Duality on gradient estimates and Wasserstein controls

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§1 Motivation

Equivalent conditions for a lower Ricci curvature bound

(von Renesse & Sturm '05, etc...)

X: complete Riemannian manifold

P_t: heat semigroup associated with Δ

(i) $\mathrm{Ric} \geq k$,

(ii) $d_p^W(P_t^* \mu, P_t^* \nu) \leq \mathrm{e}^{-kt} d_p^W(\mu, \nu)$ for some $p \in [1, \infty]$.

(iii) $|\nabla P_t f|(x) \leq \mathrm{e}^{-kt} P_t(|\nabla f|^q)(x)^{1/q}$ for some $q \in [1, \infty]$, Our goal:

Generalization of (ii) $\Leftrightarrow$ (iii), to obtain a (ii)/(iii)-type estimate from the other one.

(ii) $d_p^W(P_t^*\mu, P_t^*\nu) \le e^{-kt} d_p^W(\mu,\nu)$
(L^p-Wasserstein control)

(iii) $|\nabla P_t f|(x) \le e^{-kt} P_t(|\nabla f|^q)(x)^{1/q}$
(L^q-gradient estimate)

§2 Framework and Main Result

(X, d): Polish metric space.

- $(P_x)_{x \in X} \subset \mathcal{P}(X)$ s.t. $x \mapsto P_x$ continuous,
  $P : \mathcal{B}_b(X) \to \mathcal{B}_b(X)$

$$Pf(x) := \int_M f \, dP_x,$$

$$P^* \mu(A) := \int_X P_x(A) \mu(dx)$$

(e.g. $P = P_t$: heat semigroup)

- $\tilde{d}$: continuous distance function on X.
  (e.g. $\tilde{d} = \mathrm{e}^{-kt} d$)

For $\mu, \nu \in \mathcal{P}(X)$, $\Pi(\mu, \nu) \subset \mathcal{P}(X \times X)$ by

$$\Pi(\mu, \nu) := \left\{ \pi \;\middle|\; \begin{array}{l} \pi(A \times X) = \mu(A), \\ \pi(X \times A) = \nu(A) \end{array} \right\}.$$

<u>L^p-Wasserstein distance</u>

For $p \in [1, \infty]$,

$$d_p^W(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \|d\|_{L^p(\pi)} \in [0, \infty].$$

Gradient

$$|\nabla_d f|(x) := \lim_{r \downarrow 0} \sup_{y \in B_r(x)} \left| \frac{f(y) - f(x)}{d(y,x)} \right|,$$

$$\|\nabla_d f\|_\infty := \sup_{x \in X} |\nabla_d f|(x).$$

f: Lipschitz $\Rightarrow$ $|\nabla_d f|$ is an upper gradient of f.

i.e. for any 1-Lipschitz curve $\gamma : [a,b] \to X$ joining x and y,

$$f(y) - f(x) \leq \int_a^b |\nabla_d f|(\gamma(s))ds.$$

L^p-Wasserstein control

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu,\nu) \qquad (C_p)$$

for $p \in [1,\infty]$ and $\mu,\nu \in \mathcal{P}(X)$.

L^q-gradient estimate

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \qquad (G_q)$$

for $q \in [1,\infty)$ and $f \in C_b^{\mathrm{Lip}}(X)$,

$$|\nabla_{\tilde{d}} Pf|(x) \le \|\nabla_d f\|_\infty \qquad (G_\infty)$$

for $q = \infty$.

v: Radon measure on X with $\mathrm{supp}(v) = X$.

Assumption 1 (X, d): proper length space.

Assumption 2 (X, d, v) supports

- local (uniform) volume doubling condition,
- $(1, \rho)$-local Poincaré inequality $(\exists \rho \geq 1)$.

Assumption 3 $\tilde{d}$: geodesic distance.

Assumption 4 $P_x \ll v$, $x \mapsto \dfrac{dP_x}{dv}(y)$: continuous.

