Characterization of maximal Markovian couplings for diffusion processes

Kazumasa Kuwada

(Ochanomizu University)

§1 Introduction

M: a Polish space (state space),
$(\{X_t\}_{t\geq 0}, \{\mathrm{P}_x\}_{x\in M})$: a diffusion process on M.

Coupling of (X, P_y) and (X, P_z) :

(Y_t, Z_t): an $M \times M$-valued stochastic process defined on $(\Omega, \mathcal{F}, \mathrm{P})$,

$$\mathrm{P} \circ Y^{-1} = \mathrm{P}_y \circ X^{-1},$$
$$\mathrm{P} \circ Z^{-1} = \mathrm{P}_z \circ X^{-1}.$$

Coupling time:

$$T := \inf\{t \geq 0 \mid \forall s \geq t,\, Y_s = Z_s\}.$$

Coupling inequality

$$\mathrm{P}\,[T > t] \geq \frac{1}{2}\left\| \mathrm{P}_{y} \circ X_t^{-1} - \mathrm{P}_{z} \circ X_t^{-1} \right\|_{\mathrm{var}}.$$

(Y_t, Z_t): maximal

$\overset{\text{def}}{\Leftrightarrow}$ $\begin{cases} \text{"="}\text{ holds in the coupling inequality} \\ \text{for all } t > 0 \end{cases}$

(Y, Z): Markovian

$$\overset{\text{def}}{\Leftrightarrow} \begin{cases} \{(Y_{s+t}, Z_{s+t})\}_{t \ge 0} \text{ is a coupling} \\ \text{of } (X, \mathrm{P}_{Y_s}) \text{ and } (X, \mathrm{P}_{Z_s}) \\ \text{under } \mathrm{P}\left[\cdot \mid \sigma((Y_u, Z_u) \; ; \; 0 \le u \le s) \right]. \end{cases}$$

$\star$ $\{(Y_t, Z_t)\}_{t \ge 0}$ is a Markov process on $M \times M$

$\Rightarrow (Y, Z)$: Markovian.

Fact (cf. Sverchkov & Smirnov '90)

A maximal coupling always exists in this framework.

Question

What are sufficient/necessary conditions on the existence of a maximal Markovian coupling?

