On large deviations for current-valued processes induced from stochastic line integrals

Kazumasa Kuwada

Department of Applied Analysis and Complex Dynamical Systems

Graduate School of Informatics

Kyoto University

Abstract

We study large deviations and their rate functions in the framework of current-valued stochastic processes. The processes we consider are determined by stochastic line integrals of 1-forms on a compact Riemannian manifold. Two different types of large deviations are proved, corresponding to the conditions on the decay rate of the process. We obtain explicit expressions of their rate functions, which enable us to observe their difference more closely and give clues to know the influence of the geometric structure on the stochastic processes. Our theorems provide a unified viewpoint of various limit theorems such as the large deviation for empirical means and the Strassen law of the iterated logarithms. With the aid of these estimates, we give a probabilistic approach to the analysis on noncompact Abelian covering manifolds.

Contents

1 Introduction

Asymptotic behaviors of the diffusion processes on Riemannian manifolds have been one of the central problems on intersection of the probability theory and the geometry. In this paper, we prove several limit theorems, including the large deviation principles, for a certain vector valued functional of diffusion processes to study how the geometric structure influences the behavior of the diffusion paths.

In order to investigate the asymptotic behaviors, it is often effective to consider a functional of diffusion processes which fits for each purpose. For example, if we want to know the behavior of the diffusion paths regarded as random sets, we can observe it through the analysis of empirical measures driven by the diffusion path. Another typical example is the behavior of winding numbers of the Brownian paths, which was first studied by Manabe [22]. By the harmonic integration theory, the line integral of harmonic 1-forms along a cycle brings us homological information about the cycle. In the same way, the stochastic line integral of harmonic 1-forms along the Brownian path expresses homological behaviors of the Brownian motion.

The object of our analysis in this paper, the random current, is more general in the sense that it contains all information about functionals explained above. It is defined by embedding the diffusion process into the space of 1-currents by means of the stochastic line integrals. Here, (1-)currents mean a kind of distributions whose test functions consist of differential 1-forms. This framework provides us a unified approach to the analysis of many functionals such as empirical measures and winding numbers. Moreover, it enables us to investigate more refined asymptotic behaviors than what are obtained by individual functionals. In what follows, we give a short introduction of our framework. The precise definitions will be given in the next section.

Consider a nondegenerate diffusion process $\{z_t\}_{t\geq 0}$ on a compact Riemannian manifold M. The stochastic line integral $\int_{z[0,t]} \alpha$ of a 1-form α on M along the diffusion paths $\{z_s\}_{s\in[0,t]}$ can be defined. For each $t \geq 0$, we can regard a random functional $X_t : \alpha \mapsto \int_{z[0,t]} \alpha$ on the space of smooth 1-forms as a current-valued random variable. The current-valued process $\{X_t\}_{t\geq 0}$ is known to be a semimartingale. Let us denote the martingale part of $\{X_t\}_{t\geq 0}$ by $\{Y_t\}_{t\geq 0}$.

Such current-valued processes were introduced by Ikeda and Ochi [14, 16, 24]. They have established the law of large numbers and the central limit theorem for these processes. Let us define Y^λ by $Y_t^\lambda := \lambda^{-1/2} Y_{\lambda t}$. The central limit theorem for Y asserts that Y^λ converges in law into a current-valued Wiener process W^1 as $\lambda \to \infty$. Take another scaling $g(\lambda)$ with $\lim_{\lambda\to\infty} g(\lambda) = \infty$ and set $\tilde{Y}^\lambda := g(\lambda)^{-1} Y^\lambda$. To know more precise asymptotics than the preceding limit theorems, we shall study the asymptotic behavior of

$\tilde{Y}^\lambda$ formulated as large deviations corresponding to the conditions on the scaling parameter g, which is the main theme of this article.

As a remarkable phenomenon, large deviations with a renormalizing parameter often show a drastic change of their rate functions as the parameter arrives at some critical growth order. In order to clarify the point, consider the following simple example. Let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^d$-valued zero mean i.i.d. random vectors with a non-degenerate covariance matrix C and a sufficiently good integrability condition. Take a renormalizing parameter $\{g(n)\}_{n\in\mathbb{N}}$, an increasing sequence diverging to ∞, and define $S_n = g(n)^{-1}n^{-1/2}\sum_{i=1}^n Z_i$. When $g(n) = o(\sqrt{n})$, S_n satisfies the large deviation principle in the sense that the asymptotic behavior of $g(n)^{-2}\log\mathbb{P}[S_n \in \cdot]$ is governed by a rate function independent of g. This estimate is often called a moderate deviation. When $g(n) = \sqrt{n}$, S_n also satisfies the large deviation as above by the Cramér theorem, but the rate function is quite different in general. Indeed, the rate function corresponding to the moderate deviation is quadratic and depends only on C. As a matter of fact, it also governs the large deviation for $\{g(n)^{-1}N\}_{n\in\mathbb{N}}$ where N is a zero mean Gaussian random vector with the same covariance matrix C. Meanwhile, we can easily see that the latter rate function is more complicated; its expression by variational equality indicates that it is influenced by the law of Z_1 more sensitively.

Our main theorems give the existence of large deviations for $\tilde{Y}^\lambda$ together with an explicit distinct expression of their rate functions under the condition $g(\lambda) = o(\sqrt{\lambda})$ or $g(\lambda) = \sqrt{\lambda}$ respectively. The large deviation principle we shall show is formulated as follows:

$$\limsup_{\lambda\to\infty}\frac{1}{g(\lambda)^2}\log\left(\sup_{x\in M}\mathbb{P}_x\left[\tilde{Y}^\lambda \in \mathscr{A}\right]\right) \le -\inf_{w\in\bar{\mathscr{A}}}\mathscr{I}(w), \tag{1.1}$$

$$\liminf_{\lambda\to\infty}\frac{1}{g(\lambda)^2}\log\left(\inf_{x\in M}\mathbb{P}_x\left[\tilde{Y}^\lambda \in \mathscr{A}\right]\right) \ge -\inf_{w\in\mathscr{A}^\circ}\mathscr{I}(w). \tag{1.2}$$

When $g(\lambda) = o(\sqrt{\lambda})$, we obtain the estimates of type (1.1) and (1.2) with the rate function $\mathscr{I} = L$ (see Theorem 2.6; for the definition of L, see Definition 2.5). We also call this estimate the moderate deviation as in the case of i.i.d. random vectors. On the other hand, when $g(\lambda) = \sqrt{\lambda}$, we show the estimates of the same type with the rate function $\mathscr{I} = I$, instead of L (see Theorem 2.8; for the definition of I, see Definition 2.7).

The functional L is quadratic and characterized by the fact that it also governs the large deviation for $\{g(\lambda)^{-1}W^1\}_{\lambda>0}$. Thus, to see the definition of $\tilde{Y}^\lambda$, we can say that the effect of the convergence of $\{Y^\lambda\}_{\lambda>0}$ in law to W^1 is stronger than the effect of decay by $g(\lambda)^2$ in the case of $g(\lambda) = o(\sqrt{\lambda})$. As for I, comparing with L by using our expressions, we conclude that I is distinct from L even when M is a flat torus and $\{z_t\}_{t\ge0}$ is the Brownian motion. These observations suggest that the rate of the convergence of $\{Y^\lambda\}_{\lambda>0}$ is precisely equal to λ. That is, in some sense, a precision of the central limit theorem is derived by comparing these rate functions.

The difference between I and L comes from the fact that I is influenced by the asymptotic behavior of the mean empirical measures $\{\lambda^{-1}\int_0^\lambda \delta_{z_s}ds\}_{\lambda>0}$, while L is not. More precisely, while the value of $L(w)$ is determined by the normalized invariant measure m of the diffusion $\{z_t\}_{t\ge0}$, $I(w)$ needs another measure μ^w depending on w, instead of m, for its expression. We can interpret μ^w as a normalized invariant measure of a differential operator which is a sum of the generator of $\{z_t\}_{t\geq 0}$ and a lower-order perturbation determined by w. In addition, as we will see in section 4, this perturbation term controls the asymptotic behavior of the mean empirical measures. Thus, we can say that, in the case of $g(\lambda)=\sqrt{\lambda}$, the asymptotic behavior of $\tilde{Y}^\lambda$ is affected by the trajectory of diffusion path more sensitively. This fact means that the difference between I and L reflects the influence of the geometric structure in the asymptotic behavior of diffusion paths.

Note that we can study the asymptotic behavior of the mean empirical measures itself directly in our framework. Through a natural embedding of the mean empirical measures, our result implies the large deviation principle for the mean empirical measures as random currents. Thus, in this sense, our result generalizes the celebrated Donsker-Varadhan law [7] in the case of diffusion processes on a compact Riemannian manifold.

We now review some preceding results on this topic in order to explain the relation to the above-mentioned ones. Our theorem on the moderate deviation is regarded as a generalization of Baldi's work [3]. He has developed a large deviation estimate associated with the stochastic homogenization of periodic diffusions on $\mathbb{R}^d$. As is pointed out in [24], the stochastic homogenization of periodic diffusions on $\mathbb{R}^d$ follows from the central limit theorem of stochastic line integrals on the torus. In the case of $g(\lambda)=\sqrt{\lambda}$, Avellaneda [1] has studied the large deviation for the stochastic line integrals $\{t^{-1}X_t(\alpha)\}_{t>0}$ of a harmonic 1-form α along the Brownian motion. Manabe [23] has extended his result to the large deviation for random currents $\{t^{-1}X_t\}_{t>0}$ corresponding to the Brownian motion. Note that we mainly deal with the martingale part Y instead of X itself. The crucial difference between X and Y in asymptotics is that X degenerates on exact 1-forms by the Stokes formula: $\int_{z[0,t]} df = f(z_t)-f(z_0)$. Combining it with the fact that the measure μ^w appearing in the expression of $I(w)$ is determined by the action of w on exact 1-forms, we know that the analysis of Y has an advantage in observing the influence of the mean empirical laws. Besides, almost all limit theorems for $\{X_t\}_{t\geq 0}$, including the large deviation, are deduced as corollaries of those for $\{Y_t\}_{t\geq 0}$.