Local volume doubling condition

$\exists D > 0$, $\exists R_1 > 0$ s.t. $\forall x \in X$, $\forall r < R_1$

$$v(B_{2r}(x)) \le D v(B_r(x)).$$

$(1, \rho)$-local Poincaré inequality

$\forall R > 0$, $\exists \lambda \ge 1$, $\exists C_P > 0$ s.t. $\forall r < R$,

$$\fint_{B_r(x)} |f - f_{x,r}| dv \le C_P r \left(\fint_{B_{\lambda r}(x)} g^\rho \, dv \right)^{1/\rho}$$

for $\forall f$ and $\forall g$: upper gradient of f, where

$$f_{x,r} := \frac{1}{v(B_r(x))} \int_{B_r(x)} f \, dv =: \fint_{B_r(x)} f \, dv.$$

Theorem (K.)

For $p, q \in [1, \infty]$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q)$.

(ii) Under Assumption 1-4, $(G_q) \Rightarrow (C_p)$.

Remarks

- For $p' > p$, $\begin{cases} (G_p) \Rightarrow (G_{p'}), \\ (C_{p'}) \Rightarrow (C_p). \end{cases}$
  (without Assumption 1-4)

- $(G_\infty) \Leftrightarrow (C_1)$ is well-known.
  $\left(\begin{array}{l} \text{via Kantorovich-Rubinstein formula;} \\ \text{without Assumption 1-4} \end{array} \right)$

- $(C_\infty) \Rightarrow (G_1)$ is essentially well-known.

§3 Examples and Applications

How do we obtain $(C_{\boldsymbol{p}})$?

(A) Known derivation of (C_p)

(X: cpl. Riem. mfd., $P = P_t$, $\tilde{d} = \mathrm{e}^{-kt} d$)

- Coupling by parallel transport of B.m.'s:

$$\boxed{\mathrm{Ric} \geq k \Rightarrow (C_\infty)}$$

$\rightsquigarrow$ Extension to backward Ricci flow.

$\star$ $\tilde{d}$ is essentially different from d!

- Gradient flow formulation of the heat flow μ_t:

$\partial_t \mu_t = -\nabla E(\mu_t)$.

  - $\mathrm{Ric} \geq k \Leftrightarrow$ "$\mathrm{Hess}\, E \geq k$",
  - $\boxed{\mathrm{Hess}\, E \geq k \Rightarrow (C_2)}$ for μ_t $(= P_t^* \mu)$.

$\rightsquigarrow$ Extension to singular spaces (e.g. Alex. sp.).

Remark

To obtain (C_p), we have used some notion of lower curvature bound which is different from (G_q).

E.g. in von Renesse & Sturm '05,

(G_∞)	$\Rightarrow \mathbf{Ric} \geq k$		(Bochner)
	$\Rightarrow (C_\infty)$		(coupling method)
	$\Rightarrow (G_1)$		
	$\Rightarrow (G_\infty)$		(monotonicity)

(B) Hörmander-type operators on a Lie group

X: Lie group with a right-Haar measure v.

$\{X_i\}_{i=1}^n$: left-invariant, linearly independent vector fields generating all left-invariant vector fields in the sense of Lie algebra (Hörmander condition).

$$P_t := \mathrm{e}^{tA}, \quad A := \sum_{i=1}^{n} X_i^2.$$

$$|\nabla f|^2 := \frac{1}{2}\big(A(f^2) - 2fAf\big) = \sum_{i=1}^{n} |X_i f|^2.$$

<u>L^q-Gradient estimate</u>

$$|\nabla P_t f|(x) \le K_q(t) P_t(|\nabla f|^q)(x)^{1/q}. \quad (G_q^*)$$

Known results

- 3-dim. Heisenberg group, $K_q(t) \equiv K_q > 1$
  - $q > 1$: Driver & Melcher '05.
  - $q = 1$: H.-Q. Li '06 / Bakry et al. '08.
- X: general, $q > 1$: Melcher '08 ($K_q(t) \equiv K_q$ if X: nilpotent).
- X: group of type H, $q = 1$, $K_q(t) \equiv K_q$: Eldredge '09.
- $X = SU(2)$, $q > 1$, $K_q(t) = K_q e^{-t}$: Baudoin & Bonnefont '09.

