

**Two duality results
on gradient estimates and
Wasserstein controls**

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§1 Introduction

Equivalent conditions for a lower Ricci curvature bound

(von Renesse & Sturm '05, etc...)

X : complete Riemannian manifold

P_t : heat semigroup associated with Δ

(i) $\text{Ric} \geq k$

(ii) $d_p^W(P_t^* \mu, P_t^* \nu) \leq e^{-kt} d_p^W(\mu, \nu)$
for some $p \in [1, \infty]$

(iii) $|\nabla P_t f|(x) \leq e^{-kt} P_t(|\nabla f|^q)(x)^{1/q}$
for some $q \in [1, \infty]$

Our goal:

Generalization of (ii) \Leftrightarrow (iii)

for p, q with $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) L^p -Wasserstein control

$$d_p^W(P_t^* \mu, P_t^* \nu) \leq e^{-kt} d_p^W(\mu, \nu)$$

(iii) L^q -gradient estimate

$$|\nabla P_t f|(x) \leq e^{-kt} P_t(|\nabla f|^q)(x)^{1/q}$$

§2 Framework and main result

X : Polish space

- $(P_x)_{x \in X} \subset \mathcal{P}(X)$: Markov kernel

$$P f(x) := \int_X f dP_x,$$

$$P^* \mu(A) := \int_X P_x(A) \mu(dx)$$

Assume $P(C_b(X)) \subset C_b(X)$.

(e.g. $P = P_t$: heat semigroup)

- d, \tilde{d} : lower semi-conti. pseudo-distance on X
(e.g. $\tilde{d} = e^{-kt} d$, d : Riemannian distance)

$$\Pi(\mu, \nu) := \left\{ \pi \mid \pi \circ p_1^{-1} = \mu, \pi \circ p_2^{-1} = \nu \right\}$$

(couplings of $\mu, \nu \in \mathcal{P}(X)$)

L^p -Wasserstein distance

For $p \in [1, \infty]$,

$$d_p^W(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \|d\|_{L^p(\pi)} \in [0, \infty]$$

L^p -Wasserstein control

$$d_p^W(P^*\mu, P^*\nu) \leq \tilde{d}_p^W(\mu, \nu) \quad (C_p)$$

Gradient

$$|\nabla_d f|(x) := \lim_{r \downarrow 0} \sup_{y \in B_r(x)} \left| \frac{f(x) - f(y)}{d(x, y)} \right|,$$

$$\|\nabla_d f\|_\infty := \sup_{x \in X} |\nabla_d f|(x)$$

Subgradient

$$|\nabla_d^- f|(x) := \lim_{r \downarrow 0} \sup_{y \in B_r(x)} \left[\frac{f(x) - f(y)}{d(x, y)} \right]_+,$$

$$\|\nabla_d^- f\|_\infty := \sup_{x \in X} |\nabla_d^- f|(x)$$

L^q -gradient estimate

$$|\nabla_{\tilde{d}} P f|(x) \leq P(|\nabla_d f|^q)(x)^{1/q} \quad (G_q)$$

for $q \in [1, \infty)$ and $f \in C_b^{\text{Lip}}(X)$,

$$\|\nabla_{\tilde{d}} P f\|_{\infty} \leq \|\nabla_d f\|_{\infty} \quad (G_{\infty})$$

for $q = \infty$

(We define (G_q^-) and (G_{∞}^-) similarly
by using ∇^- instead of ∇)

The first duality result

Theorem A (K. '10 JFA)

For $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q)$

(ii) Under Assumptions A1-A4 below,
 $(G_q) \Rightarrow (C_p)$

The second duality result

Theorem B (K. '10)

For $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,

(i) $(C_p) \Rightarrow (G_q^-)$

(ii) Under Assumptions B1-B4 below,
 $(G_q^-) \Rightarrow (C_p)$

Examples satisfying A1-A4

- Cpl. Riem. mfd's with $\text{Ric} \geq k_0$
- Carnot groups
- Alexandrov spaces

($\dim X < \infty$, $P_x \ll \nu$ for a base measure ν)

Examples satisfying B1-B4

- Cpl. Riem. mfd's (no curvature assumption)
- Wiener space, d : Cameron-Martin norm

($P_x \ll \nu$ is not necessary)

Remarks

- For $p' > p$,

$$\begin{cases} (G_p) \Rightarrow (G_{p'}) / (G_p^-) \Rightarrow (G_{p'}^-) \\ (C_{p'}) \Rightarrow (C_p) \end{cases}$$

(without Assumptions A1-A4/B1-B4)

- $(G_\infty) \Leftrightarrow (C_1)$ is well known.

