# Laplace approximation for stochastic line integrals

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# §1 Introduction

$M$ : closed Riemannian manifold

$(\{z_t\}_{t\geq 0}, \{\mathrm{P}_x\}_{x\in M})$ : diffusion process on $M$

its generator : $\mathcal{L} := \frac{1}{2}\Delta + b$

$$X_t(\alpha) := \int_{z[0,t]} \alpha \text{ : stochastic line integral}$$

($\alpha$ : smooth differential 1-form)

$$X_t(\alpha) =: Y_t(\alpha) + A_t(\alpha)$$

$Y_t(\alpha)$ : martingale part of $X_t(\alpha)$

$A_t(\alpha)$ : bounded variation part of $X_t(\alpha)$

$\| \cdot \|_p$ : $p$-th order $L^2$-Sobolev norm on 1-forms;

$$\|\alpha\|_p^2 := \int_M |(1 - \Delta_1)^{p/2} \alpha|^2 dv$$

$$\mathfrak{D}_p := \overline{\{\text{smooth differential 1-forms}\}}^{\| \cdot \|_p}$$

$\star$ $\mathfrak{D}_{-p}$ is regarded as ${\mathfrak{D}_p}^*$

$p > 0$ large enough $\Longrightarrow X_t, Y_t, A_t \in \mathfrak{D}_{-p}$

Purpose : To know the asymptotic behavior of $\{Y_t\}_{t>0}$ as $t \to \infty$

Analysis for $Y_t$ $\rightsquigarrow$

- $X_t$ or $A_t$
- empirical measure $\int_0^t \delta_{z_s} ds$
- homological behavior of $\{z_s\}_{s\in[0,t]}$
- Periodic diffusion($M = \mathbf{T}^d$)

## Known results

Law of large numbers: $\bar{Y}_t := \frac{1}{t} Y_t \xrightarrow[t\to\infty]{} 0$

(Ikeda '87)

Central limit theorem: $\mathrm{P}_x \circ \left( \frac{1}{\sqrt{t}} Y_t \right)^{-1} \xrightarrow[t\to\infty]{d} \nu_{S_0}$

(Ochi '85)

Large deviation with a rate function $I$:

$$\mathrm{P}_x \left[ \bar{Y}_t \in \mathcal{A} \right] \asymp \exp \left( -t \inf_{w \in \mathcal{A}} I(w) \right)$$

(K. '03)

$\Rightarrow$ By the Varadhan lemma, for $F : \mathfrak{D}_{-p} \longrightarrow \mathrm{R}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}_x \left[\exp\left(tF(\bar{Y}_t)\right)\right] = \sup_{w\in\mathfrak{D}_{-p}} \left(F(w) - I(w)\right) =: \kappa_F$$

<u>Problem</u>:

$$e^{-t\kappa_F} \mathbf{E}_x \left[\exp\left(tF(\bar{Y}_t)\right)\right] \xrightarrow[t\to\infty]{} \quad ?$$

## §2 Rate function $I$ and its perturbation

# Heuristics on the Varadhan lemma

By the large deviation,

$$\mathbf{P}_x\left[\bar{Y}_t \in dw\right] \asymp \exp\left(-tI(w)\right)dw$$

$$\Rightarrow \mathbf{E}_x\left[\exp\left(tF(\bar{Y}_t)\right)\right]$$

$$\asymp \int \exp\Big(t\big(F(w) - I(w)\big)\Big)dw$$

$$\Rightarrow \frac{1}{t}\log \mathbf{E}_x\left[\exp\left(tF(\bar{Y}_t)\right)\right] \xrightarrow[t\to\infty]{} \kappa_F$$

$$\Rightarrow \left[ \begin{array}{c} \text{the limit of } \mathrm{e}^{-t\kappa_F} \mathbf{E}_x \left[ \exp\left( tF(\bar{Y}_t) \right) \right] \\ \updownarrow \\ \text{behavior of } F - I \text{ near } w \text{ where} \\ F(w) - I(w) = \kappa_F \end{array} \right]$$

