Duality on gradient estimates and Wasserstein controls*

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Let (X, d) be a Polish space. Let d and $\tilde{d}$ be continuous distance functions on X ($\tilde{d}$ can be different from d). For $f : X \to \mathbb{R}$ and $x \in X$, we define $|\nabla_d f|(x)$ and $\|\nabla_d f\|_\infty$ by

$$|\nabla_d f|(x) := \lim_{r \downarrow 0} \sup_{0 < d(x,y) \le r} \left| \frac{f(x) - f(y)}{d(x,y)} \right|, \qquad \|\nabla_d f\|_\infty := \sup_{x \in X} |\nabla_d f|(x).$$

Note that $\|\nabla_d f\|_\infty < \infty$ holds if and only if $f \in \mathrm{Lip}_d(X)$. Let $\mathcal{P}(X)$ be the space of all probability measures on X. For $p \in [1, \infty]$ and $\mu, \nu \in \mathcal{P}(X)$, we define L^p-Wasserstein (pseudo-) distance $d_p^W(\mu, \nu)$ by

$$d_p^W(\mu, \nu) := \inf \left\{ \|d\|_{L^p(\pi)} \;\middle|\; \pi : \text{coupling of } \mu \text{ and } \nu \right\}. \tag{I}$$

We define $|\nabla_{\tilde{d}} f|(x)$, $\|\nabla_{\tilde{d}} f\|_\infty$ and $\tilde{d}_p^W$ similarly by using $\tilde{d}$ instead of d.

Let $(P_x)_{x \in X}$ be a Markov kernel. For $p \in [1, \infty]$, we consider the following two properties:

(i) (L^q-gradient estimate (of Bakry-Émery type)) For $f \in \mathrm{Lip}_d(X) \cap C_b(X)$,

$$|\nabla_{\tilde{d}} P f|(x) \le P(|\nabla_d f|^q)(x)^{1/q} \tag{$\mathbf{G}_q$}$$

for any $x \in X$ when $q < \infty$. When $q = \infty$,

$$\left\| \nabla_{\tilde{d}} P f \right\|_\infty \le \|\nabla_d f\|_\infty . \tag{$\mathbf{G}_\infty$}$$

(ii) (L^p-Wasserstein control) For any $\mu, \nu \in \mathcal{P}(X)$,

$$d_p^W(P^* \mu, P^* \nu) \le \tilde{d}_p^W(\mu, \nu). \tag{$\mathbf{C}_p$}$$

Assumption 1

(i) d and $\tilde{d}$ are compatible with the topology on X and (X, d) is locally compact.

(ii) For any $x, y \in X$ there exists a curve $\gamma : [0, 1] \to X$ from x to y such that $d(\gamma(s), \gamma(t)) = |s - t| d(x, y)$ for any $s, t \in [0, 1]$. The same is true for $\tilde{d}$.

(iii) There exists a positive Radon measure v on X such that

(a) (X, d, v) enjoys the local uniform volume doubling condition, that is, there are constants $D, R_1 > 0$ such that $v(B_{2r}(x)) \le D v(B_r(x))$ holds for all $x \in X$ and $r \in (0, R_1)$.

(b) (X, d, v) supports a $(1, \rho)$-local Poincaré inequality for some $\rho \ge 1$, that is, for every $R > 0$, there are constants $\lambda \ge 1$ and $C_P > 0$ such that, for any $f \in \mathrm{Lip}_d(X)$,

$$\int_{B_r(x)} |f - f_{x,r}| \, dv \le C_P r \left\{ \int_{B_{\lambda r}(x)} |\nabla_d f|^\rho dv \right\}^{1/\rho} \tag{II}$$

holds for every $x \in X$ and $r \in (0, R)$, where $f_{x,r} := v(B_r(x))^{-1} \int_{B_r(x)} f \, dv$.

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(c) P_x is absolutely continuous with respect to v for all $x \in X$; $P_x(dy) = P_x(y)v(dy)$. In addition, the density $P_x(y)$ is continuous with respect to x.

Theorem 1 [3] *Let $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$. Then the following hold.*

(i) $(\mathbf{C}_p)$ *implies* $(\mathbf{G}_q)$ *(without Assumption 1).*

(ii) *Suppose that Assumption 1 holds. Then* $(\mathbf{G}_q)$ *implies* $(\mathbf{C}_p)$.

Example 1

(i) Let X be a Lie group of type H (e.g. a Heisenberg group), d the Carnot-Caratheodory distance and $P = P_t$ the heat semigroup associated with the canonical sub-Laplacian. Then $(\mathbf{G}_1)$ holds with $\tilde{d} = Kd$ for some $K > 1$ (see [2]; K must be > 1).

(ii) Let (X, d) be a Lie group with a bracket generating family $\{X_i\}_{i=1}^n$ of left-invariant vector fields, d the associated Carnot-Caratheodory distance and $P = P_t$ the heat semigroup associated with $\sum_{i=1}^n X_i^2$. Then $(\mathbf{G}_q)$ holds for $q > 1$ with $\tilde{d} = K_q(t)d$ (see [5]). If X is nilpotent, $K_q(t) \equiv K_q$ for some constant $K_q > 1$.

Assumption 1 is satisfied in the cases (i)(ii). Hence we can apply Theorem 1 to show $(\mathbf{C}_\infty)$ or $(\mathbf{C}_p)$ $(1 \le p < \infty)$ respectively.

For the proof of Theorem 1 (ii), we use a general theory of Hamilton-Jacobi semigroup developed in [1, 4] in the context of abstract metric spaces. Given $p \in [1, \infty)$, we define $Q_t f$ by

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + t \cdot \frac{1}{p} \left(\frac{d(x,y)}{t} \right)^p \right]$$

for $f \in C_b(X)$. Under Assumption 1 (iii)(a) and (iii)(b), for $t > 0$ and v-a.e. $x \in X$,

$$\lim_{s \downarrow 0} \frac{Q_{t+s} f(x) - Q_t f(x)}{s} = -\frac{1}{q} |\nabla_d Q_t f|(x)^q.$$

The Kantorovich duality provides a expression of $d_p^W(\mu, \nu)$ by means of the Hamilton-Jacobi semigroup and a sort of "the fundamental theorem of calculus" and "the Leibniz rule" implies the conclusion.

References

[1] Z.M. Balogh, A. Engoulatov, L. Hunziker, and O.E. Maasalo. Functional inequalities and Hamilton-Jacobi equations in geodesic spaces. preprint; arXiv:0906.0476.

[2] N. Eldredge. Gradient estimates for the subelliptic heat kernel on H-type groups. *J. Funct. Anal.*, 258(2):504–533, 2010.

[3] K. Kuwada. Duality on gradient estimates and Wasserstein controls. *J. Funct. Anal.*, 258(11):3758–3774, 2010.

[4] J. Lott and C. Villani. Hamilton-Jacobi semigroup on length spaces and applications. *J. Math. Pures Appl. (9)*, 88(3):219–229, 2007.

[5] T. Melcher. Hypoelliptic heat kernel inequalities on Lie groups. *Stochastic Process. Appl.*, 118(3):368–388, 2008.