Heat flow on Alexandrov spaces

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§1 Introduction

(X, d): compact Alexandrov space of curv. $\geq k$

(geodesic metric sp. of sect. curv. $\geq k$)

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- It admits singularity of "curv. $= \infty$"
- Set of singular points can be dense
- It naturally appears in geometry
- Usual differential calculus is no longer available

  $\Rightarrow$ We have to develop new techniques

  which is also new in smooth cases

Two different ways to define a "heat distribution" on a metric measure space $(X, d, \mathcal{H}^n)$

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(2) Gradient flow of relative entropy functional on a sp. of probability measures (Otto calculus)

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Q. What happens when X is an Alexandrov space?

“Thm” (1) & (2) coincide on $(X, d, \mathcal{H}^n)$

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An application:

Theorem [G.-K.-O.]

The heat kernel $p_t(x, \cdot)$ is Lipschitz continuous

(It improves the known Hölder continuity coming from the theory of Dirichlet forms)

§2 Framework and the main result

Dirichlet energy and its gradient flow

[Kuwae, Machigashira & Shioya '01]

$\exists(\mathcal{E}, W^{1,2}(X))$: (str. local, reg.) Dirichlet form

$$\mathcal{E}(u,u) := \int_X \langle \nabla u, \nabla u \rangle \, d\mathcal{H}^n$$

$$(u \in W^{1,2}(X))$$

$$(\mathcal{E}, W^{1,2}(X)) \leftrightarrow (\Delta, \mathcal{D}(\Delta)): \text{ generator}$$

$$\leftrightarrow T_t = \mathrm{e}^{t\Delta}: \text{ semigroup}$$

Properties

- $\mathrm{Lip}(X) \subset W^{1,2}(X)$ dense.

  Moreover, for $f \in \mathrm{Lip}(X)$,

$$|\nabla_d f| \left(= \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x,y)} \right)$$

$$= |\nabla f| \quad \mathcal{H}^n\text{-a.e.}$$

- $\exists$(Hölder) conti. heat kernel $p_t(x, y)$:

$$T_t f(x) = \int_X p_t(x,y) f(y) \mathcal{H}^n(dy)$$

Gradient flow of relative entropy on $\mathcal{P}(X)$

For $\mu, \nu \in \mathcal{P}(X)$,

$$d_2^W(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \|d\|_{L^2(\pi)}$$

(L^2-Wasserstein distance)

- $(\mathcal{P}(X), d_2^W)$: cpt. geodesic metric sp.,
  compatible with the weak conv.

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- $(\mathcal{P}(X), d_2^W)$: cpt. geodesic metric sp., compatible with the weak conv.

$$\mathrm{Ent}(\mu) := \begin{cases} \int_X \rho \log \rho \, d\mathcal{H}^n & \text{if } d\mu = \rho d\mathcal{H}^n \\ \infty & \text{other} \end{cases}$$

Grad. flow $(\mu_t)_{t\geq 0}$ of Ent on $(\mathcal{P}(X), d_2^W)$

$(\mu_t)_{t\geq 0}$: abs. conti., $\mathrm{Ent}(\mu_t) < \infty$,

$$\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_s) = -\frac{1}{2}\int_s^t |\dot{\mu}_r|^2 dr - \frac{1}{2}\int_s^t |\nabla_- \mathrm{Ent}(\mu_r)|^2 dr$$

for $0 \leq s \leq t$, where

$$|\dot{\mu}_r| := \limsup_{h\downarrow 0} \frac{1}{h} d_2^W(\mu_{r+h}, \mu_r)$$

$$|\nabla_- \mathrm{Ent}(\mu)| := \limsup_{\nu\to\mu} \frac{[\mathrm{Ent}(\mu) - \mathrm{Ent}(\nu)]_+}{d_2^W(\mu, \nu)}$$

Heuristically,

$$\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_s)$$

$$\text{“}{=}\text{”} \int_s^t \langle \dot{\mu}_r, \nabla \mathrm{Ent}(\mu_r) \rangle dr$$

$$\geq -\frac{1}{2}\int_s^t |\dot{\mu}_r|^2 dr - \frac{1}{2}\int_s^t |\nabla \mathrm{Ent}|^2(\mu_r) dr$$

$$\left(\because \langle u, v \rangle \geq -\frac{1}{2}(\langle u, u \rangle + \langle v, v \rangle) \right)$$

and “=” holds iff $\dot{\mu}_r = -\nabla \mathrm{Ent}(\mu_r)$

The condition CD(K,∞)