For simplicity, suppose

$$\boxed{F(w) - I(w) = \kappa_F \Leftrightarrow w = w_0}$$

$$\Rightarrow \nabla(F - I)(w_0) = 0, \ \nabla^2(F - I)(w_0) \leq 0$$

(Assume $F$ to be smooth in the Fréchet sense)

$$I(w) := \begin{cases} \frac{1}{2}\int_M |\hat{w}|^2 d\mu^w & \text{if } w \in \mathcal{H}, \\ \infty & \text{otherwise.} \end{cases}$$

$$w \in \Omega \Leftrightarrow {}^{\exists}\mu^w \in \mathcal{M}_1(M) \text{ s.t. } \sqrt{\frac{d\mu^w}{dv}} \in H_1,$$

$$\langle w, du\rangle + \int_M \mathfrak{L}u \, d\mu^w = 0.$$

$$w \in \mathcal{H} \Leftrightarrow w \in \Omega \text{ and } {}^{\exists}\hat{w} \in L_1^2(d\mu^w),$$

$$\text{s.t. } \langle w, \alpha\rangle = \int_M (\hat{w}, \alpha) d\mu^w.$$

$$\tilde{\mathfrak{L}}^{\alpha}u \;:=\; \mathfrak{L}u + (\alpha, du) + \frac{1}{2}|\alpha|^2 u$$

$\Lambda(\alpha)$ : the principal eigenvalue of $\tilde{\mathfrak{L}}^{\alpha}$

$h^{\alpha}$ : eigenfunction $\leftrightarrow \Lambda(\alpha)$

$$\overline{\alpha} \;:=\; \alpha - \frac{dh^{\alpha}}{h^{\alpha}}$$

$$\mathfrak{L}^{\alpha}u \;:=\; \mathfrak{L}u + (\overline{\alpha}, du)$$

$m_{\alpha}$ : normalized invariant measure of $\mathfrak{L}^{\alpha}$

$\alpha_0 := \nabla F(w_0)$, i.e. $\langle w, \alpha_0 \rangle = \nabla F(w_0)(w)$

<u>Lemma 1</u> $\quad \langle w_0, \beta \rangle = \int_M (\overline{\alpha_0}, \beta)\, dm_{\alpha_0}.$

**Proposition 1** $\quad L_{w_0}(w) \geq \frac{1}{2}(w, G^F_{w_0} w)_{-p}.$

$$\left( \Leftrightarrow \nabla^2 (F - I)(w_0) \leq 0 \right)$$

$$L_{w_0}(w) = \begin{cases} \frac{1}{2} \int_M |\check{w}|^2 dm_{\alpha_0} & \text{if } w \in \mathcal{H}'_{w_0}, \\ \infty & \text{otherwise.} \end{cases}$$

$$w \in \mathcal{H}'_{w_0} \quad \Leftrightarrow \quad {}^{\exists}\check{w} \in L^2_1(dm_{\alpha_0}),$$

$$\langle w, \beta \rangle = \int_M (\check{w}, \beta) dm_{\alpha_0}.$$

$$\left( 2L_{w_0} \Leftrightarrow \nabla^2 I(w_0) \right)$$

$$
\begin{aligned}
&(\eta, G^F_{w_0}\eta)_{-p} \\
&\quad = \nabla^2 F(w_0)((1-\Gamma^*_{\alpha_0})\eta, (1-\Gamma^*_{\alpha_0})\eta).
\end{aligned}
$$

$\Gamma_{\alpha_0} : \mathfrak{D}_p \longrightarrow \mathfrak{D}_p$ bounded and linear

$\Gamma_{\alpha_0}\beta = du^{\alpha_0,\beta}$

$u^{\alpha_0,\beta}$ : a solution to

$$
\mathfrak{L}^{\alpha_0} u = (\overline{\alpha_0}, \beta) - \int_M (\overline{\alpha_0}, \beta)\, dm_{\alpha_0}.
$$

**Assumption 1** $\exists \delta > 0$ s.t.

$$L_{w_0}(w) \geq \frac{1}{2}(w, G^F_{w_0} w)_{-p} + \delta \|w\|^2_{-p}.$$

$$(\Leftrightarrow \nabla^2(F - I) < 0)$$

# §3 Asymptotic behavior near equilibrium

$$J(t) := \mathrm{e}^{-t\kappa_F} \mathbf{E}_x \left[ \exp\left( tF(\bar{Y}_t) \right) \right]$$

$$\mathcal{A} := \left\{ \|\bar{Y}_t - w_0\|_{-p} \le \varepsilon \right\}$$

$$\Downarrow$$

$$J(t) \sim \mathrm{e}^{-t\kappa_F} \mathbf{E}_x \left[ \mathbf{1}_{\mathcal{A}} \exp\left( t(F(\bar{Y}_t) - \kappa_F) \right) \right]$$