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§2 Main Results

Sufficient condition: reflection structure

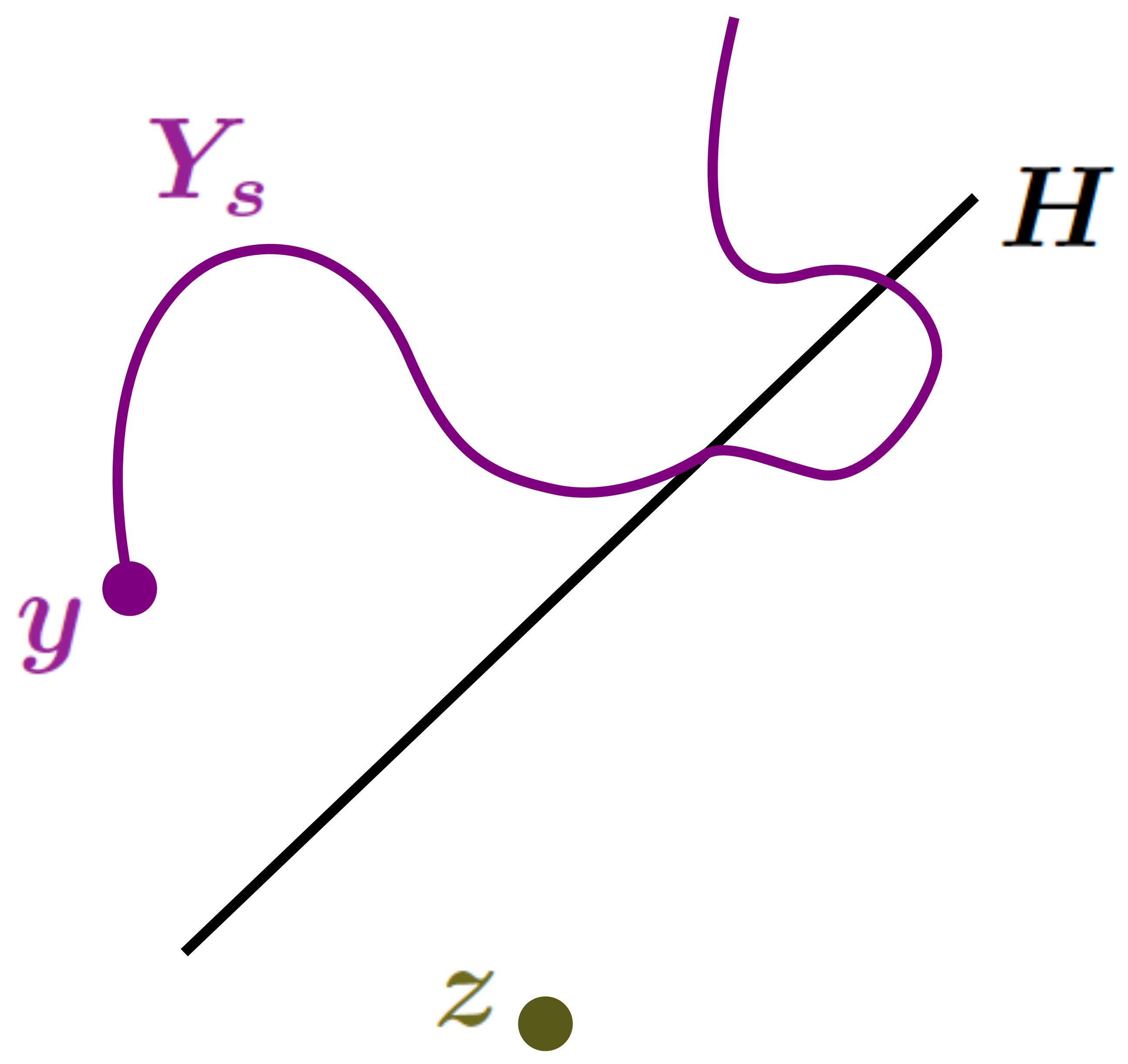
Example

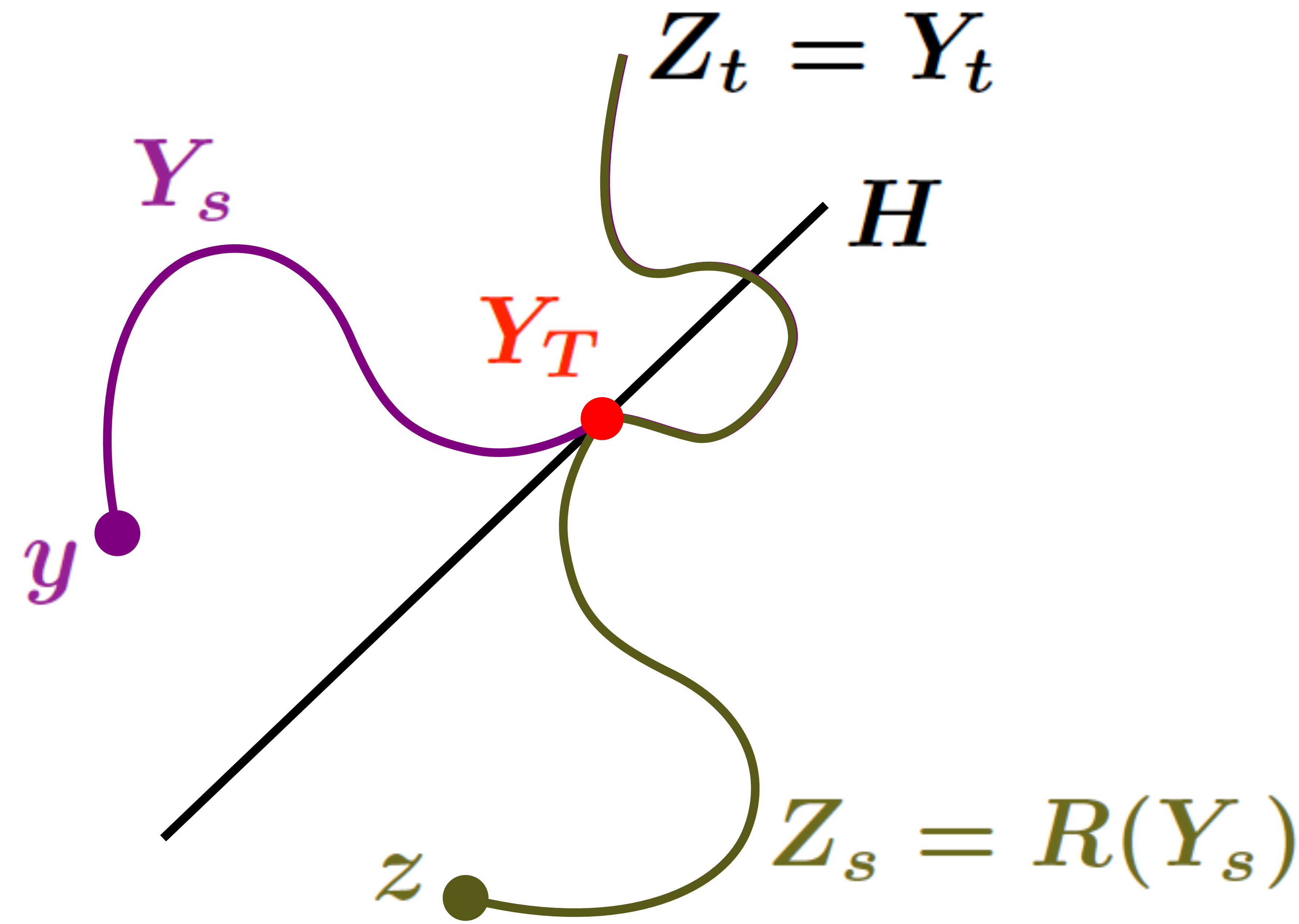
X: Brownian motion on $M = \mathbb{R}^d$.

