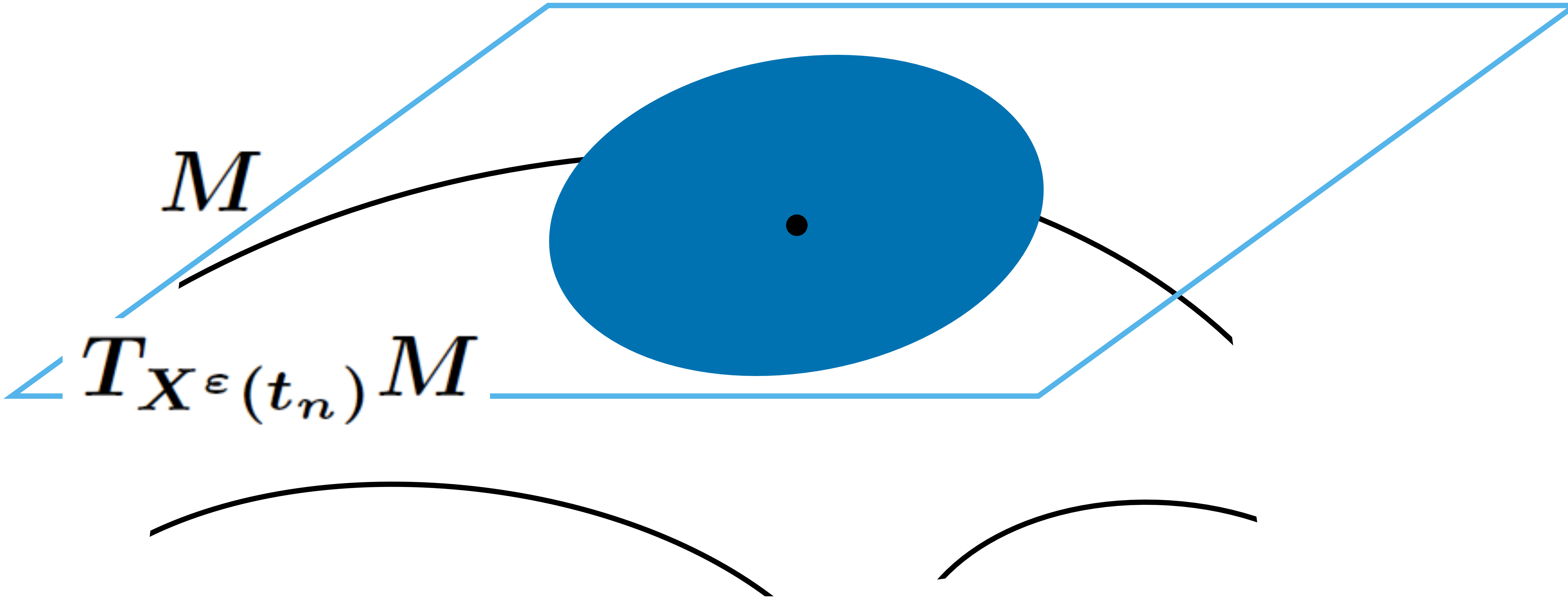
Approximation by coupled RWs $(X_1^{\varepsilon}(t), X_2^{\varepsilon}(t))$