By the large deviation for $\bar{Y}_t$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{E}_x \left[ \exp\left( tF(\bar{Y}_t) \right) ; \mathcal{A}^c \right]$$

$$\le \sup \left\{ F(w) - I(w) ; \|w - w_0\|_{-p} > \varepsilon \right\}$$

$$< \kappa_F$$

$\varepsilon \approx 0 \Rightarrow$ the Taylor expansion of $F(\bar{Y}_t)$ near $w_0$

$$\Downarrow$$

$$J(t) \sim E_x^{\alpha_0}\Big[1_{\mathcal{A}} \cdot \frac{h^{\alpha_0}(x)}{h^{\alpha_0}(z_t)} \times \exp\Big(\frac{t}{2}\nabla^2 F(w_0)(\bar{Y}_t - w_0, \bar{Y}_t - w_0)\Big)\Big]$$

$$E_x^{\alpha}[f(z_t)] = e^{-t\Lambda(\alpha)}\mathbf{E}_x\left[e^{Y_t(\alpha)}\frac{h^{\alpha}(z_t)}{h^{\alpha}(x)}f(z_t)\right]$$

( $\Lambda(\alpha)$ : principal eigenvalue of $\tilde{\mathfrak{L}}^{\alpha} \leftrightarrow h^{\alpha}$ )

## $h$-transform

| | | | |
|---|---|---|---|
| measure | $\mathbf{P}_x$ | $\Rightarrow$ | $\mathbf{P}_x^{\alpha_0}$ |
| generator | $\mathfrak{L}$ | $\Rightarrow$ | $\mathfrak{L}^{\alpha_0}$ |
| martingale | $Y_t$ | $\Rightarrow$ | $Y_t^{\alpha_0}$ |
| CLT | $\frac{1}{\sqrt{t}} Y_t \xrightarrow{d} \nu_{S_0}$ | $\Rightarrow$ | $\frac{1}{\sqrt{t}} Y_t^{\alpha_0} \xrightarrow{d} \nu_{S_{w_0}}$ |
| covariance | $S_0$ | $\Rightarrow$ | $S_{w_0}$ |

$$Y_t^{\alpha_0}(\beta) := Y_t(\beta) - \int_0^t (\overline{\alpha_0}, \beta)(z_s) ds$$

$$\langle S_{w_0}(\gamma^*), \beta \rangle := \int_M (\gamma, \beta) dm_{\alpha_0} \quad (\gamma, \beta \in \mathfrak{D}_p)$$

**Lemma 2**

$\bar{Y}_t - w_0 = (1 - \Gamma^*_{\alpha_0})\bar{Y}_t^{\alpha_0} +$(remainder term)

**Proof.**

$$\bar{Y}_t(\beta) - \langle w_0, \beta \rangle - \bar{Y}_t^{\alpha_0}(\beta)$$

$$= \frac{1}{t}\int_0^t (\overline{\alpha_0}, \beta)(z_s)ds - \int_M (\overline{\alpha_0}, \beta)\, dm_{\alpha_0}$$

$$= \frac{1}{t}\int_0^t \mathfrak{L}^{\alpha_0} u^{\alpha_0,\beta}(z_s)ds$$

$$= -\bar{Y}_t^{\alpha_0}(du^{\alpha_0,\beta})$$

$$+ t^{-1}\left(u^{\alpha_0,\beta}(z_t) - u^{\alpha_0,\beta}(z_0)\right)$$

$$J(t) \sim$$

$$\mathbf{E}_x^{\alpha_0}\left[1_{\mathcal{A}} \cdot \frac{h^{\alpha_0}(x)}{h^{\alpha_0}(z_t)} \exp\left(\frac{t}{2}(\bar{\mathbf{Y}}_t^{\alpha_0}, G_{w_0}^F \bar{\mathbf{Y}}_t^{\alpha_0})_{-p}\right)\right]$$