Now we introduce some applications. As a corollary of the moderate deviation, we obtain the Strassen law of the iterated logarithm for our current-valued processes (Theorem 4.12) by considering the case $g(\lambda)=\sqrt{\log\log\lambda}$. Another application is concerned with a covering manifold N of M whose covering transformation group is Abelian. Through the harmonic integration theory, we can apply our large deviations to the analysis of the long time asymptotics of the Brownian motion B_t on N. In particular, the Strassen law gives an estimate for the Riemannian distance between B_t and B_0 as $t\to\infty$ (Corollary 4.19):

$$\limsup_{t\to\infty}\frac{\mathrm{dist}(B_t,B_0)}{\sqrt{2t\log\log t}}=c \quad \text{a.s.}$$

The organization of this paper is as follows. The framework on which our results are based is arranged in the next section. There we review the law of large numbers and the central limit theorem and state our main theorems, the large deviation estimates for Y (Theorem 2.6 and Theorem 2.8). Section 3 is devoted to the proof of our main theorems. In the case of $g(\lambda)=o(\sqrt{\lambda})$, all part of the proof, including our expression of the rate function, is arranged in section 3.2. This is deduced from Baldi's theorem [2], which is an infinite dimensional analogue of Gärtner's theorem [11]. The rest of section 3 deals with the case $g(\lambda)=\sqrt{\lambda}$. We show the existence of large deviation in section 3.3.1.

First we prove the existence for finite dimensional distributions and extend it to the space of current-valued processes. To provide an explicit expression of the rate function, we prepare in section 3.3.2 an extension of the variational formula for the principal eigenvalue of second order differential operators on M. In section 3.3.3, we obtain an explicit representation of the rate function. There we study the Legendre transform of principal eigenvalues of perturbed Laplace operators. Indeed, it coincides with our rate function.

Some results related to large deviations for Y and their rate functions are arranged in section 4. First we deal with the large deviations for X and its bounded variation part A in section 4.1. Section 4.2 is devoted to a comparison of the rate functions. The difference between rate functions I and L is clarified there through some examples. As we will see, the essential difference between the large deviation and the moderate deviation resulted from the difference of the asymptotic behavior of A. To observe this, we give an explicit form of rate functions for A, coming from the large deviation for empirical measures of the diffusion process. Related to the large deviation for A, we mention how it is regarded as a generalization of the Donsker-Varadhan law in section 4.3. In section 4.4, we shall prove the Strassen law of the iterated logarithms for X, Y and A. In the last part of section 4, we develop an application of our moderate deviation to the analysis of Abelian covering manifolds of M.

Acknowledgment. I am deeply indebted to my advisor, Professor Masanori Hino for his thoughtful guidance and innumerable discussions. I would also like to thank Professor Shinzo Watanabe for leading me to the beautiful world of the probability theory and Professor Takashi Kumagai for his valuable comments and encouragement.

2 Framework and main results

2.1 Current-valued processes X, Y and A

Let M be a d-dimensional, oriented, compact and connected Riemannian manifold. Take a differential operator $\Delta/2 + b$ on M, where Δ is the Laplace-Beltrami operator and b a smooth vector field. The diffusion process on M associated with $\Delta/2 + b$ shall be denoted by $(\{z_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M})$. Note that the generator of any nondegenerate diffusion on M is of the form $\Delta/2 + b$ when we replace the Riemannian metric suitably. For simplicity, we omit the initial point x from the notation and denote $\mathbb{P} = \mathbb{P}_x$ when it is not so significant. Let m be the normalized invariant measure of $\{z_t\}_{t\geq 0}$.

For differential 1-forms α and β on M, $|\alpha|(x)$ (resp. $(\alpha, \beta)(x)$) means the cotangent norm of $\alpha(x)$ (resp. the inner product of $\alpha(x)$ and $\beta(x)$) on the cotangent space T_x^*M at $x \in M$. Let v be the normalized Riemannian measure on M. Let $\mathscr{D}_{1,\infty}$ be the totality of smooth 1-forms on M equipped with the L^2-Schwartz topology. This topology is determined by seminorms $\{\|\cdot\|_p\}_{p\geq 0}$ given by the power of the Hodge-Kodaira Laplacian Δ_1 acting on 1-forms. That is,

$$\|\alpha\|_p = \left\{ \int_M |(1-\Delta_1)^{p/2}\alpha|^2 dv \right\}^{1/2}.$$

This topology makes $\mathscr{D}_{1,\infty}$ a nuclear space (see [18, 24]; our seminorms are different from those in [24]. However, they induce the same topology). Let $\mathscr{D}_{1,p}$ be the Hilbert space given by the completion of $\mathscr{D}_{1,\infty}$ by $\|\cdot\|_p$. The space $\mathscr{D}_{1,-\infty}$ of 1-currents on M is the dual space of $\mathscr{D}_{1,\infty}$ and $\mathscr{D}_{1,-p}$ the dual space of $\mathscr{D}_{1,p}$. We use a symbol $\|\cdot\|_{-p}$ for the operator norm on $\mathscr{D}_{1,-p}$. The dual pairing of $\mathscr{D}_{1,p}$ and $\mathscr{D}_{1,-p}$ is denoted by $\langle\cdot,\cdot\rangle$. For each positive measure μ on M, let $L^2_1(d\mu)$ be the family of all measurable 1-forms α with $\|\alpha\|_{L^2_1(d\mu)} < \infty$, where

$$(\alpha^1, \alpha^2)_{L^2_1(d\mu)} := \int_M (\alpha^1, \alpha^2) d\mu$$

for measurable 1-forms α^1, α^2 and $\|\alpha\|_{L^2_1(d\mu)} = (\alpha, \alpha)^{1/2}_{L^2_1(d\mu)}$. Note that the Laplace-Beltrami operator Δ is expressed as $\Delta = -\delta d$ where δ is the adjoint derivative of the exterior derivative d in $L^2_1(dv)$.

For each smooth 1-form α and $t \geq 0$, we define $X_t(\alpha)$ by the stochastic line integral $\int_{z[0,t]} \alpha$ along the diffusion path $\{z_s\}_{0\leq s\leq t}$ (see [15] for the precise definition of $\int_{z[0,t]} \alpha$). For each $\alpha \in \mathscr{D}_{1,\infty}$, $\{X_t(\alpha)\}_{t\geq 0}$ is a semimartingale. We denote its martingale part by $Y_t(\alpha)$ and its bounded variation part by $A_t(\alpha)$:

$$X_t(\alpha) = Y_t(\alpha) + A_t(\alpha).$$

Let $\mathscr{C}_p = C([0,\infty) \to \mathscr{D}_{1,-p})$ be the Polish space of $\mathscr{D}_{1,-p}$-valued continuous functions equipped with the compact uniform topology. Then we can regard $Y = \{Y_t(\alpha)\}_{t\geq 0, \alpha\in\mathscr{D}_{1,\infty}}$ (resp. $X = \{X_t(\alpha)\}_{t\geq 0, \alpha\in\mathscr{D}_{1,\infty}}$ or $A = \{A_t(\alpha)\}_{t\geq 0, \alpha\in\mathscr{D}_{1,\infty}}$) as a $\mathscr{C}_p$-valued random variable for sufficiently large p as follows.

Proposition 2.1 ([16, 24])

(i) *For $p > d$, there exists a $\mathscr{C}_p$-valued random variable $\hat{Y} = \{\hat{Y}_t\}_{t\geq 0}$ such that for each $t \geq 0$*

$$\langle \hat{Y}_t, \alpha\rangle = Y_t(\alpha), \quad \alpha \in \mathscr{D}_{1,\infty} \text{ a.s.}$$

(ii) *For $p > d+1$, there exist $\mathscr{C}_p$-valued random variables $\hat{X} = \{\hat{X}_t\}_{t\geq 0}$ and $\hat{A} = \{\hat{A}_t\}_{t\geq 0}$ such that for each $t \geq 0$*

$$\langle \hat{X}_t, \alpha\rangle = X_t(\alpha), \quad \alpha \in \mathscr{D}_{1,\infty} \text{ a.s.},$$
$$\langle \hat{A}_t, \alpha\rangle = A_t(\alpha), \quad \alpha \in \mathscr{D}_{1,\infty} \text{ a.s.}$$

For simplicity we use the same symbols X, Y and A for these versions. Thus we regard X, Y and A as $\mathscr{C}_p$-valued random variables in the sense above.

Let us observe some properties which shall be used later. These are given in [15, 24].

(i) For $\alpha \in \mathscr{D}_{1,p}$, if $\alpha = du$ for some function u, then

$$X_t(\alpha) = u(z_t) - u(z_0). \tag{2.1}$$

(ii) The quadratic variation $\langle Y(\alpha)\rangle_t$ of the martingale part $Y_t(\alpha)$ is given by

$$\langle Y(\alpha)\rangle_t = \int_0^t |\alpha|(z_s)^2 ds. \tag{2.2}$$

(iii) For the bounded variation part $A_t(\alpha)$, we have

$$A_t(\alpha) = \int_0^t \left((\hat{b}, \alpha) - \frac{1}{2}\delta\alpha \right)(z_s) ds, \tag{2.3}$$

where $\hat{b}$ is a 1-form corresponding to b via the natural identification between TM and T^*M.

2.2 The law of large numbers and the central limit theorem

For each $\alpha \in \mathscr{D}_{1,p}$, $e(\alpha)$ is given as follows:

$$e(\alpha) = \int_M \left((\hat{b}, \alpha) - \frac{1}{2}\delta\alpha \right) dm. \tag{2.4}$$

Note that the mapping $e : \alpha \mapsto e(\alpha)$ belongs to $\mathscr{D}_{1,-p}$ when $p > d/2 + 1$. Then the law of large numbers is given as follows.