$X, v, \{X_i\}_{i=1}^n, P$: as before.

Carnot-Caratheodory distance

For $V \in T_x X$,

$$|V| = \begin{cases} \left(\sum_{i=1}^n {a_i}^2\right)^{1/2} & \text{if } V = \sum_{i=1}^n a_i X_i(x), \\ \infty & \text{otherwise.} \end{cases}$$

$$d(x, y) := \inf \left\{ \int_0^1 |\dot{\gamma}_s| ds \;\middle|\; \begin{array}{l} \gamma_0 = x, \\ \gamma_1 = y \end{array} \right\}.$$

Proposition

(X, d, v), $P = P_t$: as above.

(i) $(X, d, v; P)$ satisfies Assumption 1-4

(ii) "$(G_q^*) \Rightarrow (G_q)$".

Corollary

$(G_q^*) \Rightarrow (C_p)$ for $q \in [1, \infty]$.

Examples

3-dim. Heisenberg group

$X = \mathbb{R}^3$, v: Lebesgue.

$$(x, y, z) \cdot (x', y', z')$$
$$= \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')\right),$$

$$X_1 = \partial_x - \frac{y}{2}\partial_z\ , \quad X_2 = \partial_y + \frac{x}{2}\partial_z\ .$$

Associated diffusion (B_t^1, B_t^2, B_t^3) from (x, y, z):

$$B_t^1 = W_t^1, \qquad B_t^2 = W_t^2,$$

$$B_t^3 = z + \frac{1}{2}\int_0^t W_t^1 dW_t^2 - W_t^2 dW_t^1,$$

where (W_t^1, W_t^2): 2-dim. BM from (x, y).

(C_∞): For each $t > 0$, $\exists$ a coupling $(\boldsymbol{B}_t, \tilde{\boldsymbol{B}}_t)$ of (B_t^1, B_t^2, B_t^3) with initial conditions $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$ respectively s.t.

$$d(\boldsymbol{B}_t, \tilde{\boldsymbol{B}}_t) \leq K_1 d(\boldsymbol{a}, \boldsymbol{b}) \quad \mathbb{P}\text{-a.s..}$$

In this case, $\exists C_1, C_2 > 0$ s.t.

$$C_1 \|\boldsymbol{b}^{-1}\boldsymbol{a}\| \leq d(\boldsymbol{a}, \boldsymbol{b}) \leq C_2 \|\boldsymbol{b}^{-1}\boldsymbol{a}\|,$$

where $\|(x, y, z)\| = \left((x^2 + y^2)^2 + z^2\right)^{1/4}$.

Definition

X: a group of type H iff, for $\mathcal{X}$: Lie alg. associated with X with a scalar product $\langle \cdot, \cdot \rangle$,

- $\mathcal{X} = \mathcal{V} \oplus \mathcal{Z}$ with $[\mathcal{V}, \mathcal{V}] = \mathcal{Z}$, $[\mathcal{V}, \mathcal{Z}] = [\mathcal{Z}, \mathcal{Z}] = 0$.
- $J : \mathcal{Z} \longrightarrow \mathrm{End}\,\mathcal{V}$ given by

$$\langle J(Z)V_1, V_2 \rangle := \langle Z, [V_1, V_2] \rangle$$

satisfies $J(Z)^2 = -\|Z\| \mathrm{Id}$.

$\{X_i\}_{i=1}^n$ will be an ONB of $\mathcal{V}$.

Remarks

- Any group of type H is a stratified nilpotent Lie group of step 2.

- $\forall m$, the $(2m+1)$-dim. Heisenberg group is of type H.

- A free nilpotent Lie group of step 2 is of type H iff it is the 3-dim. Heisenberg group.

- Possible dimension of a group of type H is completely determined.

