Duality on gradient estimates and Wasserstein controls

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Let (X, d) be a complete, separable, proper, length metric space (hence d is a geodesic distance). Let $\tilde{d}$ be a continuous distance function on X, possibly different from d. Assume that $\tilde{d}$ is also a geodesic distance.

For a measurable function f on X and $x \in X$, we define $|\nabla_d f|(x)$ and $\|\nabla_d f\|_\infty$ by

$$|\nabla_d f|(x) := \lim_{r \downarrow 0} \sup_{0 < d(x,y) \le r} \left| \frac{f(x) - f(y)}{d(x,y)} \right|,$$
$$\|\nabla_d f\|_\infty := \sup_{x \in X} |\nabla_d f|(x).$$

Note that $\|\nabla_d f\|_\infty < \infty$ holds if and only if f is Lipschitz continuous. For two probability measures μ and ν on X, we denote the space of all couplings of μ and ν by $\Pi(\mu, \nu)$. That is, $\pi \in \Pi(\mu, \nu)$ means that π is a probability measure on $X \times X$ satisfying $\pi(A \times X) = \mu(A)$ and $\pi(X \times A) = \nu(A)$ for each Borel set A. For $p \in [1, \infty]$, we define $d_p^W(\mu, \nu)$ by

$$d_p^W(\mu, \nu) := \inf \left\{ \|d\|_{L^p(\pi)} \;\middle|\; \pi \in \Pi(\mu, \nu) \right\}. \tag{I}$$

We define $|\nabla_{\tilde{d}} f|(x)$, $\|\nabla_{\tilde{d}} f\|_\infty$ and $\tilde{d}_W^p$ similarly by using $\tilde{d}$ instead of d.

Set $\mathcal{P}(X)$ be the space of all probability measures on X equipped with the topology of weak convergence. Let $(P_x)_{x \in X}$ be a family of elements in $\mathcal{P}(X)$. Assume that $x \mapsto P_x$ is continuous as a map from X to $\mathcal{P}(X)$. Then $(P_x)_{x \in X}$ defines a bounded linear operator P on $C_b(X)$ by

$$Pf(x) := \int_X f(y) P_x(dy).$$

Let P^* be the adjoint operator of P. Note that $P^*(\mathcal{P}(X)) \subset \mathcal{P}(X)$ holds.

Assumption 1 There exists a positive Radon measure v on X such that

(i) (X, d, v) enjoys the local volume doubling condition. That is, there are constants $D, R_1 > 0$ such that $v(B_{2r}(x)) \le D v(B_r(x))$ holds for all $x \in X$ and $r \in (0, R_1)$.

(ii) (X, d, v) supports a $(1, p_0)$-local Poincaré inequality for some $p_0 \ge 1$. That is, for every $R > 0$, there are constants $\lambda \ge 1$ and $C_P > 0$ such that, for any $f \in L^1_{\mathrm{loc}}(v)$ and any upper gradient g of f,

$$\int_{B_r(x)} |f - f_{x,r}| \, dv \le C_P r \left\{ \int_{B_{\lambda r}(x)} g^{p_0} dv \right\}^{1/p_0} \tag{II}$$

holds for every $x \in X$ and $r \in (0, R)$, where $f_{x,r} := v(B_r(x))^{-1} \int_{B_r(x)} f \, dv$.

(iii) P_x is absolutely continuous with respect to v for all $x \in X$; $P_x(dy) = P_x(y) v(dy)$. In addition, the density $P_x(y)$ is continuous with respect to x.

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For $p \in [1,\infty]$, we consider the following two properties:

(i) (L^p-gradient estimate) For all bounded, Lipschitz continuous $f : X \to \mathbb{R}$,

$$|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^p)(x)^{1/p}, \qquad (\mathbf{G}_p)$$

for any $x \in X$ when $p < \infty$. When $p = \infty$,

$$\left\| \nabla_{\tilde{d}} Pf \right\|_\infty \le \|\nabla_d f\|_\infty . \qquad (\mathbf{G}_\infty)$$

(ii) (L^p-Wasserstein control) For all $\mu, \nu \in \mathcal{P}(X)$,

$$d_p^W(P^*\mu, P^*\nu) \le \tilde{d}_p^W(\mu,\nu). \qquad (\mathbf{C}_p)$$

Theorem 1 *Let $p, q \in [1,\infty]$ with $p^{-1} + q^{-1} = 1$. Then the following assertions hold.*

(i) $(\mathbf{C}_p)$ *implies* $(\mathbf{G}_q)$.

(ii) *Suppose that Assumption 1 holds. Then* $(\mathbf{G}_q)$ *implies* $(\mathbf{C}_p)$.

Example 1

(i) *Let (X,d) be a complete Riemannian manifold, $P = P_t$ the heat semigroup and $\tilde{d} = \mathrm{e}^{-kt} d$. Then $\mathbf{G}_q$ (for some $q \in [1,\infty]$) or $\mathbf{C}_p$ (for some $p \in [1,\infty]$) is equivalent to* $\mathrm{Ric} \ge k$ *[4].*

(ii) *Let X be a Lie group of type H (e.g. a Heisenberg group), d the Carnot-Caratheodory distance and $P = P_t$ the heat semigroup associated with the canonical sub-Laplacian. Then (G_1) holds for any $f \in C_c^1(X)$ with $\tilde{d} = Kd$ for some $K > 1$ [2].*

(iii) *Let (X,d) be a Lie group with a bracket generating family $\{X_i\}_{i=1}^n$ of left-invariant vector fields, d the associated Carnot-Caratheodory distance and $P = P_t$ the heat semigroup associated with $\sum_{i=1}^n X_i^2$. Then (G_p) holds for $p > 1$ for any $f \in C_c^\infty(X)$ with $\tilde{d} = K_p(t)d$ for $K_p(t) > 1$ [3]. If G is nilpotent, $K_p(t) \equiv K_p$ for some constant $K_p > 1$.*

Assumption 1 is satisfied in the cases (ii)(iii). We can apply Theorem 1 to show $(\mathbf{C}_\infty)$ *or* $(\mathbf{C}_q)$ *$(1 \le q < \infty)$ respectively.*

For the proof of Theorem 1 (ii), we use a general theory of Hamilton-Jacobi semigroup developed in [1]. Given $p \in [1,\infty)$, we define $Q_t f$ by $Q_t f(x) := \inf_{y \in X}[f(y) + p^{-1} t^{p-1} d(x,y)^p]$ for bounded continuous function f. Under Assumption 1 (i)(ii), for $t > 0$ and $\mathfrak{v}$-a.e. $x \in X$,

$$\lim_{s \downarrow 0} \frac{Q_{t+s} f(x) - Q_t f(x)}{s} = -\frac{1}{q} |\nabla_d Q_t f|(x)^q.$$

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