Theorem 2.2 (the law of large numbers [14, 23]) *For sufficiently large p, we have*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} X_\lambda = \lim_{\lambda\to\infty} \frac{1}{\lambda} A_\lambda = e, \quad \lim_{\lambda\to\infty} \frac{1}{\lambda} Y_\lambda = 0 \quad a.s.\ in\ \mathscr{D}_{1,-p}.$$

Next, we state the central limit theorem. To describe the limit law, we give the definition of $\mathscr{D}_{1,-\infty}$-valued Wiener processes following Itô [18].

Definition 2.3 ($\mathscr{D}_{1,-\infty}$-valued Wiener processes) A continuous $\mathscr{D}_{1,-\infty}$-valued process $\{W_t\}_{t\geq 0}$ with stationary independent increments and $W_0 = 0$ is called a $\mathscr{D}_{1,-\infty}$-valued Wiener process. It is characterized by its mean functional ζ and covariance functional σ given by

$$\begin{aligned} \zeta(\alpha) &= \mathbb{E}[W_1(\alpha)], \\ \sigma(\alpha^1, \alpha^2) &= \mathbb{E}[(W_1(\alpha^1) - \langle \zeta, \alpha^1 \rangle_{\mathscr{D}_{1,\infty}})(W_1(\alpha^2) - \langle \zeta, \alpha^2 \rangle_{\mathscr{D}_{1,\infty}})]. \end{aligned}$$

We consider the differential equation

$$\left(\frac{1}{2}\Delta + b \right) u = (\hat{b}, \alpha) - \frac{1}{2}\delta\alpha - e(\alpha) \tag{2.5}$$

for each $\alpha \in \mathscr{D}_{1,p}$. The equation (2.5) has a unique solution up to an additive constant (see [13], for example). We denote it by u_α and set $Q\alpha = du_\alpha$. Then $Q : \mathscr{D}_{1,p} \to \mathscr{D}_{1,p}$ becomes a continuous linear idempotent operator. Let Q^* be the adjoint operator on $\mathscr{D}_{1,-p}$. We use the same symbol Q^* for the operator naturally extended on $\mathscr{C}_p$.

For $\lambda > 0$, let us define scaled processes X^λ, Y^λ and A^λ in the following:

$$X_t^\lambda(\alpha) := \frac{1}{\sqrt{\lambda}} \left(X_{\lambda t}(\alpha) - \lambda t\, e(\alpha) \right), \quad Y_t^\lambda(\alpha) := \frac{1}{\sqrt{\lambda}} Y_{\lambda t}(\alpha), \quad A_t^\lambda(\alpha) := X_t^\lambda(\alpha) - Y_t^\lambda(\alpha).$$

Then, the central limit theorem is known in the following sense.

Theorem 2.4 (the central limit theorem [14, 16])

(i) *Take $p > d$. If λ tends to ∞, the probability law of Y^λ on $\mathscr{C}_p$ converges weakly to that of the $\mathscr{D}_{1,-\infty}$-valued Wiener process W^1 with the mean functional $\zeta^1 = 0$ and the covariance functional*

$$\sigma^1(\alpha^1, \alpha^2) = \left(\alpha^1, \alpha^2\right)_{L^2_1(dm)}.$$

(ii) *Take $p > d + 1$. If λ tends to ∞, the law of X^λ converges weakly to that of the $\mathscr{D}_{1,-\infty}$-valued Wiener process W^2 with the mean functional $\zeta^2 = 0$ and the covariance functional*

$$\sigma^2(\alpha^1, \alpha^2) = \left((1-Q)\alpha^1, (1-Q)\alpha^2\right)_{L^2_1(dm)}.$$

(iii) *Take $p > d+1$. If λ tends to ∞, the law of A^λ converges weakly to that of the $\mathscr{D}_{1,-\infty}$-valued Wiener process W^3 with the mean functional $\zeta^3 = 0$ and the covariance functional*

$$\sigma^3(\alpha^1, \alpha^2) = \left(Q\alpha^1, Q\alpha^2\right)_{L^2_1(dm)}.$$

Note that we can take W^1, W^2 and W^3 as $\mathscr{C}_p$-valued random variable for each corresponding p in a similar way as Proposition 2.1.

2.3 Statement of main results

Take a scaling parameter $g(\lambda) > 0$ with $\lim_{\lambda\to\infty} g(\lambda) = \infty$. Let us define $\{\tilde{Y}^\lambda\}_{\lambda>0}$ by $\tilde{Y}^\lambda_t := g(\lambda)^{-1} Y^\lambda_t$. Our main theorems assert the large deviations for $\tilde{Y}^\lambda$. First we state the moderate deviation, or the case $g(\lambda) = o(\sqrt{\lambda})$ as $\lambda \to \infty$, for $\{\tilde{Y}^\lambda\}_{\lambda>0}$. We provide the rate function in the following definition.

Definition 2.5 Let $L_1 : \mathscr{D}_{1,-p} \to [0, \infty]$ be given by

$$L_1(\omega) = \begin{cases} \dfrac{1}{2}\displaystyle\int_M |\check{\omega}|^2 dm & \text{if } \omega \in \mathscr{H}', \\ \infty & \text{otherwise,} \end{cases} \tag{2.6}$$

where $\omega \in \mathscr{H}'$ if and only if there exists $\check{\omega} \in L^2_1(dm)$ so that $\langle \omega, \alpha\rangle = \int_M (\check{\omega}, \alpha) dm$ for each $\alpha \in \mathscr{D}_{1,p}$. For $\omega \in \mathscr{C}_p$, let us define a rate function L by

$$L(w) = \begin{cases} \displaystyle\int_0^\infty L_1(\dot{w}_t) dt & \text{if } w_t = \displaystyle\int_0^t \dot{w}_s ds \text{ with } \dot{w}_s \in \mathscr{D}_{1,-p} \text{ for a.e.} s\ , \\ \infty & \text{otherwise,} \end{cases} \tag{2.7}$$

where the integral $\int_0^t \dot{w}_s ds$ is the Bochner integral.

Theorem 2.6 *Suppose $p > d$. We assume $g(\lambda) = o(\sqrt{\lambda})$ as $\lambda \to \infty$. Then, $\{\tilde{Y}^\lambda\}_{\lambda>0}$ satisfies the large deviation principle in $\mathscr{C}_p$ with speed $g(\lambda)^2$ and the convex good rate function L uniformly in initial point $x \in M$. That is, for any Borel sets $\mathscr{A} \subset \mathscr{C}_p$,*

$$\begin{aligned}
\limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log\left(\sup_{x\in M} \mathbb{P}_x[\tilde{Y}^\lambda \in \mathscr{A}]\right) &\leq -\inf_{w\in\bar{\mathscr{A}}} L(w),\\
\liminf_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log\left(\inf_{x\in M} \mathbb{P}_x[\tilde{Y}^\lambda \in \mathscr{A}]\right) &\geq -\inf_{w\in\mathscr{A}^\circ} L(w),
\end{aligned}$$

where $\bar{\mathscr{A}}$ (resp. $\mathscr{A}^\circ$) is the closure of $\mathscr{A}$ (resp. the interior of $\mathscr{A}$) in $\mathscr{C}_p$.

Our next result is the large deviation for $\tilde{Y}^\lambda$ when $g(\lambda) = \sqrt{\lambda}$. In this case, we use the symbol $\bar{Y}^\lambda$ instead of $\tilde{Y}^\lambda$, namely,

$$\bar{Y}^\lambda_t := \frac{1}{\sqrt{\lambda}} Y^\lambda_t = \frac{1}{\lambda} Y_{\lambda t}.$$

In order to state the theorem, we prepare some notations. Let $(\mathscr{E}, \mathrm{Dom}(\mathscr{E}))$ be a Dirichlet form given by the closure of the following pre-Dirichlet form $(\tilde{\mathscr{E}}, C^\infty(M))$:

$$\tilde{\mathscr{E}}(f,g) = \int_M (df, dg)\, dv.$$

When $f = g$, the symbol $\mathscr{E}(f)$ indicates $\mathscr{E}(f,f)$. Let $\mathscr{W}$ be the set of all functions f in $\mathrm{Dom}(\mathscr{E})$ such that $f \geq 0$ a.e. and $\int_M f^2\, dv = 1$. Let $\mathscr{M}_1$ be the totality of Borel probability measures on M. For $\omega \in \mathscr{D}_{1,-p}$, we say $\omega \in \tilde{\Omega}$ if and only if there is $\mu^\omega \in \mathscr{M}_1$ so that

$$\langle \omega, du \rangle + \int_M \left(\frac{1}{2}\Delta + b\right) u\, d\mu^\omega = 0 \tag{2.8}$$

for all $u \in C^\infty(M)$. As we will show in section 3.3.3, such $\mu^\omega \in \mathscr{M}_1$ is unique for each given $\omega \in \tilde{\Omega}$. For $\omega \in \mathscr{D}_{1,-p}$, we say $\omega \in \Omega$ if and only if $\omega \in \tilde{\Omega}$ and $d\mu^\omega = f^2 dv$ for some $f \in \mathscr{W}$. We define a map $\chi : \Omega \to \mathscr{W}$ by $\chi(\omega) = f$.

Definition 2.7 Let us define a functional I_1 on $\mathscr{D}_{1,-p}$ by

$$I_1(\omega) = \begin{cases} \dfrac{1}{2}\displaystyle\int_M |\hat{\omega}|^2\, d\mu^\omega & \text{if } \omega \in \mathscr{H}, \\ \infty & \text{otherwise,} \end{cases} \tag{2.9}$$

where $\omega \in \mathscr{H}$ if and only if $\omega \in \Omega$ and there is $\hat{\omega} \in L^2_1(d\mu^\omega)$ so that

$$\langle \omega, \alpha \rangle = \int_M (\hat{\omega}, \alpha) d\mu^\omega \tag{2.10}$$

for all $\alpha \in \mathscr{D}_{1,p}$.