$\left(\begin{array}{l} \text{via Kantorovich-Rubinstein formula;} \\ \text{without Assumptions A1-A4/B1-B4} \end{array} \right)$

- $(C_\infty) \Rightarrow (G_1)$ is essentially well known.

§3 Applications

(0) Review: How did we obtain (C_p)?

- Coupling by parallel transport of B.m.'s:

$$\text{Ric} \geq k \Rightarrow (C_\infty)$$

- Gradient flow formulation of the heat flow μ_t :

$$\partial_t \mu_t = -\nabla E(\mu_t)$$

- $\text{Ric} \geq k \Leftrightarrow \text{“Hess } E \geq k\text{”}$

- $\text{Hess } E \geq k \Rightarrow (C_2)$ for $\mu_t (= P_t^* \mu)$

⇒ Extensions to a backward Ricci flow or singular spaces (e.g. Alex. sp.).

Remark

“(a lower Ricci bound) $\Rightarrow (C_p)$ ” in the literature.

No direct way “ $(G_q) \Rightarrow (C_p)$ ” was known.

E.g. in von Renesse & Sturm '05,

$$\text{Ric} \geq k$$

\Downarrow coupling method

$$(C_\infty) \Rightarrow (C_p) \Rightarrow (C_1)$$

\Downarrow

\Downarrow

$$(G_1) \Rightarrow (G_q) \Rightarrow (G_\infty) \Rightarrow \text{Ric} \geq k$$

Bochner

(1) Manifolds with a time-dependent metric

$(M, g(t))$: cpl. Riem. mfds, $t \in [0, T]$

B_t : $g(t)$ -B.m. (\leftrightarrow generator $\partial_t + \frac{1}{2}\Delta_{g(\cdot)}$)

$P := P_{s \rightarrow t}$: heat semigroup associated with B_t .

$d := d_{g(t)}$, $\tilde{d} := e^{-k(t-s)/2} d_{g(s)}$

(\tilde{d} is essentially different from d !)

$$\partial_t g(t) \leq \text{Ric}_{g(t)} - kg(t) \quad (\star)$$

- McCann & Topping '08: X : cpt, $k = 0$

$$(\star) \Leftrightarrow [(C_2) \forall s < t] \Leftrightarrow [(G_\infty) \forall s < t]$$

- Arnaudon, Coulibaly & Thalmaier '09:

$$(\star) \Rightarrow [(C_\infty) \forall s < t]$$

(cf. K. & Philipowski '09: B_t is non-explosive)

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- Theorem B $\Rightarrow \forall s < t, [(C_p) \Leftrightarrow (G_q^-)]$

(When B_t is non-explosive)

**(2) Hörmander-type operators
on a Lie group**

X : Lie group

$\{X_i\}_{i=1}^n$: left-invariant, lin. indep. vector fields
satisfying the Hörmander condition

$$P_t := e^{tA}, \quad A := \sum_{i=1}^n X_i^2,$$

$$|\Gamma f| := \frac{1}{2} (A(f^2) - 2fAf) = \sum_{i=1}^n |X_i f|^2$$

L^q -gradient estimate

$$|\Gamma P_t f|(x) \leq K_q(t) P_t(|\Gamma f|^{q/2})(x)^{2/q} \quad (G_q^*)$$

Known results

- 3-dim. Heisenberg group, $K_q(t) \equiv K_q > 1$
 - $q > 1$: Driver & Melcher '05
 - $q = 1$: H.-Q. Li '06 / Bakry, Baudoin, Bonnefont & Chafaï '08
- X : general, $q > 1$: Melcher '08
($K_q(t) \equiv K_q$ if X : nilpotent)
- X : group of type H, $q = 1$, $K_q(t) \equiv K_q$:
Eldredge '10
- $X = SU(2)$, $q > 1$, $K_q(t) = K_q e^{-t}$:
Baudoin & Bonnefont '09