R: reflection w.r.t. a hyperplane H s.t. $Ry = z$.

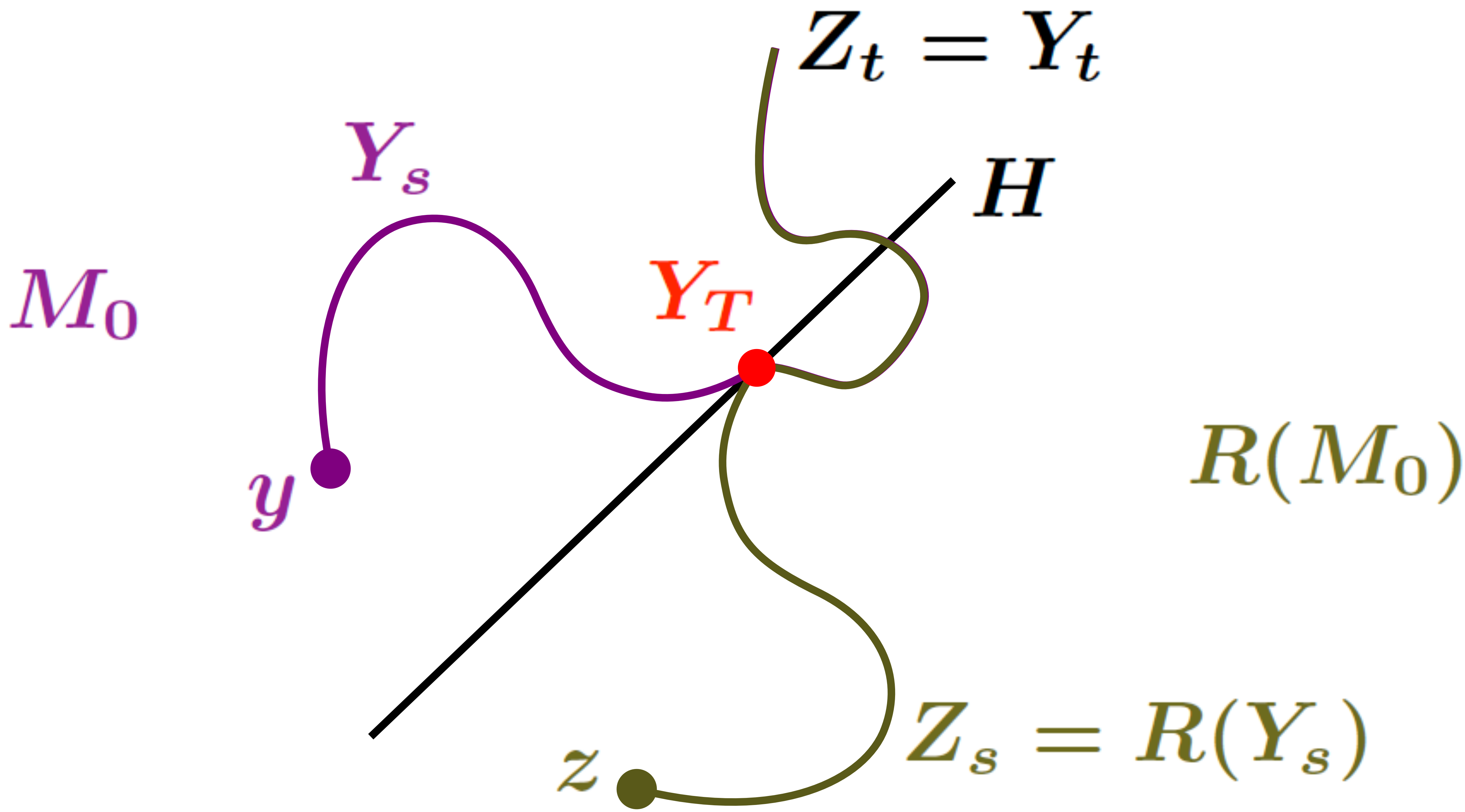
(Y_t, Z_t): mirror coupling, i.e.

$$Z_t := \begin{cases} RY_t & \text{if } t < T, \\ Y_t & \text{if } t \geq T. \end{cases}$$


Y_s
H
y
z



$Z_t = Y_t$
Y_s
H
Y_T
y
z
$Z_s = R(Y_s)$



- ⋆ $(RX, \mathrm{P}_y) \stackrel{d}{=} (X, \mathrm{P}_z)$.
- ⋆ $M = M_0 \sqcup H \sqcup R(M_0)$.