Also let us define $I : \mathscr{C}_p \to [0,\infty]$ by

$$I(w) = \begin{cases} \displaystyle\int_0^\infty I_1(\dot{w}_t) dt & \text{if } w_t = \displaystyle\int_0^t \dot{w}_s ds \text{ with } \dot{w}_s \in \mathscr{D}_{1,-p} \text{ for a.e.} s, \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 2.8 *Suppose $p > d$. Then the large deviation principle for $\{\bar{Y}^\lambda\}_{\lambda>0}$ holds as $\lambda \to \infty$ in $\mathscr{C}_p$ uniformly in $x \in M$ with speed λ and the convex good rate function $I : \mathscr{C}_p \to [0, \infty]$. That is, for each Borel set $\mathscr{A} \subset \mathscr{C}_p$, we have*

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \left(\sup_{x\in M} \mathbb{P}_x \left[\bar{Y}^\lambda \in \mathscr{A}\right] \right) \le - \inf_{w\in\bar{\mathscr{A}}} I(w),$$
$$\liminf_{\lambda\to\infty} \frac{1}{\lambda} \log \left(\inf_{x\in M} \mathbb{P}_x \left[\bar{Y}^\lambda \in \mathscr{A}\right] \right) \ge - \inf_{w\in\mathscr{A}^\circ} I(w).$$

We remark that the large deviations for X and A will be shown in section 4 (Theorem 4.1) as corollaries of Theorem 2.6 and Theorem 2.8.

Remark 2.9 We would like to take p as small as possible since the estimate becomes more precise as p becomes smaller. Indeed, if we take $p' < p$, the $\mathscr{C}_{p'}$-topology is stronger than the $\mathscr{C}_p$-topology. Note that the lower bound of p in Theorem 2.6 or Theorem 2.8 essentially comes from the assumption of Proposition 2.1.

On the other hand, as a direct consequence of Theorem 2.6 or Theorem 2.8, we obtain the large deviation estimates subordinate to the compact uniform topology of $C([0,\infty) \to \mathscr{D}_{1,-\infty})$ since the topology of $C([0,\infty) \to \mathscr{D}_{1,-\infty})$ is weaker than that of $\mathscr{C}_p$.

Remark 2.10 In almost the same way as Theorem 2.6, we can prove the large deviation principle for the current-valued Wiener processes $\{g(\lambda)^{-1}W^1\}_{\lambda>0}$, where W^1 appears in Theorem 2.4. Then its rate function coincides with L.

It is intuitively obvious that the coincidence occurs when the decay parameter g of $\tilde{Y}^\lambda$ increases much slower than λ, which causes the weak convergence. However, if we tried to prove this fact as a consequence of Theorem 2.4, we needed to investigate the precise rate of the weak convergence. Thus Theorem 2.6 asserts that when $g(\lambda) = o(\sqrt{\lambda})$ as $\lambda \to \infty$ the growth of g is sufficiently slow to cause the coincidence of rate functions. As we stated, when $\sqrt{\lambda} = g(\lambda)$, $\tilde{Y}^\lambda$ satisfies the large deviation with another rate function I. As is seen in section 4, I is actually different from L even when M is a flat torus.

3 Proof of main theorems

Throughout this section, we fix $p > d$.

3.1 Basic lemmas

First we prepare some lemmas which play an important role in this section.

Lemma 3.1 *We can take p_0 and q with $0 < q < p_0 < p$ which satisfy the following:*

(i) *Y takes its values in $\mathscr{C}_{p_0}$,*

(ii) *there exists $C > 0$ such that $\sup_{x\in M} |\alpha|(x) \le C\|\alpha\|_q$ for all $\alpha \in \mathscr{D}_{1,\infty}$,*

(iii) *the canonical embedding $\mathscr{D}_{1,-p_0} \to \mathscr{D}_{1,-p}$ is compact,*

(iv) *there is an orthonormal basis* $\{\alpha_n\}_{n=1}^{\infty}$ *of* $\mathscr{D}_{1,p_0}$ *so that* $\sum_{n=1}^{\infty} n^{\gamma}\|\alpha_n\|_q^2 < \infty$ *holds for some* $\gamma > 0$.

Proof. By Proposition 2.1, (i) follows for $p_0 > d$. The Sobolev lemma on compact manifolds implies that if $q > d/2$ then (ii) holds. From the compactness of the power of the resolvent operator $(1-\Delta_1)^{p_0-p}$, (iii) holds for any $p > p_0$.

As for (iv), let λ_n be the n-th eigenvalue of $-\Delta_1$ and $\tilde{\alpha}_n \in \mathscr{D}_{1,\infty}$ an eigenform corresponding to λ_n which makes a complete orthonormal system of $L_1^2(dv)$. Then for $\alpha \in \mathscr{D}_{1,\infty}$ and $r > 0$,

$$\|\alpha\|_r = \left\{ \sum_{n=1}^{\infty} (1+\lambda_n)^r \, (\alpha, \tilde{\alpha}_n)^2_{L_1^2(dv)} \right\}^{1/2}.$$

Thus $\alpha_n = (1+\lambda_n)^{-p_0}\tilde{\alpha}_n$ forms a complete orthonormal system of $\mathscr{D}_{1,p_0}$. The Weyl asymptotic formula (see [10]) implies that $\lambda_n^{d/2}/n$ is bounded as $n \to \infty$. Thus, if we take $p_0 - q > d/2$, which is consistent with the condition for (i)-(iii), then (iv) holds. $\langle$**q.e.d.**$\rangle$

Lemma 3.2 *There is a constant* $C_1 > 0$ *such that*

$$\sup_{x\in M} \mathbb{P}_x \left[\sup_{0\le s\le t} \|\tilde{Y}_s^{\lambda}\|_{-p_0} \ge \rho \right] \le C_1 \exp\left(-\frac{g(\lambda)^2\rho^2}{C_1 t} \right), \quad t > 0.$$

Proof. For each $\alpha \in \mathscr{D}_{1,\infty}$, the martingale representation theorem implies $Y_t(\alpha) = B_{\langle Y(\alpha)\rangle_t}$ where $B.$ is a Brownian motion on $\mathbb{R}$. Thus, by using (2.2) and (ii) of Lemma 3.1, we have for any $\rho > 0$,

$$\begin{aligned}
\mathbb{P}\left[\sup_{0\le s\le t} |\tilde{Y}_s^{\lambda}(\alpha)| > \rho \right] &= \mathbb{P}\left[\sup_{0\le s\le t} |B_{\langle \tilde{Y}^{\lambda}(\alpha)\rangle_s}| > \rho \right] \\
&\le \mathbb{P}\left[\sup_{0\le s\le C^2\|\alpha\|_q^2 t/g(\lambda)^2} |B_s| > \rho \right] \le 2\exp\left(-\frac{g(\lambda)^2\rho^2}{2C^2\|\alpha\|_q^2 t} \right).
\end{aligned}$$

The last inequality follows from the exponential inequality of the Brownian motion. Take $N := \sum_{n=1}^{\infty} n^{\gamma}\|\alpha_n\|_q^2 < \infty$ and $b_n := n^{\gamma}\|\alpha_n\|_q^2/N$, where γ, q and $\{\alpha_n\}_{n\in\mathbb{N}}$ are as in (iv) of Lemma 3.1. Letting $r = g(\lambda)^2\rho^2/2C^2Nt$, we have

$$\begin{aligned}
\mathbb{P}_x\left[\sup_{0\le s\le t} \|\tilde{Y}_s^{\lambda}\|_{-p_0} > \rho \right] &= \mathbb{P}_x\left[\sup_{0\le s\le t} \left(\sum_{n=1}^{\infty} \tilde{Y}_s^{\lambda}(\alpha_n)^2 \right) > \rho^2 \right] \\
&\le \sum_{n=1}^{\infty} \mathbb{P}_x\left[\sup_{0\le s\le t} \tilde{Y}_s^{\lambda}(\alpha_n)^2 > \rho^2 b_n \right] \\
&\le 2\sum_{n=1}^{\infty} \exp\left(-\frac{g(\lambda)^2\rho^2 b_n}{2C^2\|\alpha_n\|_q^2 t} \right) \\
&= 2\sum_{n=1}^{\infty} \mathrm{e}^{-rn^{\gamma}} \\
&\le 2\mathrm{e}^{-r} + 2\int_1^{\infty} \mathrm{e}^{-rs^{\gamma}}\, ds \\
&= 2\mathrm{e}^{-r}\left(1 + \frac{1}{\gamma r}\int_0^{\infty} \left(\frac{s}{r} + 1 \right)^{(1-\gamma)/\gamma} \mathrm{e}^{-s} ds \right).
\end{aligned}$$

Since the right-hand side of the last equality is independent of the choice of x, we obtain the desired result for large r. As for small r, we can take C_1 large enough to obtain desired estimate since the left-hand side of the stated inequality is less than 1. $\langle$**q.e.d.**$\rangle$

Here we give a remark about the uniformity of our large deviations in initial data.

Remark 3.3 The large deviation principle we want to show is uniform in initial data. Though most theorems concerning the existence of the large deviation usually take no account of such uniformity, we can extend them by adding a uniformity assumption. For example, we will give the notion of uniform exponential tightness in the next definition. It is well-known that the usual exponential tightness and the weak large deviation imply the full large deviation (see Lemma 1.2.18 of [4]). In the same way, the uniform exponential tightness and the uniform weak large deviation imply the uniform full large deviation. In this paper, we use such an extension many times. Since we can prove these extended theorems along the same line as usual one, we omit proofs.

Definition 3.4 (Uniform exponential tightness) Let $\mathscr{X}$ be a Hausdorff topological space. Let $\{\mu_{\lambda,\gamma}\}_{\lambda>0,\gamma\in\Gamma}$ be a 2-parameter family of Borel probability measures on $\mathscr{X}$. Then, $\{\mu_{\lambda,\gamma}\}_{\lambda>0}$ is said to be exponentially tight with speed $g(\lambda)^2$ uniformly in $\gamma \in \Gamma$ if for each $R > 0$ there exists a compact set $\mathscr{K} \subset \mathscr{X}$ such that

$$\limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log\left(\sup_{\gamma\in\Gamma} \mu_{\lambda,\gamma}(\mathscr{K}^c)\right) \le -R.$$

By using Lemma 3.2, we shall prove the uniform exponential tightness.