$\Downarrow$ $z_t$ is strongly mixing

$$J(t) \sim h^{\alpha_0}(x)\mathbf{E}_x^{\alpha_0}\left[\frac{1}{h^{\alpha_0}(z_t)}\right]$$

$$\times\, \mathbf{E}_x^{\alpha_0}\left[1_{\mathcal{A}} \cdot \exp\left(\frac{1}{2}\left\|\sqrt{G_{w_0}^F}\left(\frac{1}{\sqrt{t}}\mathbf{Y}_t^{\alpha_0}\right)\right\|_{-p}^2\right)\right]$$

$$\xrightarrow[t\to\infty]{\text{CLT}} h^{\alpha_0}(x)\int_M \frac{1}{h^{\alpha_0}}dm_{\alpha_0}$$

$$\times \int \exp\left(\frac{1}{2}(w, G^F_{w_0}w)_{-p}\right) d\nu_{S_{w_0}}(w)$$

$$= \frac{h^{\alpha_0}(x)}{\det(1 - G^F_{w_0} \circ S_{w_0})^{1/2}} \int_M \frac{1}{h^{\alpha_0}}dm_{\alpha_0}$$

$$\left( d\nu_{S_{w_0}} = \frac{1}{Z}\exp\left(-\frac{1}{2}(w, S^{-1}_{w_0}w)_{-p}\right) dw \right)$$

Now remove the assumption

$$F(w) - I(w) = \kappa_F \Leftrightarrow w = w_0$$

$$\mathcal{K}_F := \{w \in \mathcal{D}_{-p} \, ; \, F(w) - I(w) = \kappa_F\}$$

$\mathcal{K}_F$ is a nonempty compact set

**Assumption 1'**

For $\forall w \in \mathcal{K}_F$, $\exists \delta_w > 0$ so that

$$L_w(\eta) \geq \frac{1}{2}(\eta, G_w^F \eta)_{-p} + \delta_w \|\eta\|_{-p}^2$$

for all $\eta \in \mathcal{D}_{-p}$.

**Theorem 1** Under Assumption 1',

$\#\mathcal{K}_F < \infty$ and

$$\lim_{t\to\infty} e^{-t\kappa_F}\mathbf{E}_x\left[\exp\left(tF(\bar{Y}_t)\right)\right]$$

$$=\sum_{w\in\mathcal{K}_F}\frac{h^{\alpha_w}(x)}{\det(1-G_w^F\circ S_w)^{1/2}}\int_M\frac{1}{h^{\alpha_w}}dm_{\alpha_w},$$

where $\alpha_w = \nabla F(w)$.

$$\left(\begin{array}{l}\text{Assumption 1} \Leftrightarrow \nabla^2(F-I)<0 \\ \quad \Rightarrow \#\mathcal{K}_F<\infty\end{array}\right)$$

# §4 Related results

## Asymptotic behavior of $X_t$ and $A_t$

$$e(\alpha) := \int_M \left( (\hat{b}, \alpha) - \frac{1}{2} d^* \alpha \right) dm_0$$

$$\bar{X}_t := \frac{1}{t} X_t - e, \quad \bar{A}_t := \frac{1}{t} A_t - e$$

$$\star \lim_{t \to \infty} \bar{X}_t = \lim_{t \to \infty} \bar{A}_t = 0 \quad \text{a.s.}$$

$$Q\alpha := du^{\alpha}, \quad \mathfrak{L} u^{\alpha} = (\hat{b}, \alpha) - \frac{1}{2} d^* \alpha - e(\alpha)$$

$$\Rightarrow \bar{A}_t\big((1 - Q)\alpha\big) = 0$$

$$\Rightarrow \quad \bar{X}_t(\alpha) = \bar{Y}_t((1-Q)\alpha) + t^{-1}\left(u^{\alpha}(z_t) - u^{\alpha}(z_0)\right)$$

$$\bar{A}_t(\alpha) = \bar{Y}_t((-Q)\alpha) + t^{-1}\left(u^{\alpha}(z_t) - u^{\alpha}(z_0)\right)$$