For $\forall(\nu_t)_{t\in[0,1]}$: d_2^W-min. geod.,

$$\mathrm{Ent}(\nu_\lambda) \le (1-\lambda)\mathrm{Ent}(\nu_0) + \lambda\mathrm{Ent}(\nu_1) - \frac{K}{2}\lambda(1-\lambda)d_2^W(\nu_0,\nu_1)^2$$

(K-convexity of Ent w.r.t. d_2^W)

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- [von Renesse & Sturm '05]
  CD$(K, \infty) \Leftrightarrow \mathrm{Ric} \ge K$ if X: Riem. mfd

- [Petrunin]
  $(X, d, \mathcal{H}^n)$ satisfies CD$((n-1)k, \infty)$

Existence and uniqueness of gradient flow

Under $\mathsf{CD}(K, \infty)$, $\exists!$ grad. flow of Ent for $\forall$ initial $\mu \in \mathcal{P}(X)$ with $\mathrm{Ent}(\mu) < \infty$

[Ambrosio, Gigli & Savaré '05, Ohta '09, Gigli '10]

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L^2-Wasserstein contraction

For grad. flows μ_t and $\tilde{\mu}_t$,

$$d_2^W(\mu_t, \tilde{\mu}_t) \le \mathrm{e}^{-Kt} d_2^W(\mu_0, \tilde{\mu}_0)$$

[Savaré '07, Ohta '09, Gigli & Ohta '10]

$\Rightarrow$ Grad. flow for any initial $\mu \in \mathcal{P}(X)$

Theorem 1 [G.-K.-O.]

For any $\mu \in \mathcal{P}(X)$,

$T_t\mu$ is a gradient flow of Ent on $(\mathcal{P}(X), d_2^W)$

§3 Sketch of the proof

Suppose $\mathrm{Ent}(\mu) < \infty$. $\mu_t := T_t\mu$, $\rho_t := \dfrac{d\mu_t}{d\mathcal{H}^n}$.

<u>Goal</u>

$$\mathrm{Ent}(\mu_t) - \mathrm{Ent}(\mu_s) = -\frac{1}{2}\int_s^t |\dot{\mu}_r|^2 dr - \frac{1}{2}\int_s^t |\nabla_{-}\mathrm{Ent}(\mu_r)|^2 dr$$

- "$\geq$" is always true
- Sufficient to show: for a.e. t,

$$\partial_t \mathrm{Ent}(\mu_t) + \frac{1}{2}|\dot{\mu}_t|^2 + \frac{1}{2}|\nabla_{-}\mathrm{Ent}(\mu_t)|^2 \leq 0$$

Claims

(i) $\partial_t \mathbf{Ent}(\mu_t) = -I(\mu_t)$

(ii) $|\nabla_- \mathbf{Ent}(\mu_t)|^2 \leq I(\mu_t)$

(iii) $|\dot{\mu}_t|^2 \leq I(\mu_t)$ a.e.t

$$\left(I(\mu_t) := \int_X \frac{|\nabla \rho_t|^2}{\rho_t} d\mathcal{H}^n \text{: Fisher information} \right)$$

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- Integration by parts $\Rightarrow$ (i)
- [Villani '09]: $\mathrm{CD}(K, \infty) \Rightarrow$ (ii)

Remark:

$|\nabla_d f| = |\nabla f| = |\nabla_- f|$ a.e. for $f \in \mathrm{Lip}(X)$

Recall: $|\dot{\mu}_t| := \limsup_{h \downarrow 0} \frac{1}{h} d_2^W(\mu_{t+h}, \mu_t)$

$$\frac{1}{2} d_2^W(\mu_{t+h}, \mu_t)^2$$

$$= \sup_{\varphi \in \mathrm{Lip}(X)} \left[\int_X Q_1\varphi \, d\mu_{t+h} - \int_X \varphi \, d\mu_t \right],$$

(Kantorovich duality)

where $Q_t\varphi(x) := \inf_{y \in X} \left[\varphi(y) + \frac{d(x,y)^2}{2t} \right]$

(Hamilton-Jacobi semigroup)

- [Lott-Villani '07]: $\partial_t Q_t\varphi = -\frac{1}{2}|\nabla Q_t\varphi|^2$ a.e.