Proposition 3.5 *Under $\mathbb{P}_x$, $\{\tilde{Y}^\lambda\}_{\lambda>0}$ is exponentially tight in $\mathscr{C}_p$ with speed $g(\lambda)^2$ uniformly in $x \in M$.*

Proof. Given $N \in \mathbb{N}$ and a sequence $\{c_k\}_{k\in\mathbb{N}}$ decreasing to 0, we set

$$D_k := \left\{ w \in \mathscr{C}_{p_0} \; ; \; \sup_{\substack{|t-s|\le c_k \\ 0\le t,s\le k}} \|w_t - w_s\|_{-p_0} \le \frac{1}{k},\ w_0 = 0 \right\}, \quad k \in \mathbb{N}$$

and $D = \bigcap_{k=N}^\infty D_k$. For each $T > 0$, let $\iota_T : \mathscr{C}_{p_0} \to \mathscr{C}^T_{p_0} := C([0,T] \to \mathscr{D}_{1,-p_0})$ be the canonical restriction. Then $\iota_T(D)$ is uniformly bounded, equicontinuous family of functions in $\mathscr{C}^T_{p_0}$. Thus, if we take arbitrary $\varepsilon > 0$, then there exists $\eta > 0$ with $\eta^{-1}T \in \mathbb{N}$ such that if $|t-s| < \eta$ then $\|w_t - w_s\|_{-p_0} < \varepsilon$ for all $w \in \iota_T(D)$. Let $h := \sup\{\|w_s\|_{-p_0} \; ; \; w \in D, s \in [0,T]\}$ and $B_h := \{\xi \in \mathscr{D}_{1,-p_0} \; ; \; \|\xi\|_{-p_0} \le h\}$. Then, by (iii) of Lemma 3.1, there exists a finite set $\{\xi_1, \ldots, \xi_n\} \subset B_h$ such that for each $\xi \in B_h$ there is $j \in \{1, \ldots, n\}$ which satisfies $\|\xi_j - \xi\|_{-p} < \varepsilon$. Then we can easily show that there is a constant C_2 such that a family

$$\bigcup_{\substack{i_\ell\in\{1,\ldots,n\} \\ \ell=0,\ldots,\eta^{-1}T}} \left\{ w_t = (1-\frac{\tau}{\eta})\xi_{i_k} + \frac{\tau}{\eta}\xi_{i_{k+1}} \quad \begin{matrix} \text{if } t = k\eta + \tau,\ \tau \in [0,\eta), \\ k = 0, \ldots, \eta^{-1}T - 1 \end{matrix} \right\}$$

forms $C_2\varepsilon$-net of $\iota_T(D)$ in $\mathscr{C}^T_p$.

Consequently, $\iota_T(D)$ is precompact in $\mathscr{C}_p^T$ and therefore so is D in $\mathscr{C}_p$ by diagonal method. Thus it suffices to show

$$\limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\sup_{x\in M} \mathbb{P}_x[\tilde{Y}^\lambda \in D^c] \right) \le -R$$

for each $R > 0$ by taking N and c_k suitably.

Let $E_{t,k} := \{ w \in \mathscr{C}_{p_0} \; ; \; \sup_{t\le s\le t+c_k} \|w_s - w_t\|_{-p_0} > (3k)^{-1} \}$. Then we have $D_k^c \subset \bigcup_{\ell\in\mathbb{Z}, 0\le \ell < c_k^{-1}k} E_{\ell c_k, k}$. Thus, the Markov property and Lemma 3.2 imply

$$\sup_{x\in M} \mathbb{P}_x[\tilde{Y}^\lambda \in E_{\ell c_k, k}] = \sup_{x\in M} \mathbb{P}_x[\mathbb{P}_{z_s}[\tilde{Y}^\lambda \in E_{0,k}]] \le C_1 \; \exp\left(-\frac{g(\lambda)^2}{9C_1 k^2 c_k} \right).$$

Therefore, we have

$$\sup_{x\in M} \mathbb{P}_x[\tilde{Y}^\lambda \in D_k^c] \le \frac{kC_1}{c_k} \exp\left(-\frac{g(\lambda)^2}{9C_1 k^2 c_k} \right).$$

Now we take $c_k = k^{-3}$ and fix N sufficiently large such that for all $k \ge N$

$$\exp\left(-\frac{g(\lambda)^2 k}{9C_1} \right) \le \frac{1}{k^2} \exp\left(-\frac{g(\lambda)^2 k}{18C_1} \right) \le \frac{1}{k^2} \exp\left(-Rg(\lambda)^2 \right)$$

for sufficiently large λ. Consequently,

$$\begin{aligned} \limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\sup_{x\in M} \mathbb{P}_x[\tilde{Y}^\lambda \in D^c] \right) &\le \limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\sum_{k=N}^{\infty} \sup_{x\in M} \mathbb{P}_x[\tilde{Y}^\lambda \in D_k^c] \right) \\ &\le -R. \end{aligned}$$

$\langle$**q.e.d.**$\rangle$

It should be noted that we need no assumption on the divergence rate of $g(\lambda)$ for the proof of Lemma 3.2 and Proposition 3.5.

3.2 Proof of Theorem 2.6

For the proof of Theorem 2.6, we shall use the following theorem due to Baldi [2, 4].

Theorem 3.6 ([2, 4]) *Let $\{\mu_\lambda\}_{\lambda>0}$ be a family of probability measures on a topological vector space $\mathscr{V}$. We denote by $\mathscr{V}'$ the dual topological vector space of $\mathscr{V}$. Take an increasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\lambda\to\infty} g(\lambda) = +\infty$ and assume the following properties:*

(i) *There exists*

$$\Lambda(\beta) := \lim_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\int_{\mathscr{V}} \exp\left(g(\lambda)^2 \,{}_{\mathscr{V}'}\langle \beta, x\rangle_{\mathscr{V}} \right) d\mu_\lambda(x) \right)$$

for each $\beta \in \mathscr{V}'$ and it is finite in some neighborhood of 0.

(ii) $\{\mu_\lambda\}_{\lambda>0}$ *is exponentially tight.*

(iii) *Let $\mathscr{L}$ be the Legendre transform of Λ, given by*

$$\mathscr{L}(x) = \sup_{\beta \in \mathscr{V}'} \left({}_{\mathscr{V}'}\langle \beta, x \rangle_{\mathscr{V}} - \Lambda(\beta) \right).$$

The totality of points where $\mathscr{L}$ is strictly convex is denoted by $\mathscr{F}$. Namely, $x \in \mathscr{V}$ is an element of $\mathscr{F}$ if and only if there exists $\alpha = \alpha(x) \in \mathscr{V}'$ so that

$$\mathscr{L}(y) > \mathscr{L}(x) + {}_{\mathscr{V}'}\langle \alpha, y - x \rangle_{\mathscr{V}}$$

holds for all $y \neq x$. Then, $\inf_{x \in G \cap \mathscr{F}} \mathscr{L}(x) = \inf_{x \in G} \mathscr{L}(x)$ holds for any open sets G.

Then, for every Borel set $\mathscr{A} \subset \mathscr{V}$,

$$\begin{aligned} \limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \mu_\lambda(\mathscr{A}) &\leq -\inf_{x \in \bar{\mathscr{A}}} \mathscr{L}(x), \\ \liminf_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \mu_\lambda(\mathscr{A}) &\geq -\inf_{x \in \mathscr{A}^\circ} \mathscr{L}(x). \end{aligned}$$

We will apply Theorem 3.6 to the case that μ_λ is the law of $\tilde{Y}^\lambda$ under $\mathbb{P}_x$ and $\mathscr{V} = \mathscr{C}_p$. To obtain the uniform large deviation in initial distribution, we need to verify that the assumption of Theorem 3.6 holds uniformly in $x \in M$ (cf. Remark 3.3).

Let us define a functional $H : \mathscr{C}_p' \to \mathbb{R}$ as follows:

$$H(\mu) := \lim_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda \rangle_{\mathscr{C}_p} \right) \right] \right),$$

where $\langle \cdot, \cdot \rangle_{\mathscr{C}_p}$ means the dual pairing between $\mathscr{C}_p'$ and $\mathscr{C}_p$. We shall calculate $H(\mu)$. First, we treat the case $\mu = \alpha\delta_t$, or μ is a vector-valued Dirac measure for some $\alpha \in \mathscr{D}_{1,p}$ and $t \in [0, \infty)$. That is, $\langle \mu, w \rangle_{\mathscr{C}_p} = \langle w_t, \alpha \rangle$ holds for any $w \in \mathscr{C}_p$. The case $t = 0$ is trivial, we assume $t > 0$.

Let $\Xi_t = t^{-1} \int_0^t \delta_{z_s} ds$ be the mean empirical law of $z_\cdot$. We can regard $\{\Xi_t\}_{t>0}$ as probability measures on M. Then, by (2.2), $\langle Y^\lambda(\alpha) \rangle_t = t \int_M |\alpha|^2 d\Xi_{\lambda t}$ holds. In order to estimate the asymptotic behavior of the quadratic variation, we shall use the upper estimate of the following Donsker-Varadhan large deviation principle.

Lemma 3.7 ([7]) *For any Borel sets $\mathscr{A}$ in $\mathscr{M}_1$,*

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda t} \log \left(\sup_{x \in M} \mathbb{P}_x[\Xi_{\lambda t} \in \mathscr{A}] \right) \leq -\inf_{\nu \in \bar{\mathscr{A}}} J(\nu).$$

Note that the good rate function J attains its minimum 0 only at m.

By using this lemma, we shall prove the following.