$\left(\{X_t\}_{t\geq 0}, \{\mathrm{P}_x\}_{x\in M}\right)$ has a reflecton structure

w.r.t. (y, z)

$\Updownarrow$ def

$\exists R : M \to M$ continuous, $R^2 = \mathrm{id}$ s.t.

(i) $(RX, \mathrm{P}_y) \overset{d}{=} (X, \mathrm{P}_z)$

(ii) $\exists M_0$: open s.t. $M = M_0 \sqcup H \sqcup R(M_0)$,

(H: fixed points of R)

⋆ $\exists$ reflection structure $\Rightarrow$ $\exists$ mirror coupling.

Theorem 1 (K. '07)

M : a complete Riemannian manifold,

X : the Brownian motion on M.

$\exists$ a reflection structure w.r.t. (y, z)

$$\Downarrow$$

the mirror coupling is a unique maximal Markovian coupling of (X, P_y) and (X, P_z).

The reflection structure is also necessary in the following case:

Theorem 2 (K.)

M : a Riemannian homogeneous space,

X : the Brownian motion on M.

∃ a maximal Markovian coupling

of (X, P_y) and (X, P_z)

$$\Downarrow$$

∃ a reflection structure w.r.t. (y, z).

§3 Examples and Applications

1) Examples

M: irreducible global Riemannian symmetric space.

Then

$\exists$ a reflection structure $\Rightarrow M$ is of constant curvature.

i.e.,

M has a non-constant curvature $\Downarrow$ $\nexists$ maximal Markovian coupling of Brownian motions on M.

Constant curvature cases:

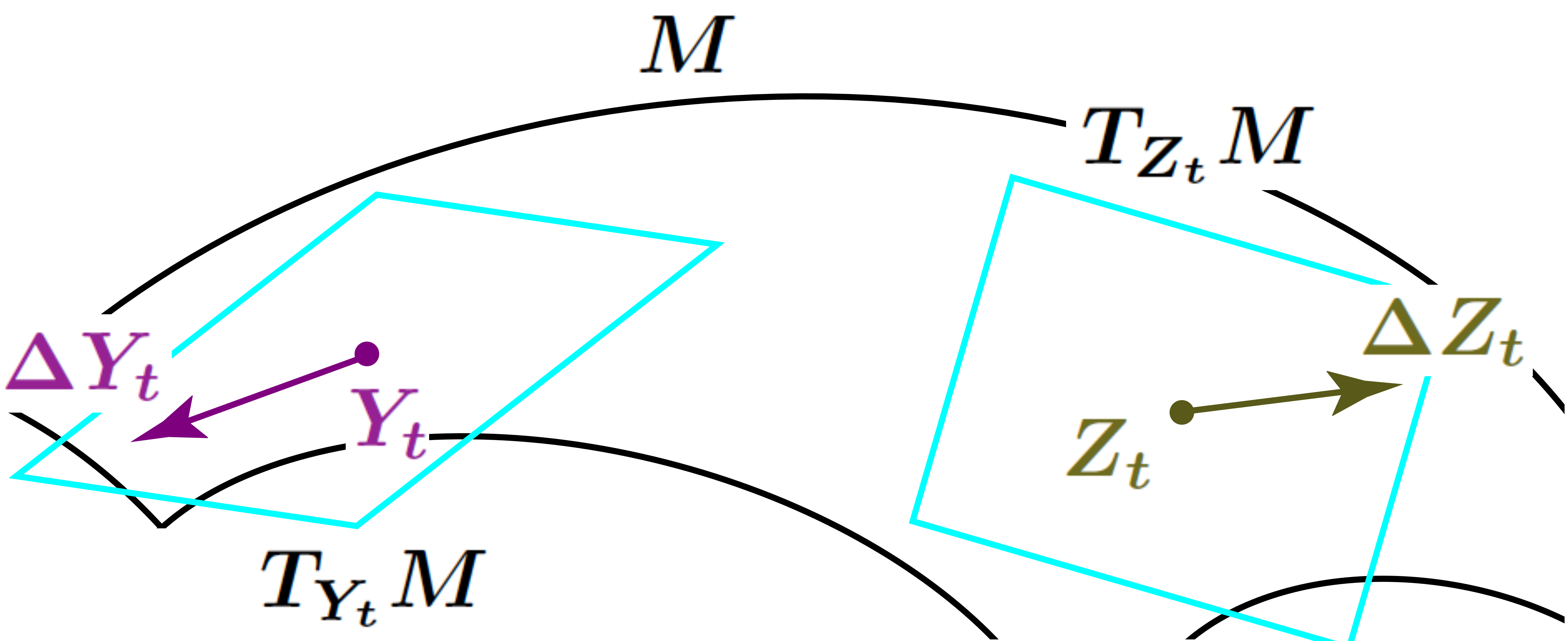
$$\exists \text{ a reflection structure w.r.t. } (y, z)?$$