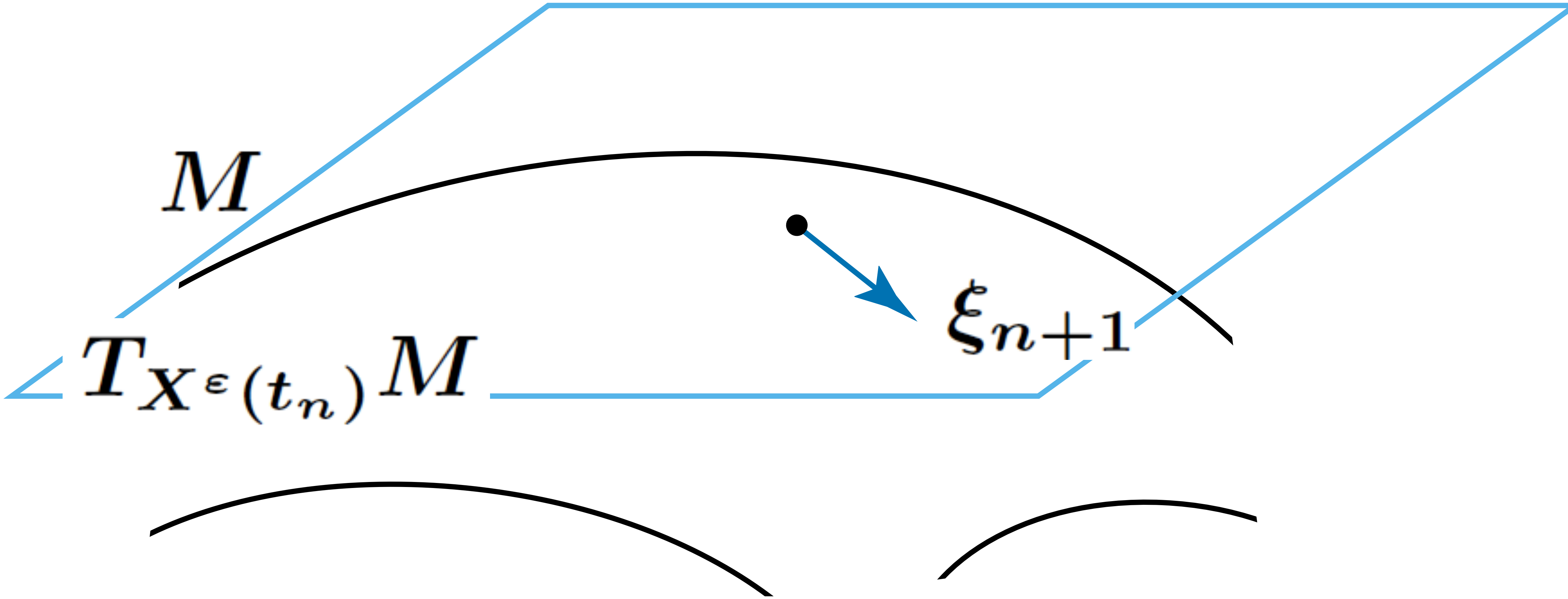
Approximation by coupled RWs $(X_1^\varepsilon(t), X_2^\varepsilon(t))$



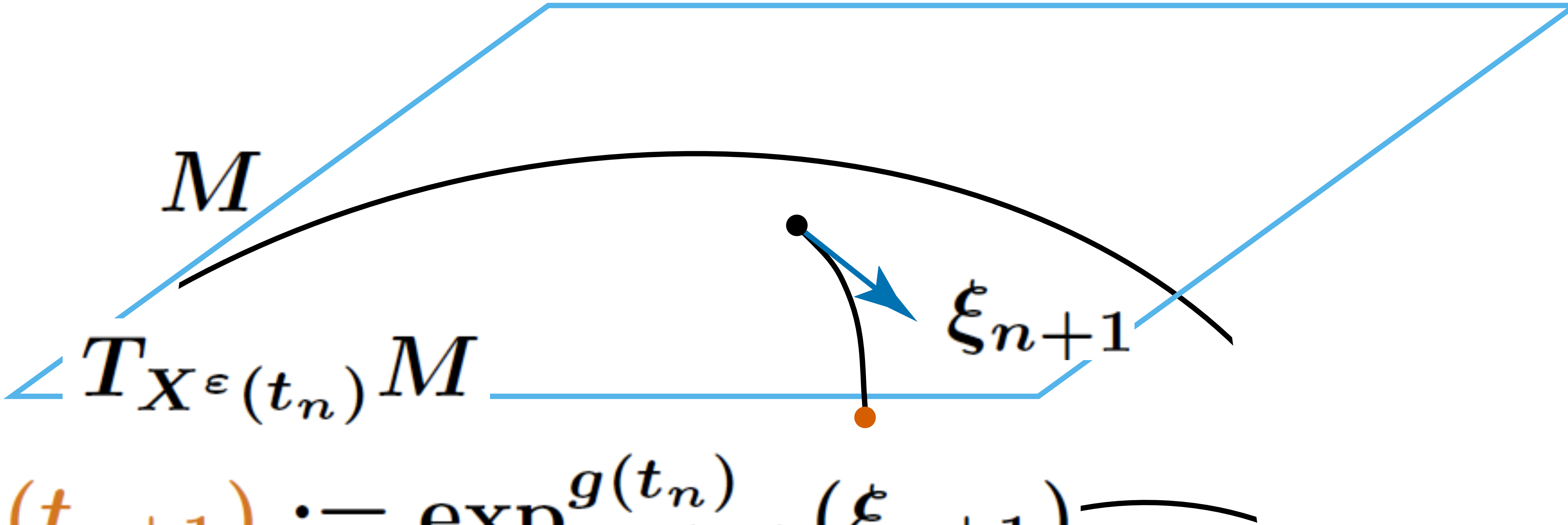
Approximation by coupled RWs $(X_1^\varepsilon(t), X_2^\varepsilon(t))$