Proposition 3.8 *If $\mu = \alpha\delta_t$ for some $\alpha \in \mathscr{D}_{1,p}$ and $t > 0$, then we have*

$$\begin{aligned} H(\mu) &= \lim_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\sup_{x \in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda \rangle_{\mathscr{C}_p} \right) \right] \right) \\ &= \lim_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\inf_{x \in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda \rangle_{\mathscr{C}_p} \right) \right] \right) = \frac{1}{2} \|\alpha\|_{L_1^2(dm)}^2 t. \end{aligned}$$

Proof. Take $\varepsilon > 0$. We set $\mathscr{A}_{\varepsilon,\lambda} := \left\{ \left| \langle Y^\lambda(\alpha) \rangle_t - \|\alpha\|^2_{L^2_1(dm)} t \right| < \varepsilon \right\}$ and divide the expectation into two parts, the main term and the remainder term.

As a first step, we estimate the remainder term. By using the Schwarz inequality, we have

$$\begin{aligned} \mathbb{E}\left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda \rangle_{\mathscr{C}_p}\right) \; ; \; \mathscr{A}^c_{\varepsilon,\lambda}\right] &= \mathbb{E}\left[\exp\left(g(\lambda) Y^\lambda_t(\alpha)\right) \; ; \; \mathscr{A}^c_{\varepsilon,\lambda}\right] \\ &\le \mathbb{E}\left[\exp\left(2g(\lambda) Y^\lambda_t(\alpha)\right)\right]^{1/2} \mathbb{P}[\mathscr{A}^c_{\varepsilon,\lambda}]^{1/2} \\ &\le \mathbb{E}\left[\exp\left(2g(\lambda) Y^\lambda_t(\alpha) - 2g(\lambda)^2 \langle Y^\lambda(\alpha) \rangle_t\right)\right]^{1/2} \\ &\qquad \times \exp\left(g(\lambda)^2 t C^2 \|\alpha\|^2_q\right) \mathbb{P}[\mathscr{A}^c_{\varepsilon,\lambda}]^{1/2} \\ &= \exp\left(g(\lambda)^2 t C^2 \|\alpha\|^2_q\right) \mathbb{P}[\mathscr{A}^c_{\varepsilon,\lambda}]^{1/2}. \end{aligned}$$

Lemma 3.7 implies that there exists $c > 0$ such that $\log\left(\sup_{x \in M} \mathbb{P}_x[\mathscr{A}^c_{\varepsilon,\lambda}]\right) \le -c\lambda$ for sufficiently large λ. Since $\sqrt{\lambda}/g(\lambda) \to \infty$ holds as $\lambda \to \infty$ by assumption, we have

$$\limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log\left(\sup_{x \in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda_t \rangle_{\mathscr{C}_p}\right) \; ; \; \mathscr{A}^c_{\varepsilon,\lambda}\right]\right) = -\infty. \tag{3.1}$$

Next let us turn to the estimate of the main term. Apparently we have

$$\begin{aligned} &\liminf_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log\left(\inf_{x \in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda_t \rangle_{\mathscr{C}_p}\right) \; ; \; \mathscr{A}_{\varepsilon,\lambda}\right]\right) \\ &\quad \ge \liminf_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log\left(\inf_{x \in M} \mathbb{E}_x \left[\exp\left(g(\lambda) Y^\lambda_t(\alpha) - \frac{g(\lambda)^2}{2} \langle Y^\lambda(\alpha) \rangle_t\right) \; ; \; \mathscr{A}_{\varepsilon,\lambda}\right]\right) \\ &\qquad + \frac{1}{2} \|\alpha\|^2_{L^2_1(dm)} t - \frac{\varepsilon}{2} \end{aligned} \tag{3.2}$$

and

$$\limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log\left(\sup_{x \in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda_t \rangle_{\mathscr{C}_p}\right) \; ; \; \mathscr{A}_{\varepsilon,\lambda}\right]\right) \le \frac{1}{2} \|\alpha\|^2_{L^2_1(dm)} t + \frac{\varepsilon}{2}. \tag{3.3}$$

Now the following lemma, which follows from easy calculation, gives the final touch of this estimate.

Lemma 3.9 *Suppose that $\{a(\lambda)\}_{\lambda>0} \subset \mathbb{R}_+$ and $\{a'(\lambda)\}_{\lambda>0} \subset \mathbb{R}$ satisfy the following properties:*

(i) *There exist $\eta > 0$ and $a \in \mathbb{R}$ such that $|g(\lambda)^{-2} \log a(\lambda) - a| < \eta$ holds for sufficiently large λ,*

(ii) $\lim_{\lambda\to\infty} g(\lambda)^{-2} \log |a'(\lambda)| = -\infty$.

Then, $g(\lambda)^{-2} \log(a(\lambda) + a'(\lambda)) \in (a - 2\eta, a + 2\eta)$ for sufficiently large λ.

Indeed, by applying Lemma 3.9 for $a(\lambda) \equiv 1$ and

$$a'(\lambda) = -\sup_{x\in M} \mathbb{E}_x \left[\exp\left(g(\lambda) Y_t^\lambda(\alpha) - \frac{g(\lambda)^2}{2} \langle Y^\lambda(\alpha) \rangle_t \right) \; ; \; \mathscr{A}_{\varepsilon,\lambda}^c \right],$$

we conclude that

$$\lim_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\inf_{x\in M} \mathbb{E}_x \left[\exp\left(g(\lambda) Y_t^\lambda(\alpha) - \frac{g(\lambda)^2}{2} \langle Y^\lambda(\alpha) \rangle_t \right) \; ; \; \mathscr{A}_{\varepsilon,\lambda} \right] \right) = 0$$

since we can prove $\lim_{\lambda\to\infty} g(\lambda)^{-2} \log |a'(\lambda)| = -\infty$ in the similar way as (3.1). Thus the right-hand side in (3.2) is equal to $(\|\alpha\|_{L_1^2(dm)}^2 t - \varepsilon)/2$. Finally, we use Lemma 3.9 again with

$$\begin{aligned} a(\lambda) &= \inf_{x\in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}_t^\lambda \rangle_{\mathscr{C}_p} \right) \; ; \; \mathscr{A}_{\varepsilon,\lambda} \right], \\ a'(\lambda) &= \inf_{x\in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}_t^\lambda \rangle_{\mathscr{C}_p} \right) \; ; \; \mathscr{A}_{\varepsilon,\lambda}^c \right] \end{aligned}$$

and with

$$\begin{aligned} a(\lambda) &= \sup_{x\in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}_t^\lambda \rangle_{\mathscr{C}_p} \right) \; ; \; \mathscr{A}_{\varepsilon,\lambda} \right], \\ a'(\lambda) &= \sup_{x\in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}_t^\lambda \rangle_{\mathscr{C}_p} \right) \; ; \; \mathscr{A}_{\varepsilon,\lambda}^c \right]. \end{aligned}$$

Note that $a(\lambda)$ and $a'(\lambda)$ satisfy the assumption of Lemma 3.9 in each case by (3.1), (3.2) and (3.3). Then we conclude

$$\begin{aligned} \frac{1}{2} \|\alpha\|_{L_1^2(dm)} t - 2\varepsilon &\le \liminf_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\inf_{x\in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}_t^\lambda \rangle_{\mathscr{C}_p} \right) \right] \right) \\ &\le \limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\sup_{x\in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}_t^\lambda \rangle_{\mathscr{C}_p} \right) \right] \right) \\ &\le \frac{1}{2} \|\alpha\|_{L_1^2(dm)}^2 t + 2\varepsilon. \end{aligned}$$

$\langle$**q.e.d.**$\rangle$

By using the Markov property, we obtain a similar estimate when μ is written by finite sum of vector-valued Dirac measures.

Corollary 3.10 *If $\mu = \sum_{k=1}^n \alpha_k \delta_{t_k}$ for $\alpha_k \in \mathscr{D}_{1,p}, k = 1, \ldots, n$ and $0 \le t_1 \le \cdots \le t_n$, then we have*

$$\begin{aligned} H(\mu) &= \lim_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\sup_{x\in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda \rangle_{\mathscr{C}_p} \right) \right] \right) \\ &= \lim_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \left(\inf_{x\in M} \mathbb{E}_x \left[\exp\left(g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda \rangle_{\mathscr{C}_p} \right) \right] \right) = \frac{1}{2} \int_0^\infty \|\mu_s\|_{L_1^2(dm)}^2 \, ds, \end{aligned}$$

where $\mu_s = \sum_{t_k > s} \alpha_k$.

Take $\mu \in \mathscr{C}_p'$. We show that μ is a $\mathscr{D}_{1,p}$-valued measure on $[0,\infty)$. Recall that $\mathscr{C}_p^T = C([0,T] \to \mathscr{D}_{1,-p})$ and ι_T is the canonical restriction mapping from $\mathscr{C}_p$ to $\mathscr{C}_p^T$. Since $\mathscr{C}_p$ is the projective limit of $\mathscr{C}_p^T$ as $T \to \infty$, we can regard μ as an element of $(\mathscr{C}_p^T)'$ for some T. That is, there exists $\tilde{\mu} \in \mathscr{C}_p^T$ for some T such that the equality $\langle \mu, w\rangle_{\mathscr{C}_p} = \langle \tilde{\mu}, \iota_T(w)\rangle_{\mathscr{C}_p^T}$ holds. Hence we can identify μ with $\tilde{\mu}$. We can verify the continuity of the bilinear mapping from $C([0,T] \to \mathbb{R}) \times \mathscr{D}_{1,-p}$ to $\mathscr{C}_p$ given by $(\varphi, \beta) \mapsto \varphi\beta$ for $\varphi \in C([0,T] \to \mathbb{R})$ and $\beta \in \mathscr{D}_{1,-p}$. Thus μ is considered as a continuous bilinear functional on $C([0,T] \to \mathbb{R}) \times \mathscr{D}_{1,-p}$, or a continuous linear operator from $C([0,T] \to \mathbb{R})$ to $\mathscr{D}_{1,p}$. Therefore, μ is identified with a $\mathscr{D}_{1,p}$-valued measure whose support is contained in $[0,T]$ by Theorem 2 in VI 7.2 of [8].