- $\mathbf{S}^d, \mathbf{R}^d, \mathbf{H}^d$: $\exists$ for any (y, z) $(y \neq z)$.
- $\mathbf{RP}^d$ $(d \geq 2)$: $\nexists$ for any (y, z).
- $\mathbf{T}^d$ $(d \geq 2)$:

  $\exists$ for $(y, z) \Leftrightarrow \begin{cases} \text{only one coordinate of} \\ y \text{ and } z \text{ is distinct.} \end{cases}$

2) Kendall-Cranston couplings

K.-C. coupling: "infinitesimally mirror" coupling of Brownian motions on a complete Riemannian manifold.



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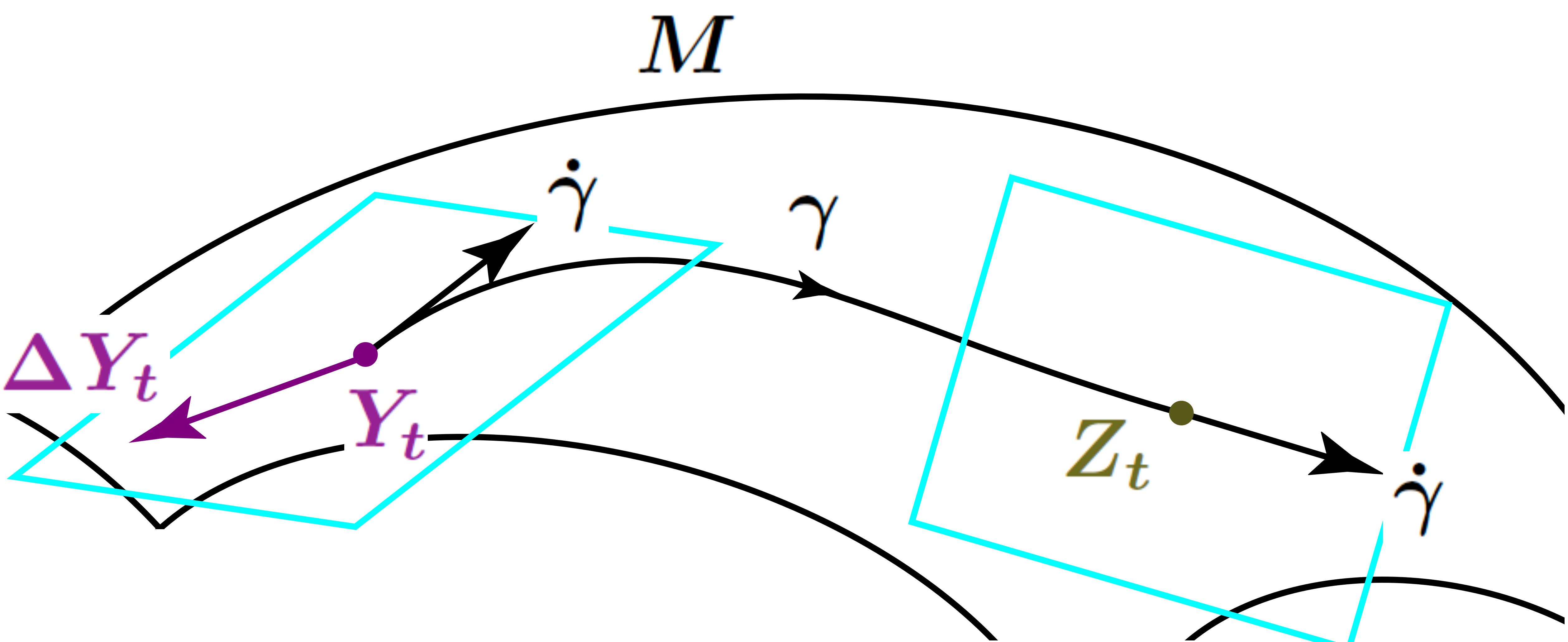