Approximation by coupled RWs $(X_1^\varepsilon(t), X_2^\varepsilon(t))$

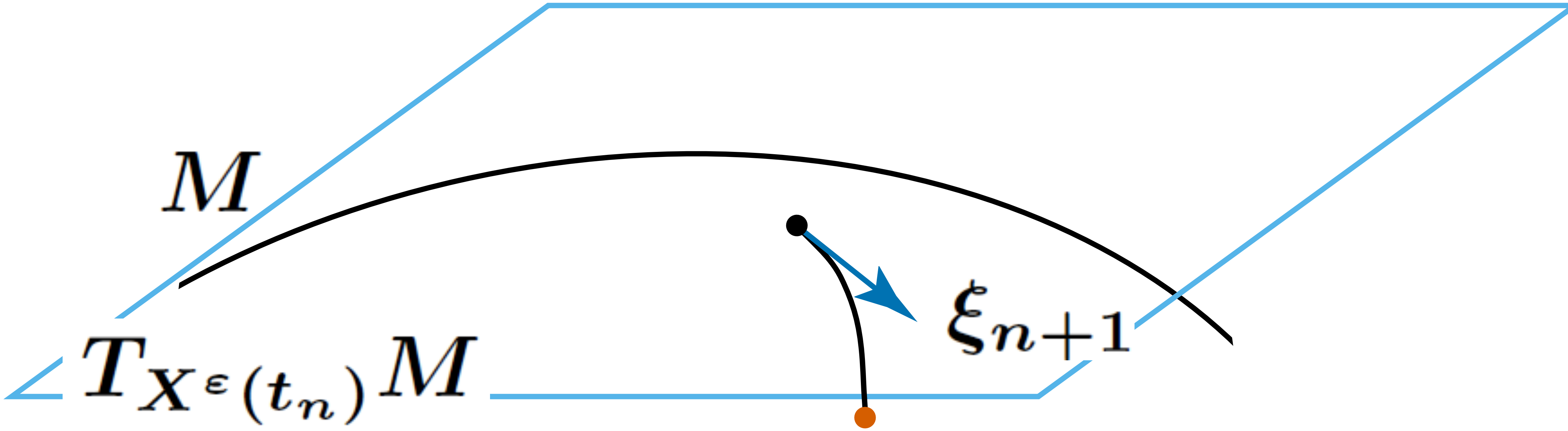


Approximation by coupled RWs $(X_1^\varepsilon(t), X_2^\varepsilon(t))$



$$X^\varepsilon(t_{n+1}) := \exp^{g(t_n)}_{X^\varepsilon(t_n)}(\xi_{n+1})$$

Approximation by coupled RWs $(X_1^{\varepsilon}(t), X_2^{\varepsilon}(t))$



$$X^{\varepsilon}(t_{n+1}) := \exp^{g(t_n)}_{X^{\varepsilon}(t_n)}(\xi_{n+1})$$

$$\Downarrow$$

$$d\rho^{\varepsilon}(t) \leq 2\sqrt{2}\,dW^{\varepsilon}(t) - K d\rho^{\varepsilon}(t)dt + o(1)$$

with high probability ($\longrightarrow 1$ as $\varepsilon \longrightarrow 0$)