Carnot-Caratheodory distance

For $V \in T_x X$,

$$|V| := \begin{cases} \left(\sum_{i=1}^n a_i^2 \right)^{1/2} & \text{if } V = \sum_{i=1}^n a_i X_i(x), \\ \infty & \text{otherwise.} \end{cases}$$

$$d(x, y) := \inf \left\{ \int_0^1 |\dot{\gamma}_s| ds \mid \begin{array}{l} \gamma_0 = x, \\ \gamma_1 = y \end{array} \right\}$$

ν : right-Haar measure, $P := P_t$.

Proposition

- (i) $(X, d, \nu; P)$ satisfies Assumption A1-A4
- (ii) $(G_q^*) \Rightarrow (G_q)$

Corollary

$(G_q^*) \Rightarrow (C_p)$ for $q \in [1, \infty]$.

Examples

3-dim. Heisenberg group

$X := \mathbb{R}^3$, ν : Lebesgue

$$(x, y, z) \cdot (x', y', z') \\ := (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx'))$$

$$X_1 := \partial_x - \frac{y}{2}\partial_z, \quad X_2 := \partial_y + \frac{x}{2}\partial_z$$

Associated diffusion (B_t^1, B_t^2, B_t^3) from (x, y, z) :

$$B_t^1 = W_t^1, \quad B_t^2 = W_t^2,$$

$$B_t^3 = z + \frac{1}{2} \int_0^t W_t^1 dW_t^2 - W_t^2 dW_t^1,$$

where (W_t^1, W_t^2) : 2-dim. BM from (x, y)

(C_∞) : For each $t > 0$,

\exists a coupling $(\mathbf{B}_t, \tilde{\mathbf{B}}_t)$ of (B_t^1, B_t^2, B_t^3) s.t.

$$d(\mathbf{B}_t, \tilde{\mathbf{B}}_t) \leq K_1 d(\mathbf{B}_0, \mathbf{B}_0) \quad \mathbb{P}\text{-a.s.}$$

Definition

X : a group of type H iff, for \mathcal{X} : Lie alg.
associated with X with a scalar product $\langle \cdot, \cdot \rangle$,

- $\mathcal{X} = \mathcal{V} \oplus \mathcal{Z}$ with $[\mathcal{V}, \mathcal{V}] = \mathcal{Z}$,
 $[\mathcal{V}, \mathcal{Z}] = [\mathcal{Z}, \mathcal{Z}] = 0$.

- $J : \mathcal{Z} \rightarrow \text{End } \mathcal{V}$ given by

$$\langle J(\mathcal{Z})V_1, V_2 \rangle := \langle \mathcal{Z}, [V_1, V_2] \rangle$$

satisfies $J(\mathcal{Z})^2 = -\|\mathcal{Z}\|\text{Id}$.

$\{X_i\}_{i=1}^n$ will be an ONB of \mathcal{V} .

Remarks

- (Heisenberg) \subset (type H)

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- (Heisenberg) \subset (type H)
- (type H) \subset (stratified, step 2 nilpotent)
- (type H) \cap (free step 2 nilpotent)
= {3-dim. Heisenberg}
- For type H,
possible dimension is completely determined

§4 Sketch of the proof of $(G_q) \Rightarrow (C_p)$

Recall:

$$d_p^W(P^*\mu, P^*\nu) \leq \tilde{d}_p^W(\mu, \nu) \quad (C_p)$$

$$|\nabla_{\tilde{d}} P f|(x) \leq P(|\nabla_d f|^q)(x)^{1/q} \quad (G_q)$$

Recall:

$$d_p^W(P^*\mu, P^*\nu) \leq \tilde{d}_p^W(\mu, \nu) \quad (C_p)$$

$$|\nabla_{\tilde{d}} P f|(x) \leq P(|\nabla_d f|^q)(x)^{1/q} \quad (G_q)$$

- $p = 1$ ($q = \infty$) : well-known
- (C_p) for $\forall p < \infty \Rightarrow (C_\infty)$
 - \rightsquigarrow We may assume $p \in (1, \infty)$