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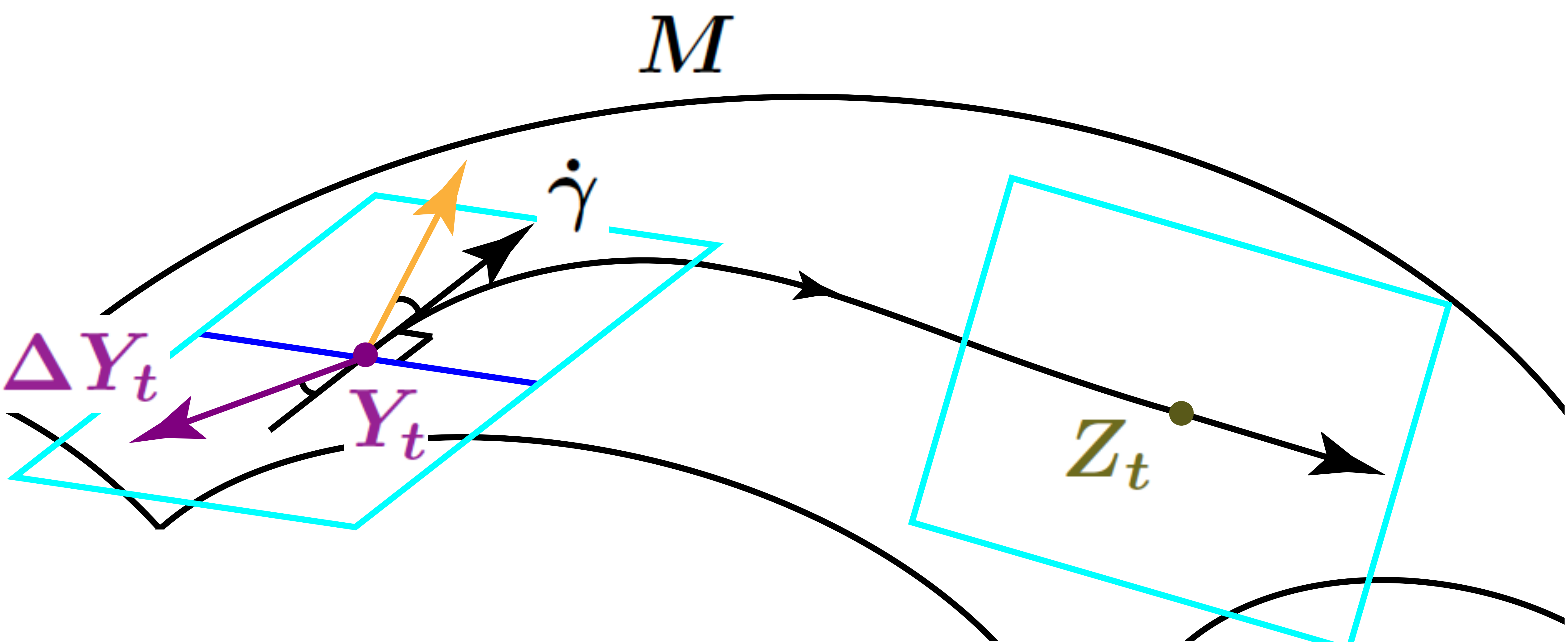

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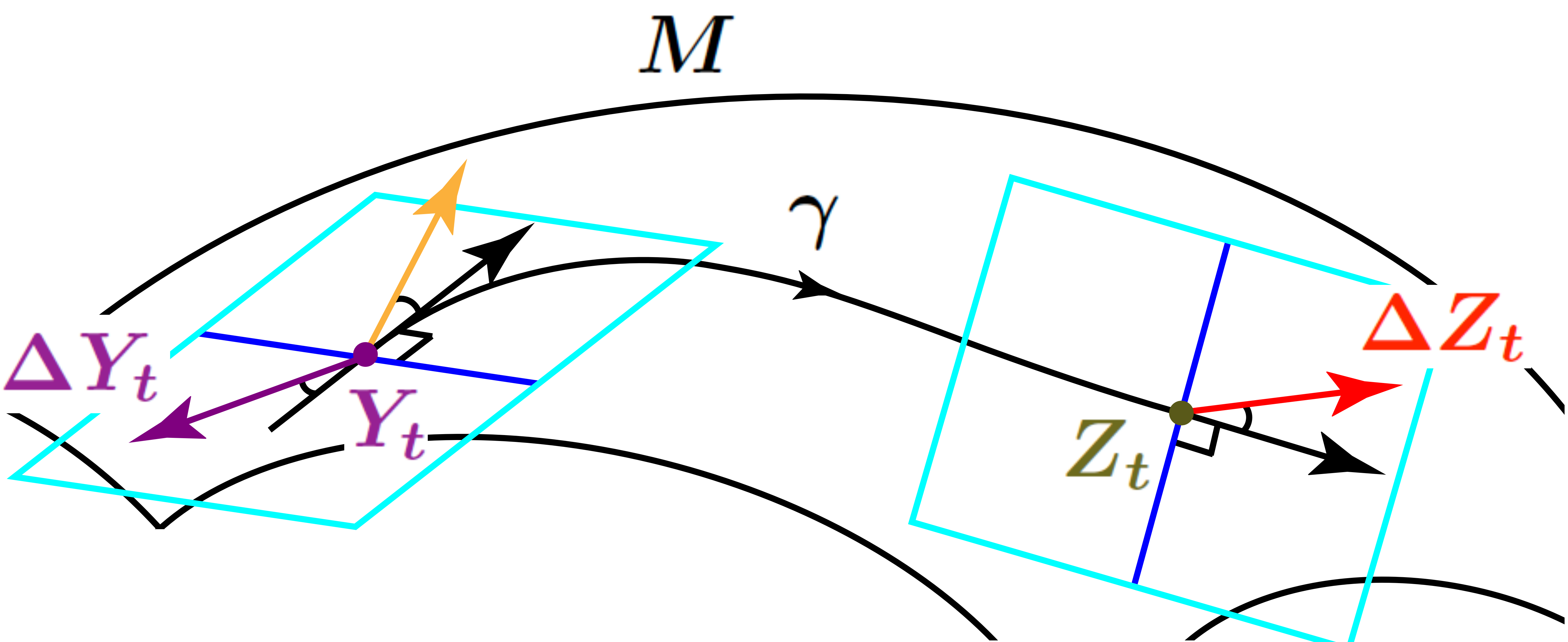