Let us define $\mathscr{C}_{p,\Pi}'$ by

$$\mathscr{C}_{p,\Pi}' := \left\{ \mu = \sum_{k=1}^n \alpha_k \delta_{t_k} \quad \begin{array}{c} \text{for some } n \in \mathbb{N},\ \{\alpha_k\}_{k=1}^n \subset \mathscr{D}_{1,p} \\ \text{and } \{t_k\}_{k=1}^n \subset [0,\infty) \end{array} \right\}.$$

The following lemma is a slight modification of that in the basic measure theory on $[0,\infty)$:

Lemma 3.11 *For each $\mu \in \mathscr{C}_p'$, there exists a sequence $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathscr{C}_{p,\Pi}'$ such that μ_s^n converges to μ_s in $\mathscr{D}_{1,p}$ as $n \to \infty$ uniformly in $s \in [0,\infty)$. In particular, we obtain $\lim_{n\to\infty} \sup_{x\in M, s\in[0,T]} |\mu_s^n - \mu_s|(x) = 0$ for all $T > 0$.*

The last assertion of Lemma 3.11 is a consequence of (ii) of Lemma 3.1.

Now we complete the calculation of $H(\mu)$.

Proposition 3.12

$$\begin{aligned} H(\mu) &= \lim_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log\left(\sup_{x\in M} \mathbb{E}_x \left[\exp\left\{ g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda\rangle_{\mathscr{C}_p} \right\}\right]\right) \\ &= \lim_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log\left(\inf_{x\in M} \mathbb{E}_x \left[\exp\left\{ g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda\rangle_{\mathscr{C}_p} \right\}\right]\right) = \frac{1}{2}\int_0^\infty \|\mu_s\|_{L_1^2(dm)}^2\, ds \end{aligned}$$

for all $\mu \in \mathscr{C}_p'$, where $\mu_s = \mu((s,\infty)) \in \mathscr{D}_{1,p}$.

Proof. First we remark on some upper estimate. That is,

$$\frac{1}{g(\lambda)^2} \log\left(\sup_{x\in M} \mathbb{E}_x \left[\exp\left\{ g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda\rangle_{\mathscr{C}_p} \right\}\right]\right) \le \tilde{H}(\mu), \tag{3.4}$$

where

$$\tilde{H}(\mu) = \frac{1}{2}\int_0^\infty \sup_{x\in M} |\mu_s|(x)^2 ds.$$

In order to obtain (3.4), take the orthonormal basis $\{\alpha_n\}_{n\in\mathbb{N}}$ of $\mathscr{D}_{1,p_0}$ given in (iv) of Lemma 3.1 and its dual basis $\{\beta_n\}_{n\in\mathbb{N}}$ in $\mathscr{D}_{1,-p_0}$. Then, the integration-by-parts formula for semimartingales implies

$$\begin{aligned} \langle \mu, Y\rangle_{\mathscr{C}_p} = \langle \mu, \sum_{n=1}^\infty Y(\alpha_n)\beta_n\rangle_{\mathscr{C}_p} &= \sum_{n=1}^\infty \int_0^\infty Y_s(\alpha_n) d\mu^{\beta_n}(s) \\ &= \sum_{n=1}^\infty \int_0^\infty \mu_s^{\beta_n} dY_s(\alpha_n), \end{aligned}$$

where μ^β is an $\mathbb{R}$-valued signed measure determined by $\mu^\beta(A) = \langle \mu(A), \beta\rangle_p$ for any Borel sets A and $\mu_s^\beta = \mu^\beta((s,\infty))$. By using it,

$$
\begin{aligned}
\mathbb{E}\left[\exp\left\{g(\lambda)^2\langle \mu, \tilde{Y}^\lambda\rangle_{\mathscr{C}_p}\right\}\right] &= \mathbb{E}\left[\exp\left\{g(\lambda)\sum_{n=1}^\infty \int_0^\infty \mu_s^{\beta_n} dY_s^\lambda(\alpha_n)\right\}\right] \\
= \mathbb{E}\Bigg[\exp\Bigg\{\frac{g(\lambda)^2}{2}\sum_{n,k=1}^\infty \int_0^\infty &\mu_s^{\beta_n}\mu_s^{\beta_k} d\langle Y^\lambda(\alpha_n), Y^\lambda(\alpha_k)\rangle_s\Bigg\} \\
&\times \exp\Bigg\{g(\lambda)\sum_{n=1}^\infty \int_0^\infty \mu_s^{\beta_n} dY_s^\lambda(\alpha_n) \\
&\qquad - \frac{g(\lambda)^2}{2}\sum_{n,k=1}^\infty \int_0^\infty \mu_s^{\beta_n}\mu_s^{\beta_k} d\langle Y^\lambda(\alpha_n), Y^\lambda(\alpha_k)\rangle_s\Bigg\}\Bigg].
\end{aligned}
$$

Note that the infinite series $\sum_{n=1}^\infty \mu_s^{\beta_n}\alpha_n$ converges to μ_s uniformly on M and in $\mathscr{D}_{1,p_0}$ for each fixed s. Indeed, by virtue of (ii) and (iv) of Lemma 3.1, we have

$$
\begin{aligned}
\lim_{k\to\infty}\sup_{N\geq k}\sum_{n=k}^N |\mu_s^{\beta_n}| \sup_{x\in M}|\alpha_n|(x) &\leq C\lim_{k\to\infty}\sup_{N\geq k}\sum_{n=k}^N |\mu_s^{\beta_n}| \|\alpha_n\|_q \\
&\leq \lim_{k\to\infty}\left\{\sum_{n=k}^\infty |\mu_s^{\beta_n}|^2\right\}^{1/2}\left\{\sum_{n=k}^\infty \|\alpha_n\|_q^2\right\}^{1/2} = 0.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\sum_{n,k=1}^\infty \int_0^\infty \mu_s^{\beta_n}\mu_s^{\beta_k} d\langle Y^\lambda(\alpha_n), Y^\lambda(\alpha_k)\rangle_s &= \int_0^\infty \sum_{n,k=1}^\infty \mu_s^{\beta_n}\mu_s^{\beta_k}(\alpha_n,\alpha_k)(z_{\lambda s})ds \\
&= \int_0^\infty |\mu_s|^2(z_{\lambda s})ds \leq \int_0^\infty \sup_{x\in M}|\mu_s|^2(x)ds.
\end{aligned}
$$

Thus we conclude

$$
\begin{aligned}
\mathbb{E}\left[\exp\left\{g(\lambda)^2\langle \mu, \tilde{Y}^\lambda\rangle_{\mathscr{C}_p}\right\}\right] &\leq \exp\left\{\frac{g(\lambda)^2}{2}\int_0^\infty \sup_{x\in M}|\mu_s(x)|^2 ds\right\} \\
&= \exp\left\{g(\lambda)^2\tilde{H}(\mu)\right\}.
\end{aligned}
$$

Take $\hat{\mu} \in \mathscr{C}'_\Pi$. For $a_1 > 1$ and $a_2 > 1$ with $a_1^{-1} + a_2^{-1} = 1$, the Hölder inequality and (3.4) imply

$$
\begin{aligned}
&\frac{1}{g(\lambda)^2}\log\left(\sup_{x\in M}\mathbb{E}_x\left[\exp\left\{g(\lambda)^2\langle\mu,\tilde{Y}^\lambda\rangle_{\mathscr{C}_p}\right\}\right]\right) \\
&\quad\leq \frac{1}{a_1 g(\lambda)^2}\log\left(\sup_{x\in M}\mathbb{E}_x\left[\exp\left\{a_1 g(\lambda)\langle\mu-\hat{\mu}, Y^\lambda\rangle_{\mathscr{C}_p}\right\}\right]\right) \\
&\qquad\qquad + \frac{1}{a_2 g(\lambda)^2}\log\left(\sup_{x\in M}\mathbb{E}_x\left[\exp\left\{a_2 g(\lambda)\langle\hat{\mu}, Y^\lambda\rangle_{\mathscr{C}_p}\right\}\right]\right) \\
&\quad\leq a_1\tilde{H}(\mu-\hat{\mu}) + a_2\frac{1}{(a_2 g(\lambda))^2}\log\left(\sup_{x\in M}\mathbb{E}_x\left[\exp\left\{a_2 g(\lambda)\langle\hat{\mu}, Y^\lambda\rangle_{\mathscr{C}_p}\right\}\right]\right).
\end{aligned}
$$

Applying Corollary 3.10 by using $a_2 g$ instead of g, we have

$$\lim_{\lambda\to\infty} \frac{1}{(a_2 g(\lambda))^2} \log\left(\sup_{x\in M} \mathbb{E}_x \left[\exp\left\{ a_2 g(\lambda) \langle \hat{\mu}, Y^\lambda \rangle_{\mathscr{C}_p} \right\} \right] \right) = H(\hat{\mu}).$$

Note that

$$\sup_{s\in[0,\infty)} \|\mu_s\|_{L^2_1(dm)} \le C \sup_{s\in[0,\infty)} \|\mu_s\|_p < \infty$$

holds and the support of μ_s is compact. Hence, approximating μ by $\hat{\mu}$, we conclude that Lemma 3.11 implies that $\tilde{H}(\mu - \hat{\mu})$ tends to 0 and $H(\hat{\mu})$ tends to $H(\mu)$. Thus we obtain the upper bound by taking $a_2 \downarrow 1$.

