Laplace approximation for stochastic line integrals

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Let M be a closed Riemannian manifold. Consider a nondegenerate diffusion process $(\{z_t\}_{t\ge 0}, \{\mathbb{P}_x\}_{x\in M})$ on M with the generator $\mathscr{L} = \Delta/2 + b$, where Δ is the Laplace-Beltrami operator and b a smooth vector field. We endow a family of L^2-Sobolev norms $\{\|\cdot\|_p\}_{p\in\mathbb{R}}$ on the space of smooth differential 1-forms by using powers of the Hodge-Kodaira Laplacian. For each $p \in \mathbb{R}$, $\mathscr{D}_p$ stands for the complition by $\|\cdot\|_p$.

We can define the stochastic line integral $\int_{z[0,t]} \alpha$ of a smooth 1-form α along a diffusion path $\{z_s\}_{s\in[0,t]}$. The martingale part of $\int_{z[0,t]} \alpha$ is denoted by $Y_t(\alpha)$. By choosing $p > 0$ sufficiently large, we regard a random mapping $Y_t : \alpha \mapsto Y_t(\alpha)$ as a $\mathscr{D}_{-p}$-valued random variable.

In this talk, we will deal with the Laplace approximation for $\bar{Y}_t := t^{-1}Y_t$ as $t \to \infty$.

Definition 1 *We define the good rate function $I : \mathscr{D}_{-p} \to [0,\infty]$ as follows:*

$$I(w) = \begin{cases} \dfrac{1}{2}\displaystyle\int_M |\hat{w}|^2 d\mu^w & \text{if } w \in \mathscr{H}, \\ \infty & \text{otherwise,} \end{cases}$$

where we say $w \in \mathscr{H}$ when $w \in \mathscr{D}_{-p}$ satisfies the following three conditions:

(i) *There exists a probability measure μ^w on M so that, for each $u \in C^\infty(M)$,*

$$\langle \omega, du\rangle + \int_M \mathscr{L}u \, d\mu^\omega = 0.$$

(ii) *There exists $\hat{\omega} \in L^2_1(d\mu^\omega)$ so that, for each $\alpha \in \mathscr{D}_p$, $\langle \omega, \alpha\rangle = \displaystyle\int_M (\hat{\omega}, \alpha)\, d\mu^\omega$ holds.*

(iii) *The probability measure μ^ω appeared in (i) is absolutely continuous with respect to the Riemannian measure v and $\sqrt{d\mu^w/dv} \in H_1$. Here H_1 stands for the first order L^2-Sobolev space.*

Our starting point is the fact that $\bar{Y}_t$ satisfies the large deviation principle with the rate function I as $t \to \infty$ (see [2]). Take a continuous function $F : \mathscr{D}_{-p} \to \mathbb{R}$. Then the Varadhan lemma asserts that, under a good integrability condition for $F(\bar{Y}_t)$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp\left(tF(\bar{Y}_t)\right)\right] = \sup_{w\in\mathscr{D}_{-p}} \{F(w) - I(w)\} =: \kappa_F.$$

In what follows, we assume that F is third-times Fréchet differentiable.

Now we introduce some notations to state our main theorem.

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(I) Let $\mathscr{K}_F := \{w \in \mathscr{D}_{-p}\ ;\ F(w) - I(w) = \kappa_F\}$. Note that $\mathscr{K}_F$ is a nonempty compact set. We denote k-th order Fréchet derivative at w by $\nabla^k F(w)$. (we will abbreviate k when $k = 1$). For $w \in \mathscr{D}_{-p}$, we write $\alpha_w := \nabla F(w) \in \mathscr{D}_p$.

(II) Consider a differential operator $u \mapsto \mathscr{L}u + (\alpha, du) + |\alpha|^2 u/2$ on $L^2(dv)$. We denote by h^α the eigenfunction corresponding to its principal eigenvalue. h^α becomes an element of H_{p+1}, $(p+1)$-th order L^2-Sobolev space. By using h^α, we define a differential operator $\mathscr{L}^\alpha\ :\ u \mapsto \mathscr{L}u + (\alpha - dh^\alpha/h^\alpha, du)$. We denote the normalized invariant measure of $\mathscr{L}^\alpha$ by m_α. Note that $m_{\alpha_w} = \mu^w$ holds when $w \in \mathscr{K}_F \subset \mathscr{H}$.

(III) Take $\alpha \in \mathscr{D}_p$. By using a solution $u^{\alpha,\beta}$ to the following defferential equation

$$\mathscr{L}u + \left(\alpha - \frac{dh^\alpha}{h^\alpha}, du\right) = \left(\alpha - \frac{dh^\alpha}{h^\alpha}, \beta\right) - \int_M \left(\alpha - \frac{dh^\alpha}{h^\alpha}, \beta\right) dm_\alpha,$$

we define a operator Γ_α by $\Gamma_\alpha \beta := du^{\alpha,\beta}$. Then Γ_α becomes a bounded linear operator on $\mathscr{D}_p$. Let $G^F_w\ :\ \mathscr{D}_{-p} \to \mathscr{D}_{-p}$ be a bounded symmetric linear operator given by

$$(\eta, G^F_w \eta)_{-p} = \nabla^2 F(w)((1 - \Gamma^*_{\alpha_w})\eta, (1 - \Gamma^*_{\alpha_w})\eta).$$

(IV) Let $\beta^* \in \mathscr{D}_{-p}$ be the adjoint element of $\beta \in \mathscr{D}_p$. For $w \in \mathscr{H}$, we define a symmetric, positive definite operator $S_w\ :\ \mathscr{D}_{-p} \to \mathscr{D}_{-p}$ of trace class by

$$\langle S_w(\beta^*), \gamma \rangle = \int_M (\beta, \gamma) d\mu^w.$$

(V) Take $w \in \mathscr{D}_{-p}$. Let us define a functional $L_w\ :\ \mathscr{D}_{-p} \to [0, \infty]$ by

$$L_w(\eta) := \begin{cases} \dfrac{1}{2}\displaystyle\int_M |\breve{w}|^2 dm_{\alpha_w} & \text{if } \langle w, \beta \rangle = \displaystyle\int_M (\breve{w}, \beta) dm_{\alpha_w} \text{ for some } \breve{w} \in L^2_1(dm_{\alpha_w}), \\ \infty & \text{otherwise.} \end{cases}$$

Then we have $2L_w(\eta) = \inf\left\{\|\eta'\|_{-p}\ ;\ \eta = \sqrt{S_w}\eta'\right\}$.

Assumption 1 For each $w \in \mathscr{K}_F$, there is a constant $\delta_w > 0$ so that

$$L_w(\eta) \geq \frac{1}{2}(\eta, G^F_w \eta)_{-p} + \delta_w \|\eta\|^2_{-p}$$

holds for every $\eta \in \mathscr{D}_{-p}$.

Theorem 1 [1] *Under Assumption 1, $\mathscr{K}_F$ is a finite set and we have*

$$\lim_{t \to \infty} \mathrm{e}^{-t\kappa_F} \mathbb{E}_x \left[\exp\left(tF(\bar{Y}_t)\right)\right] = \sum_{w \in \mathscr{K}_F} \frac{1}{\det\left(1 - G^F_w \circ S_w\right)^{1/2}} h^{\alpha_w}(x) \int_M \frac{1}{h^{\alpha_w}} d\mu^w.$$

References

[1] K. Kuwada. Laplace approximation for stochastic line integrals. preprint.

[2] K. Kuwada. On large deviations for random currents induced from stochastic line integrals. to appear in Forum Math.