K.-C. coupling is Markovian

$\Downarrow$

M: homogeneous, no reflection structure

$\Rightarrow$ K.-C. coupling cannot be maximal

$\rightsquigarrow$ Many examples of non-maximal K.-C. coupling!!

$\left(\right.$Non-maximal K.-C. coupling is first found in [K.-Sturm '07].$\left.\right)$

3) The case for Markov chains

The existence of a maximal coupling:

[Griffeath '75], [Pitman '76], [Goldstein '79], etc.

(shown by construction)

$\rightsquigarrow$

It has been believed that, in general,

there is no maximal Markovian coupling .

Theorem 3 (K.)

There exists a Markov chain admitting no maximal Markovian coupling for specified starting points.

Idea of the proof of Theorem 3:

$X^{(n)}$: (nice) Markov chains $\xrightarrow{d}$ X: BM on $\mathbf{T}^d$.

$y, z \in \mathbf{T}^d$ s.t. $\nexists$ reflection structure w.r.t. (y, z).

$(y_n, z_n) \to (y, z)$.

$\forall n, \exists (Y^{(n)}, Z^{(n)})$: maximal Markovian coupling of $(X^{(n)}, P_{y_n})$ and $(X^{(n)}, P_{z_n})$.

$\Downarrow$

So is a (subsequential) limit of them.

It contradicts with the choice of (y, z)!

§4 Proof of the Main Theorem

Theorem 2 (K.)

M : a Riemannian homogeneous space,

X : the Brownian motion on M.

$\exists$ a maximal Markovian coupling (Y, Z)

of (X, P_y) and (X, P_z)

$\Downarrow$

$\exists$ a reflection structure w.r.t. (y, z).

Moreover, (Y, Z) is the mirror coupling w.r.t. the reflection structure.

Outline of the proof

Step 1: $\exists$ of a (moving) deterministic mirror H_t;

H_t "=" $\{w \mid d(Y_t, w) = d(Z_t, w)\}$.

Step 2: $\forall t, s > 0$, $H_t = H_{t+s}$

(H_t doesn't move).

Step 3: $\exists$ of a reflection map R w.r.t. H_0,

R provides a reflecton structure.

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R provides a reflecton structure.

A generalization of the framework in step 1 and 2

- M: a locally compact, complete geodesic space, minimal geodesics on M cannot branch.
- $(X_t, \mathbf{P}_x)$: a diffusion process on M.

★ $\exists p_t(x, y)$: transition density of X_t.

★ p_t is symmetric and continuous.

★

$$p_{t_n}(x, v) \geq p_{t_n}(w, v) \text{ for some } t_n \downarrow 0$$
$$\Updownarrow$$
$$d(x, v) \leq d(w, v).$$

Step 1: $\exists$ of a mirror H_t

$$\varphi_t(x,w) := \frac{1}{2} \left\| \mathrm{P}_x \circ X_t^{-1} - \mathrm{P}_w \circ X_t^{-1} \right\|_{\mathrm{var}} .$$

Lemma 1

(Y_t, Z_t): maximal Markovian coupling of

$$(X_t, \mathrm{P}_y) \ \& \ (X_t, \mathrm{P}_z)$$

$$\Rightarrow \mathbf{E}[\varphi_s(Y_t, Z_t)] = \varphi_{s+t}(y,z) .$$

Observation:

$$\varphi_t(x,w) = \sup_S \left(\mathbf{P}_x[X_t \in S] - \mathbf{P}_w[X_t \in S]\right).$$

Take $A \subset M$ s.t.