(We can avoid to extract "L_t")

Invariance principle for X^ε

- Tightness
- Uniqueness of the mart. pbm. for $\partial_t + \Delta_{g(\cdot)}$

Key estimate for tightness

$$\overline{\lim_{R\to\infty}} \sup_{\varepsilon} \mathbb{P}_x \left[\sup_{0\le s\le t} d_{g(s)}(o, X^\varepsilon(s)) > R \right] = 0$$

(cf. [K.-Philipowski '09] for $X(t)$)

Idea of the Proof

- (discrete) Itô formula for $d_{g(t)}(o, X^\varepsilon(t))$

Related results

- The case $\partial_t g(t) \equiv 0$
  - SDE approach: [Kendall '86], [Cranston '91], [F.-Y. Wang '94,'05]
  - Approx. by RWs: [von Renesse '04], [K. '10]
- Coupling by parallel transport
  - [McCann & Topping '10] $K = 0$, M: cpt. via optimal transport
  - [Arnaudon, Coulibaly & Thalmaier '10] via stoch. diff. geom.

§4 Coupling by $\mathcal{L}$-parallel transport

(joint work with R. Philipowski)

Perelman's $\mathcal{L}$-distance

$$\gamma : [\tau_1, \tau_2] \to M,\ [\tau_1, \tau_2] \subset [T_1, T_2]$$

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left(|\dot\gamma(\tau)|^2_{g(\tau)} + R_{g(\tau)}(\gamma(\tau)) \right) d\tau$$

$$L(\tau_1, x; \tau_2, y) := \inf \left\{ \mathcal{L}(\gamma) \;\middle|\; \begin{array}{l} \gamma(\tau_1) = x, \\ \gamma(\tau_2) = y \end{array} \right\}$$

Normalization

Given $T_1 \leq \bar\tau_1 < \bar\tau_2 \leq T_2$,

$$\Theta(t, x, y) := 2(\sqrt{\bar\tau_2 t} - \sqrt{\bar\tau_1 t}) L(\bar\tau_1 t, x; \bar\tau_2 t, y) - 2m(\sqrt{\bar\tau_2 t} - \sqrt{\bar\tau_1 t})^2$$

Theorem 2 [K. & Philipowski '10]

Suppose $\partial_t g(t) = 2\,\mathrm{Ric}_{g(t)}$,

$$\sup_{x \in M, \tau \in [T_1, T_2]} |\mathrm{Rm}_{g(\tau)}|_{g(\tau)}(x) < \infty$$

$\Rightarrow$ $\exists (X_1(\tau), X_2(\tau))$: coupling of $g(\tau)$-BMs s.t. $(\Theta(t, X_1(\bar{\tau}_1 t), X_2(\bar{\tau}_2 t)))_{t \in [1, T_2/\bar{\tau}_2]}$ is a supermartingale

$\Rightarrow$ recover the monotonicity of the normalized $\mathcal{L}$-optimal transport in [Topping '09]

Strategy of the Proof

- Properties of $\mathcal{L}$-distance
  being analogous to the Riem. distance
  - $\mathcal{L}$-geodesic, 1st & 2nd variation of $\mathcal{L}$-length, $\mathcal{L}$-index lemma, $\mathcal{L}$-cut locus

- Approximation by RWs

- Coupling of $dX_{\mathbf{1}}^{\varepsilon}(\bar{\tau}_1 t)$ and $dX_{\mathbf{2}}^{\varepsilon}(\bar{\tau}_2 t)$ by
  spacetime-parallel transport along $\mathcal{L}$-geodesic