$$\int_X Q_1\varphi\, d\mu_{t+h} - \int_X \varphi\, d\mu_t$$

$$= \int_0^1 \partial_r \left(\int_X Q_r\varphi\, d\mu_{t+hr} \right) dr$$

HJ IbP

$$\boxed{=} \int_0^1 dr \int_X d\mu_{t+hr}$$

$$\left(-\frac{1}{2}|\nabla Q_r\varphi|^2 - h \left\langle \nabla Q_r\varphi, \frac{\nabla \rho_{t+hr}}{\rho_{t+hr}} \right\rangle \right)$$

$$\leq \frac{h^2}{2} \boxed{\int_0^1 I(\mu_{t+hr}) dr} \qquad \blacksquare$$

§4 Applications (under $\mathsf{CD}(K, \infty)$)

L^2-Wasserstein control for T_t

$$d_2^W(T_t\mu, T_t\nu) \le \mathrm{e}^{-Kt} d_2^W(\mu, \nu)$$

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$$d_2^W(T_t\mu, T_t\nu) \le e^{-Kt} d_2^W(\mu,\nu)$$

$\Downarrow$ [K.'10]

L^2-gradient estimate for $f \in \mathrm{Lip}(X)$

$$|\nabla_d T_t f|^2 \le e^{-2Kt} T_t(|\nabla_d f|^2)$$

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$$|\nabla_d T_t f|^2 \le e^{-2Kt} T_t(|\nabla_d f|^2)$$

$\Downarrow$ $\exists p_t$: conti.

L^2-gradient estimate for $f \in W^{1,2}(X)$

$$|\nabla_d T_t f|^2 \le e^{-2Kt} T_t(|\nabla f|^2)$$

Theorem 2 [G.-K.-O.]

(i) $T_t f \in \mathrm{Lip}(X)$ for $f \in W^{1,2}(X)$

(ii) For $\forall f$: L^2-eigenfn. of Δ, $f \in \mathrm{Lip}(X)$

(iii) $p_t(x, \cdot) \in \mathrm{Lip}(X)$

- Theorem 2(ii) provides another proof of [Petrunin '03]

- Recently, [Zhang & Zhu] gave another proof of Theorem 2(iii) based on [Petrunin '03]

Theorem 3 [G.-K.-O.]

For $K_0 \in \mathbb{R}$, the following are equivalent:

(i) $d_2^W(T_t\mu, T_t\nu) \le \mathrm{e}^{-K_0 t} d_2^W(\mu,\nu)$

(ii) For $f \in W^{1,2}(X)$,

$$|\nabla T_t f|^2 \le \mathrm{e}^{-2K_0 t} T_t(|\nabla f|^2) \text{ a.e.}$$

(iii) For $g \in D(\Delta) \cap L^\infty, g \ge 0, \Delta g \in L^\infty$ and $f \in D(\Delta) \cap L^\infty, \Delta f \in W^{1,2}(X)$,

$$\int_X \left(\frac{1}{2} \Delta g \langle \nabla f, \nabla f \rangle - g \langle \nabla f, \nabla \Delta f \rangle \right) d\mathcal{H}^n \ge K_0 \int_X g \langle \nabla f, \nabla f \rangle d\mathcal{H}^n$$

Remarks

(a) Theorem 3 (iii) is nothing but a weak form of Bakry-Émery's Γ_2-condition:

$$\frac{1}{2}\Delta\langle\nabla f,\nabla f\rangle - \langle\nabla f,\nabla\Delta f\rangle \geq K_0\langle\nabla f,\nabla f\rangle$$

(b) $f = T_t\varphi$, $g = T_t\psi$ for some $\varphi, \psi \in L^2$, $\psi \geq 0$ satisfies all requirements on f and g in Theorem 3 (iii).

(c) When X: Riem. mfd, Theorem 3(i)-(iii) are all equivalent to $\mathrm{Ric} \geq K_0$ (or $\mathrm{CD}(K_0,\infty)$)

[von Renesse & Sturm '05]

Boundedness of Riesz transform on $(X, d, \mathcal{H})$

[Kawabi & Miyokawa '07]

Theorem 4 [G.-K.-O.]

Let $2 \le p < \infty$, $q > 1$, and $\alpha > (-K) \vee 0$.

Then

$$\|\nabla(\alpha - \Delta_p)^{-q/2} f\|_{L^p} \le C \|f\|_{L^p}$$