$$\varphi_{s+t}(y,z)$$

$$= \mathbf{P}_y[X_{s+t} \in A] - \mathbf{P}_z[X_{s+t} \in A]$$

$$\Downarrow$$

$$\mathbf{E}\left[\varphi_s(Y_t, Z_t)\right]$$

$$\geq \mathbf{E}\left[\mathbf{P}_{Y_t}[X_s \in A] - \mathbf{P}_{Z_t}[X_s \in A]\right]$$

$$= \varphi_{s+t}(x,y)$$

Consequence of Lemma 1:

$$\mathbf{E}\big[\varphi_s(Y_t, Z_t)\big]$$
$$= \mathbf{E}\left[\mathbf{P}_{Y_t}[X_s \in A] - \mathbf{P}_{Z_t}[X_s \in A]\right].$$

$$\Downarrow$$

$$\varphi_s(Y_t, Z_t) = \mathbf{P}_{Y_t}[X_s \in A] - \mathbf{P}_{Z_t}[X_s \in A]$$
a.s.

Remark

The last equation is nontrivial only on $\{Y_t \neq Z_t\}$.

Observation:

$$\varphi_u(x,w) = \mathrm{P}_x[X_t \in S] - \mathrm{P}_w[X_t \in S]$$

$$\Rightarrow S \text{ ``$=$'' } \{v \mid p_u(x,v) \geq p_u(w,v)\}.$$

$$\Downarrow$$

- A "=" $\{x \mid p_{t+s}(y,x) \geq p_{t+s}(z,x)\}$
- $\varphi_s(Y_t, Z_t)$

$$= \mathrm{P}_{Y_t}[X_s \in A] - \mathrm{P}_{Z_t}[X_s \in A]$$

$$\Rightarrow A \text{ ``$=$'' } \{x \mid p_s(Y_t,x) \geq p_s(Z_t,x)\}$$

on $\{Y_t \neq Z_t\}$.

$$\{x \mid p_{t+s}(y,x) = p_{t+s}(z,x)\}$$

$$\text{“}=\text{”} \ \{x \mid p_s(Y_t,x) = p_s(Z_t,x)\}$$

$$\text{on } \{Y_t \neq Z_t\}.$$

$$\Downarrow \quad s \downarrow 0$$

$$\{x \mid p_t(y,x) = p_t(z,x)\}$$

$$\text{“}=\text{”} \ \{x \mid d(Y_t,x) = d(Z_t,x)\}.$$

Mirror: H_t “$=$” $\{x \mid p_t(y,x) = p_t(z,x)\}$.

Step 2: $\forall t, s > 0$, $H_t = H_{t+s}$

Additional assumption

$\forall x, y \in M$, $\forall t > 0$,

$$p_t(x, y) \leq p_t(x, x). \qquad (*)$$

Remark

$$p_t(x, y) \leq p_t(x, x)^{1/2} p_t(y, y)^{1/2}$$

generally holds by the symmetry of p_t.

($\Rightarrow (*)$ holds under homogeneity.)

Notation:

$$\begin{cases} E_t := \{x \mid p_t(y,x) \geq p_t(z,x)\}, \\ E_t^* := \{x \mid p_t(y,x) \leq p_t(z,x)\}. \end{cases}$$

Remark

(i) $\begin{cases} \text{supp}\,[\mathrm{P}[Y_t \in \cdot \mid T > t]] \text{``}{=}\text{''} E_t, \\ \text{supp}\,[\mathrm{P}[Z_t \in \cdot \mid T > t]] \text{``}{=}\text{''} E_t^* \end{cases}$

($\because$ maximality)

(ii) $H_t \text{``}{=}\text{''} E_t \cap E_t^*$.

A consequence of step 1:

$$\begin{cases} E_{t+s} \text{“}=\text{”} \{x \mid p_s(Y_t, x) \geq p_s(Z_t, x)\}, \\ E^*_{t+s} \text{“}=\text{”} \{x \mid p_s(Y_t, x) \leq p_s(Z_t, x)\} \end{cases}$$

on $\{T > t\}$.

$\Downarrow$ Assumption

$$\boxed{Y_t \in E_{t+s},\ Z_t \in E^*_{t+s}}$$

$\Downarrow$ Rem.(i)

$$\boxed{E_t \subset E_{t+s},\ E^*_t \subset E^*_{t+s}} \Rightarrow \boxed{H_t \subset H_{t+s}}$$

Rem.(ii)

- H_t: closed & separates E_t and E_t^*
- If $H_{t+s} \setminus H_t \neq \emptyset$, then
  $\exists$ a path from E_t to E_t^* through $H_{t+s} \setminus H_s$
  without intersecting H_t.

$$\Downarrow$$

$$H_t \subset H_{t+s} \Rightarrow H_t = H_{t+s}$$