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- $p = 1$ ($q = \infty$): well-known
- (C_p) for $\forall p < \infty \Rightarrow (C_\infty)$
 - \rightsquigarrow We may assume $p \in (1, \infty)$
- (C_p) for $\mu = \delta_x, \nu = \delta_y \Rightarrow (C_p)$

$$\rightsquigarrow \text{We show } \frac{d_p^W(P_x, P_y)^p}{p} \leq \frac{\tilde{d}(x, y)^p}{p}$$

Hamilton-Jacobi semigroup

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + t \cdot \frac{1}{p} \left(\frac{d(x, y)}{t} \right)^p \right]$$

We expect (under our assumptions):

- $Q \cdot f(x)$: Lipschitz, $Q_t f(\cdot)$: d -Lipschitz
- Hamilton-Jacobi equation

$$\partial_t Q_t f = -\frac{1}{q} |\nabla_d Q_t f|^q$$

(Note: $q^{-1} u^q = \sup_{s \geq 0} (us - p^{-1} s^p)$)

Kantorovich duality

$$d_p^W(\mu, \nu)^p = \sup_{f \in C_b^{\text{Lip}}} \left[\int_X f^* d\mu - \int_X f d\nu \right],$$

$$\begin{aligned} f^*(x) &:= \inf_{y \in X} [f(y) + d(x, y)^p] \\ &= p Q_1(p^{-1} f)(x) \end{aligned}$$



$$\frac{d_p^W(\mu, \nu)^p}{p} = \sup_f \left[\int_X Q_1 f d\mu - \int_X f d\nu \right]$$

$$\left\{ \begin{array}{l} \tilde{\gamma} : [0, 1] \rightarrow X \quad \tilde{d}\text{-minimal geodesic,} \\ \tilde{\gamma}_0 = y, \quad \tilde{\gamma}_1 = x, \\ \tilde{d}(\tilde{\gamma}_s, \tilde{\gamma}_t) = |t - s| \tilde{d}(x, y) \end{array} \right.$$



$$\frac{d_p^W(P_x, P_y)^p}{p} = \sup_f [PQ_1 f(x) - P f(y)]$$

interpolation $\boxed{=}$ $\sup_f \left[\int_0^1 \partial_t (PQ_t f(\tilde{\gamma}_t)) dt \right]$

$$\partial_t(PQ_t f(\tilde{\gamma}_t))$$

$$\left(\text{“=” } P(\partial_t Q_t f)(\tilde{\gamma}_t) + \langle \nabla P Q_t f(\tilde{\gamma}_t), \dot{\tilde{\gamma}}_t \rangle \right)$$

HJ eq. \leq up. grad. $-\frac{1}{q} P(|\nabla_d Q_t f|^q)(\tilde{\gamma}_t)$

$$+ \tilde{d}(x, y) |\nabla_{\tilde{d}} P Q_t f|(\tilde{\gamma}_t)$$

$$(G_q) \leq \tilde{d}(x, y) \sigma - \frac{1}{q} \sigma^q \leq \frac{\tilde{d}(x, y)^p}{p}$$

$$\left(\sigma := P(|\nabla_d Q_t f|^q)(\tilde{\gamma}_t)^{1/q} \right)$$



§5 Assumptions

(1) Assumptions A1-A4

ν : Radon measure on X with $\text{supp}(\nu) = X$.

Assumption A1 (X, d) : proper length space,
compatible with the original topology

Assumption A2 (X, d, ν) supports

- local (uniform) volume doubling condition
- $(1, \rho)$ -local Poincaré inequality ($\exists \rho \geq 1$)

Assumption A3 \tilde{d} : continuous geodesic distance

Assumption A4 $P_x \ll \nu, x \mapsto \frac{dP_x}{d\nu}(y)$: conti.