§4 Sketch of the Proof of $(G_q) \Rightarrow (C_p)$

Recall:

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu,\nu), \qquad (C_p)$$

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q}. \qquad (G_q)$$

- The case $p = 1$ ($q = \infty$) is well-known.
- $d_p^W(\mu,\nu) \overset{p\to\infty}{\longrightarrow} d_\infty^W(\mu,\nu) \in [0,\infty]$.
  $\Rightarrow$ We may assume $p < \infty$.
- (C_p) for $\mu = \delta_x, \nu = \delta_y \Rightarrow (C_p)$.
  (disintegration of measures)

General theory of the Hamilton-Jacobi semigroup

(Lott & Villani '07, Balogh et al. '09)

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + t \cdot \frac{1}{p} \left(\frac{d(x,y)}{t} \right)^p \right].$$

- Under Assumption 1,
  $Q_\cdot f \in C_b^{\mathrm{Lip}}([0,\infty) \times X)$ if $f \in C_b^{\mathrm{Lip}}(X)$.

- Under Assumption 1-2, for $\forall t > 0$, ν-a.e.

$$\partial_t Q_t f(x) = -\frac{1}{q} |\nabla_d Q_t f|(x)^q \ .$$

(Note: $q^{-1} u^q = \sup_{s \geq 0} \left(us - p^{-1} s^p \right)$)

Kantorovich duality

$$d_p^W(\mu,\nu)^p = \sup_{f\in C_b^{\mathrm{Lip}}} \left[\int_X f^*\, d\mu - \int_X f\, d\nu\right],$$

$$f^*(x) := \inf_{y\in X} \left[\, f(y) + d(x,y)^p \,\right]$$

$$= p\, Q_1(p^{-1} f)(x).$$

$$\Downarrow$$

$$\frac{d_p^W(\mu,\nu)^p}{p} = \sup_f \left[\int_X Q_1 f\, d\mu - \int_X f\, d\nu\right].$$

$$\begin{cases} \gamma : [0,1] \to X \quad \tilde{d}\text{-minimal geodesic}, \\ \gamma_0 = y, \ \gamma_1 = x, \\ \tilde{d}(\gamma_s, \gamma_t) = |t-s|\tilde{d}(x,y). \end{cases}$$

(Assumption 3)

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$$\frac{d_p^W(P_x, P_y)^p}{p} = \sup_f [PQ_1 f(x) - Pf(y)]$$

$$\text{“} = \text{”} \quad \sup_f \left[\int_0^1 \partial_t (PQ_t f(\gamma_t)) dt \right].$$

interpolation

$$\partial_t (P Q_t f(\gamma_t))$$

$$\text{“}{=}\text{”}\ P(\partial_t Q_t f)(\gamma_t) + \langle \nabla P Q_t f(\gamma_t), \dot{\gamma}_t \rangle$$

HJ eq.
up. grad.

$$\boxed{\leq} -\frac{1}{q} P(|\nabla_d Q_t f|^q)(\gamma_t)$$

$$+ \tilde{d}(x,y) \left|\nabla_{\tilde{d}} P Q_t f\right| (\gamma_t)$$

$$(G_q) \boxed{\leq} \tilde{d}(x,y)\sigma - \frac{1}{q}\sigma^q \leq \frac{\tilde{d}(x,y)^p}{p}.$$

$$\left(\sigma := P(|\nabla_d Q_t f|^q)(\gamma_t)^{1/q}\right) \qquad \blacksquare$$

Questions

(i) When does $(C_p) \Rightarrow (C_{p'})$ / $(G_{p'}) \Rightarrow (G_p)$ occur for $p' > p$?
(OK if X: Riem., $P = P_t$)

(ii) When does $(C_p) \Rightarrow$ "pathwise control" occur?
(in the case $P = P_t$)

(iii) Relation between Bakry-Émery's Γ_2-criterion and (G_q) (in the case $P = P_t$, $\tilde{d} = \mathrm{e}^{-kt} d$)
(When does $|\nabla_d f| = \Gamma(f, f)^{1/2}$ hold?).

(iv) Relation with other "lower curvature bounds"...