Characterization of maximal Markovian couplings for diffusion processes

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§1 Introduction

M: a Polish space (state space),

$(\{X_t\}_{t\geq 0}, \{\mathrm{P}_x\}_{x\in M})$: a diffusion process on M.

Coupling of (X, P_y) and (X, P_z) :

(Y_t, Z_t): an $M \times M$-valued stochastic process defined on $(\Omega, \mathcal{F}, \mathrm{P})$,

$$\mathrm{P} \circ Y^{-1} = \mathrm{P}_y \circ X^{-1},$$

$$\mathrm{P} \circ Z^{-1} = \mathrm{P}_z \circ X^{-1}.$$

$\boldsymbol{T} := \inf\{\boldsymbol{t} \geq 0 \mid \boldsymbol{Y_t} = \boldsymbol{Z_t}\}$: coupling time.

Coupling inequality

$$\mathrm{P}[\boldsymbol{T} > \boldsymbol{t}] \geq \left\| \mathbf{P}_{\boldsymbol{y}} \circ \boldsymbol{X}_{\boldsymbol{t}}^{-1} - \mathbf{P}_{\boldsymbol{z}} \circ \boldsymbol{X}_{\boldsymbol{t}}^{-1} \right\|_{\mathbf{var}}.$$

$(\boldsymbol{Y_t}, \boldsymbol{Z_t})$: maximal

$$\overset{\text{def}}{\Leftrightarrow} \begin{cases} \text{“}=\text{” holds in the coupling inequality} \\ \text{for all } t > 0 \end{cases}$$

$(\boldsymbol{Y}, \boldsymbol{Z})$: Markovian

$$\overset{\text{def}}{\Leftrightarrow} \begin{cases} \{(\boldsymbol{Y}_{s+t}, \boldsymbol{Z}_{s+t})\}_{t \geq 0} \text{ is a coupling} \\ \text{of } (\boldsymbol{X}, \mathrm{P}_{\boldsymbol{Y}_s}) \text{ and } (\boldsymbol{X}, \mathrm{P}_{\boldsymbol{Z}_s}) \\ \text{under } \mathrm{P}\left[\,\cdot \mid \boldsymbol{\sigma}((\boldsymbol{Y}_u, \boldsymbol{Z}_u) \;;\; \boldsymbol{0} \leq \boldsymbol{u} \leq \boldsymbol{s})\right]. \end{cases}$$

$\star$ $\{(\boldsymbol{Y}_t, \boldsymbol{Z}_t)\}_{t \geq 0}$ is a Markov process on $M \times M$

$\Rightarrow (\boldsymbol{Y}, \boldsymbol{Z})$: Markovian.

Fact (cf. Sverchkov & Smirnov '90)

A maximal coupling always exists in this framework.

Question

What are sufficient/necessary conditions on the existence of a maximal Markovian coupling?

§2 Main results

Sufficient condition: reflection structure

Example $\quad X$: Brownian motion on $M = \mathbb{R}^d$.

R: reflection w.r.t. a hyperplane H s.t. $Ry = z$.

(Y_t, Z_t): mirror coupling, i.e.

$$Z_t := \begin{cases} RY_t & \text{if } t < T, \\ Y_t & \text{if } t \geq T. \end{cases}$$

$\star$ $(RX, \mathrm{P}_y) \stackrel{d}{=} (X, \mathrm{P}_z)$.

$\star$ $M = M_0 \sqcup H \sqcup R(M_0)$.

$\left(\{X_t\}_{t\geq 0}, \{\mathrm{P}_x\}_{x\in M}\right)$ has a reflecton structure

w.r.t. (y, z)

$\Updownarrow$ def

$\exists R : M \to M$ continuous, $R^2 = \mathrm{id}$ s.t.

(i) $(RX, \mathrm{P}_y) \overset{d}{=} (X, \mathrm{P}_z)$

(ii) $\exists M_0$: open s.t. $M = M_0 \sqcup H \sqcup R(M_0)$,

(H: fixed points of R)

⋆ $\exists$ reflection structure $\Rightarrow$ $\exists$ mirror coupling.

Theorem 1 (K. '07)

M : a complete Riemannian manifold,

X : the Brownian motion on M.

$\exists$ a reflection structure w.r.t. (y, z)

$$\Downarrow$$

the mirror coupling is a unique maximal Markovian coupling of (X, P_y) and (X, P_z).

The reflection structure is also necessary in the following case:

Theorem 2 (K.)

M : a Riemannian homogeneous space,

X : the Brownian motion on M.

∃ a maximal Markovian coupling

of (X, P_y) and (X, P_z)

$$\Downarrow$$

∃ a reflection structure w.r.t. (y, z).

§3 Applications

1) Examples

M: irreducible global Riemannian symmetric space.

Then

> $\exists$ a reflection structure
>
> $\Rightarrow M$ is of constant curvature.

i.e.,

> M has a non-constant curvature
>
> $\Downarrow$
>
> $\nexists$ maximal Markovian coupling of Brownian motions on M.

Constant curvature cases

$\exists$ a reflection structure w.r.t. (y, z)?

- $\mathbf{S}^d, \mathbf{R}^d, \mathbf{H}^d$: $\exists$ for any (y, z) $(y \neq z)$.
- $\mathbf{RP}^d$ $(d \geq 2)$: $\nexists$ for any (y, z).
- $\mathbf{T}^d$ $(d \geq 2)$:

  $\exists$ for $(y, z) \Leftrightarrow \begin{cases} \text{only one coordinate of} \\ y \text{ and } z \text{ is distinct.} \end{cases}$

2) Kendall-Cranston couplings

K.-C. coupling: "infinitesimally mirror" coupling of Brownian motions on a complete Riemannian manifold. ($\Rightarrow$ K.-C. coupling is Markovian)

$$\Downarrow$$

M: homogeneous, no reflection structure

$\Rightarrow$ K.-C. coupling cannot be maximal

$\rightsquigarrow$ Many examples of non-maximal K.-C. coupling!!

(Non-maximal K.-C. coupling is first found in K.-Sturm '07.)

3) The case for Markov chains

The existence of a maximal coupling:

Griffeath '75, Pitman '76, Goldstein '79, etc.

(shown by construction)

$\rightsquigarrow$

It has been believed that, in general,

there is **no maximal Markovian coupling** .

Theorem 3 (K.)

There exists a Markov chain admitting no maximal Markovian coupling for specified starting points.

Idea of the proof of Theorem 3:

$X^{(n)}$: (nice) Markov chains $\xrightarrow{d}$ X: BM on $\mathbb{T}^d$.

$y, z \in \mathbb{T}^d$ s.t. $\nexists$ reflection structure w.r.t. (y, z).

$(y_n, z_n) \longrightarrow (y, z)$.

$\forall n, \exists (Y^{(n)}, Z^{(n)})$: maximal Markovian coupling of $(X^{(n)}, P_{y_n})$ and $(X^{(n)}, P_{z_n})$.

$\Downarrow$

So is a (subsequential) limit of them.

It contradicts with the choice of (y, z)!