LDP: $I^{(T)}(w) = \inf_{T\eta = w} I(\eta) \quad (T : \mathfrak{D}_{-p} \longrightarrow \mathfrak{D}_{-p})$

| r.v. | | rate function |
|---|---|---|
| $\bar{X}_t$ | $\longleftrightarrow$ | $I^{(1-Q^*)}$ |
| $\bar{A}_t$ | $\longleftrightarrow$ | $I^{(-Q^*)}$ |

$$F_1 := F \circ (1 - Q^*)$$

**Theorem 2** Assume $\nabla^2(F_1 - I) < 0$ on $\mathcal{K}_{F_1}$. Then $\#\mathcal{K}_{F_1} < \infty$ and

$$\lim_{t \to \infty} e^{-t\kappa_{F_1}} \mathrm{E}_x \left[ \exp \left( tF(\bar{X}_t) \right) \right]$$

$$= \sum_{w \in \mathcal{K}_{F_1}} \frac{h^{\alpha_w}(x) \mathrm{e}^{-u^{\tilde{\alpha}_w}(x)}}{\det(1 - G_w^{F_1} \circ S_w)^{1/2}} \int_M \frac{\mathrm{e}^{u^{\tilde{\alpha}_w}}}{h^{\alpha_w}} dm_{\alpha_w}$$

$$\left( \alpha_w = \nabla F_1(w),\ \tilde{\alpha}_w = \nabla F((1 - Q^*)w) \right)$$

$F_2 := F \circ (-Q^*)$

**Theorem 3** Assume $\nabla^2(F_2 - I) < 0$ on $\mathcal{K}_{F_2}$. Then $\#\mathcal{K}_{F_2} < \infty$ and

$$\lim_{t\to\infty} e^{-t\kappa_{F_2}} \mathrm{E}_x \left[\exp\left(tF(\bar{A}_t)\right)\right]$$

$$= \sum_{w\in\mathcal{K}_{F_2}} \frac{h^{\alpha_w}(x)\mathrm{e}^{-u^{\tilde{\alpha}_w}(x)}}{\det(1 - G_w^{F_2} \circ S_w)^{1/2}} \int_M \frac{\mathrm{e}^{u^{\tilde{\alpha}_w}}}{h^{\alpha_w}} dm_{\alpha_w}$$

$$\left(\alpha_w = \nabla F_2(w),\ \tilde{\alpha}_w = \nabla F((-Q^*)w)\right)$$

# Empirical measures

$$A_t(\alpha) = \int_0^t \left( (\hat{b}, \alpha) - \frac{1}{2} d^* \alpha \right) (z_s) ds$$

$$\Rightarrow A_t(du) = \int_0^t \mathfrak{L} u(z_s) ds$$

$$\iota : \mathfrak{D}_{-p} \longrightarrow H_{-p-1}, \langle \iota(w), u \rangle_{H_{p+1}} = \langle w, du \rangle$$

$\mathcal{G} = \mathfrak{L}^{-1}$ : Green operator

$$\Rightarrow \mathcal{G}^* \circ \iota(\bar{A}_t) + m_0 = \frac{1}{t} \int_0^t \delta_{z_s} ds =: \bar{l}_t$$

$$\hat{F} : H_{-p+1} \longrightarrow \mathrm{R}$$

Apply Theorem 3 for $F(\cdot) = \hat{F}(\mathcal{G}^* \circ \iota(\cdot) + m_0)$

$$\Downarrow$$

$$\lim_{t\to\infty} \mathrm{e}^{-t\hat{\kappa}_{\hat{F}}} \mathbf{E}_x \left[ \exp\left( t\hat{F}(\bar{l}_t) \right) \right]$$

$$= \sum_{\mu \in \hat{\mathcal{K}}_{\hat{F}}} \frac{\hat{h}_\mu(x)}{\det\left(1 - \nabla^2 \hat{F} \circ \hat{S}_\mu\right)^{1/2}} \int_M \frac{1}{\hat{h}_\mu} d\mu$$

(cf. Bolthausen-Deuschel-Tamura '95)

## Comparison with Theorem 1

Nonexistence of a term corresponding to $(1 - \Gamma^{*}_{\alpha_w})$

$\uparrow$

† $Y_t$(or $X_t$, $A_t$) depends on $\mathrm{P}_x$ in its definition

$\Longrightarrow$ We need a correction term under the transformation of measure $\mathrm{P}_x \longrightarrow \mathrm{P}^{\alpha_0}_x$

† The definition of $l_t$ is independent of $\mathrm{P}_x$

$\Longrightarrow$ We need no correction term