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Assumption A1 (X, d) : proper length space,
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Assumption A2 (X, d, ν) supports

- local (uniform) volume doubling condition
- $(1, \rho)$ -local Poincaré inequality ($\exists \rho \geq 1$)

Assumptions A1-A2 \Rightarrow HJ eq. for $Q_t f$ ν -a.e.

Lott & Villani '07

Balogh, Engoulatov, Hunziker & Maasalo '09

Local volume doubling condition

$$\exists D > 0, \exists R_1 > 0 \text{ s.t. } \forall x \in X, \forall r < R_1$$
$$v(B_{2r}(x)) \leq Dv(B_r(x)).$$

(1, ρ)-local Poincaré inequality

$$\forall R > 0, \exists \lambda \geq 1, \exists C_P > 0 \text{ s.t. } \forall r < R,$$
$$\int_{B_r(x)} |f - f_{x,r}| dv \leq C_P r \left(\int_{B_{\lambda r}(x)} g^\rho dv \right)^{1/\rho}$$

for $\forall f$ and ∇g : upper gradient of f , where

$$f_{x,r} := \frac{1}{v(B_r(x))} \int_{B_r(x)} f dv =: \int_{B_r(x)} f dv.$$

(2) Assumptions B1-B4

Assumption B1

For $\forall x, y \in X$ with $d(x, y) < \infty$,
 \exists a minimal geodesic γ from x to y .
The same is also true for \tilde{d} .

Assumption B2

$Q_t f$ is measurable for $\forall f \in C_b(X)$

Assumption B3

$$\lim_{r \downarrow 0} \sup_{y; \tilde{d}(x, y) \leq r} P Q_t f(y) \leq P Q_t f(x)$$

“Assumption B4”

The following holds locally:

- $\exists D \geq 0$ s.t. for $\forall \gamma$: d -min. geod.,
 $d(x, \gamma(\lambda))^2$ is D -semiconcave, i.e.:

$$\begin{aligned} d(x, \gamma(\lambda))^2 &\geq (1 - \lambda)d(x, \gamma(0))^2 + \lambda d(x, \gamma(1))^2 \\ &\quad - D\lambda(1 - \lambda)d(\gamma(0), \gamma(1))^2 \end{aligned} \quad (\dagger(D))$$

- $\forall y, x \in X$, with $d(x, y)$: small,
 y is on a min. geod. from x of given length

Assumption B4

For $\forall K \subset X$ cpt., $\exists D_K \geq 0$, $\exists \eta_K > 0$ s.t.

- (i) for $\forall \gamma$: d -min. geod. with $\gamma(0) \in K$,
 $d(\gamma(0), \gamma(1)) < \eta_K$, $d(\gamma(0), x) < \eta_K$ and
 $\lambda \in (0, 1)$, $(\dagger(D_K))$ holds.
- (ii) For $\forall x \in K$, $\forall y \in X$ with $d(x, y) < \eta_K$,
 $\exists \gamma$: min. geod. with $d(x, \gamma(1)) = \eta_K$ and
 $\gamma(\lambda) = y$ for some $\lambda \in (0, 1)$.

Example B

- X : cpl. Riem. mfd.
 - Ass. B4 (i) \Leftarrow local lower sect. curv. bound
 - Ass. B4 (ii) \Leftarrow local positivity of inj. radius
- X : Wiener space, d : Cameron-Martin norm
 - Ass. B4 (i) with $D_K \equiv 1$ (“=” holds!)
 - Ass. B4 (ii) is obvious

§6 Questions

(i) When (weak) \Rightarrow (strong) ?

i.e. $(C_p) \Rightarrow (C_{p'})$ or $(G_{p'}) \Rightarrow (G_p)$

for $p' > p$

(OK if X : Riem., $P = P_t$)

(ii) When “ $(C_p) \Rightarrow$ (pathwise control)”

in the case $P = P_t$?

(iii) “Bakry-Émery’s Γ_2 -criterion $\Leftrightarrow (G_q)$ ”

in the case $P = P_t$, $\tilde{d} = e^{-kt} d$?

(When $|\nabla_d f| = |\Gamma f|^{1/2}$?).

(iv) Relation with other “lower curvature bounds” ...