As to lower bound, the estimate

$$\begin{aligned} &\frac{a_2^2}{g(\lambda)^2} \log\left(\inf_{x\in M} \mathbb{E}_x \left[\exp\left\{ \frac{g(\lambda)}{a_2} \langle \hat{\mu}, Y^\lambda \rangle_{\mathscr{C}_p} \right\} \right] \right) \\ &\qquad \le a_1 \tilde{H}(\hat{\mu} - \mu) + \frac{a_2}{g(\lambda)^2} \log\left(\inf_{x\in M} \mathbb{E}_x \left[\exp\left\{ g(\lambda) \langle \mu, Y^\lambda \rangle_{\mathscr{C}_p} \right\} \right] \right) \end{aligned}$$

given by the Hölder inequality implies the conclusion in a similar way. $\langle$**q.e.d.**$\rangle$

To complete the proof of Theorem 2.6, we need to calculate the Legendre transform of H. We call $w \in \mathscr{C}_p$ absolutely continuous when for each $\varepsilon > 0$ there exists $\rho > 0$ such that, for each partition $0 \le a_1 < b_1 \le \cdots \le a_n < b_n$ with $\sum_{i=1}^n (b_i - a_i) < \rho$, $\sum_{i=1}^n \|w_{b_i} - w_{a_i}\|_{-p} < \varepsilon$ holds. Note that, for $w \in \mathscr{C}_p$ with $w_0 = 0$, there exists a Bochner-integrable function $\dot{w} : [0,\infty) \to \mathscr{D}_{1,-p}$ so that $w_t = \int_0^t \dot{w}_s ds$ if and only if w is absolutely continuous. This fact comes from the Radon-Nikodym theorem for vector-valued measures (see [6]). Since $\mathscr{D}_{1,-p}$ is a Hilbert space, the Radon-Nikodym theorem is valid in this case.

Proposition 3.13 *Define* $H^*(w) := \sup_{\mu\in\mathscr{C}_p'} \left(\langle \mu, w \rangle_{\mathscr{C}_p} - H(\mu) \right)$. *Then* $H^* = L$ *holds.*

Proof. Recall that L is given by (2.7). First, if w is *not* absolutely continuous then $H^*(w) = \infty$ holds. Indeed, assume that there exists $\varepsilon > 0$ so that for any $\rho > 0$ we can take a partition $0 \le a_1 < b_1 \le \cdots \le a_n < b_n$ with $\sum_{i=1}^n (b_i - a_i) \le \rho$ and $\sum_{i=1}^n \|w_{b_i} - w_{a_i}\|_{-p} \ge \varepsilon$. Let us define $\mu \in \mathscr{C}_p'$ as follows:

$$\mu = \sum_{i=1}^n \frac{\theta_i}{\sqrt{\rho}} (\delta_{b_i} - \delta_{a_i}),$$

where $\theta_i \in \mathscr{D}_{1,p}$ is the dual element of $(w_{b_i} - w_{a_i})/\|w_{b_i} - w_{a_i}\|_{-p}$. Note that $\|\theta_i\|_{L^2_1(dm)} \le C$ holds for some constant C. Then we have

$$\begin{aligned} \int_0^\infty \|\mu_s\|^2_{L^2_1(dm)} ds &= \frac{1}{\rho} \sum_{i=1}^n (b_i - a_i) \le C, \\ \langle \mu, w \rangle_{\mathscr{C}_p} &= \frac{1}{\sqrt{\rho}} \sum_{i=1}^n \langle w_{b_i} - w_{a_i}, \theta_i \rangle \ge \frac{\varepsilon}{\sqrt{\rho}}. \end{aligned}$$

Since we can take ρ arbitrary small for fixed $\varepsilon > 0$, we conclude that $H^*(w) = \infty$. Thus we may assume that w is absolutely continuous with respect to $\|\cdot\|_{-p}$. We denote the Radon-Nikodym density of w by $\dot{w}$.

Take an orthonormal basis $\{\eta_n\}_{n\in\mathbb{N}}$ of $L^2_1(dm)$ which consists of elements in $\mathscr{D}_{1,\infty}$. Take $N > 0$ and take $\mu \in \mathscr{C}'_{p,N} := \left\{\nu \in \mathscr{C}'_p \, ; \, \nu^{\eta_n} = 0 \text{ for all } n > N\right\}$. Then we have

$$H^*(w) \geq \sup_{\mu\in\mathscr{C}'_{p,N}} (\langle \mu, w\rangle_{\mathscr{C}_p} - H(\mu)).$$

Note that the calculation of the right-hand side of the last inequality has been reduced to the classical finite-dimensional case. If $w_0 \neq 0$, there exists $n_0 \in \mathbb{N}$ which satisfies $\langle w_0, \eta_{n_0}\rangle \neq 0$. Accordingly, by taking $N > n_0$, $H^*(w) = \infty$ follows. In the case of $w_0 = 0$, we have

$$\sup_{\mu\in\mathscr{C}'_{p,N}} (\langle \mu, w\rangle_{\mathscr{C}_p} - H(\mu)) = \frac{1}{2}\sum_{n=1}^{N}\int_0^\infty |\langle \dot{w}_s, \eta_n\rangle|^2 \, ds,$$

where the right-hand side may diverge. Thus letting $N \to \infty$ we obtain

$$H^*(w) \geq \frac{1}{2}\int_0^\infty \sum_{n=1}^{\infty} |\langle \dot{w}_s, \eta_n\rangle|^2 \, ds. \tag{3.5}$$

Thus $H^*(w) = \infty$ holds if the right hand side of (3.5) diverges. Note that $\dot{w}_s \in \mathscr{H}'$ holds if and only if $\sum_{i=1}^\infty |\langle \dot{w}_s, \eta_n\rangle|^2 < \infty$ and we have $\sum_{i=1}^\infty |\langle \dot{w}_s, \eta_n\rangle|^2 = 2L_1(\dot{w}_s)$ for $\dot{w}_s \in \mathscr{H}'$. Finally, we consider the case $\int_0^\infty L_1(\dot{w}_s)ds < \infty$. For $\mu \in \mathscr{C}'_p$ whose support is contained in $[0, T]$, the integration-by-parts formula implies

$$\begin{aligned}\langle \mu, w\rangle_{\mathscr{C}_p} - H(\mu) &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\int_0^T |\langle \dot{w}_s, \eta_n\rangle|^2 \, ds - \frac{1}{2}\int_0^T |\langle \dot{w}_s, \eta_n\rangle - \mu_s^{\eta_n}|^2 \, ds\right) \\ &\leq \frac{1}{2}\int_0^T \sum_{n=1}^{\infty} |\langle \dot{w}_s, \eta_n\rangle|^2 \, ds.\end{aligned}$$

Hence, combining with (3.5), we obtain the desired result. **⟨q.e.d.⟩**

Now we call $w \in \mathscr{F}$ if and only if $L(w) < \infty$, the support of w is compact, $\dot{w}$ is absolutely continuous with respect to $\|\cdot\|_{-p}$, and the Radon-Nikodym derivative $\ddot{w}$ of $\dot{w}$ takes its values in $\mathscr{D}_{1,p}$ for a.e.s. Then L is strictly convex at w for all $w \in \mathscr{F}$. Indeed, for $w \in \mathscr{F}$, take $\mu \in \mathscr{C}_p$ which is determined by $\int_M (\mu_s, \alpha)dm = \langle \dot{w}_s, \alpha\rangle$. Then, for any $w' \in \mathscr{C}_p$ with $L(w') < \infty$, we obtain

$$L(w') - L(w) + \langle \mu, w' - w\rangle_{\mathscr{C}_p} = \int_0^\infty L_1(\dot{w}'_s - \dot{w}_s)ds \geq 0$$

by easy calculation. Obviously the equality holds if and only if $w' = w$.

In order to complete the proof of Theorem 2.6, we show $\inf_{x\in G\cap\mathscr{F}} L(x) = \inf_{x\in G} L(x)$ for all open set $G \subset \mathscr{C}_p$. It is sufficient to prove that, for all $w \in \mathscr{C}_p$ with $L(w) < \infty$, there exists a sequence $\{w^n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ such that w^n tends to w in $\mathscr{C}_p$ and $L(w^n)$ tends to $L(w)$ as n goes to ∞. We prepare the following lemma for the proof.

Lemma 3.14 *Let S be a separable Hilbert space and $\tilde{S}$ a dense subspace of S. We define a subset $\mathscr{S}$ of $L^2([0,\infty) \to S)$ such that $w \in \mathscr{S}$ if and only if $w_0 = 0$, w is absolutely continuous with respect to $\|\cdot\|_S$, the support of w is compact, and $w_s \in \tilde{S}$ for almost every s. Then $\mathscr{S}$ is dense in $L^2([0,\infty) \to S)$.*

Proof. We prove the orthogonal complement $\mathscr{S}^\perp$ of $\mathscr{S}$ is equal to $\{0\}$. Take $w \in \mathscr{S}^\perp$. For $0 < a < b < \infty$ and $\alpha \in \tilde{S}$, we take $\tilde{\eta}^n$ such that

$$\tilde{\eta}^n_t = \begin{cases} n\alpha & t \in [a - \frac{1}{n}, a), \\ 0 & t \in [0, a - \frac{1}{n}) \cup [a,b) \cup [b + \frac{1}{n}, \infty), \\ -n\alpha & t \in [b, b + \frac{1}{n}). \end{cases}$$

Then $\eta^n \in \mathscr{S}$ is defined by $\eta^n_t = \int_0^t \tilde{\eta}^n_s ds$. Thus we have $\int_0^\infty (w_s, \eta^n_s)_S \, ds = 0$, and letting $n \to \infty$, we obtain $\int_a^b (w_s, \alpha)_S \, ds = 0$. Since a and b are arbitrarily taken, $(w_s, \alpha)_S = 0$ for almost all s. Since S is separable, we can take a countable dense subset $\{\alpha_n\}_{n\in\mathbb{N}} \subset \tilde{S}$ and we conclude that $(w_s, \alpha_n)_S = 0$ holds for all n for almost all s. Hence $w = 0$. $\langle$**q.e.d.**$\rangle$

For $w, \tilde{w} \in \mathscr{C}_p$ with $L(w) < \infty$ and $L(\tilde{w}) < \infty$, we have

$$\begin{aligned} \|\tilde{w}_t - w_t\|_{-p} &\le \sup_{\substack{\alpha \in \mathscr{D}_{1,p} \\ \|\alpha\|_p \le 1}} \langle \tilde{w}_t - w_t, \alpha \rangle \\ &\le C \sup_{\substack{\alpha \in \mathscr{D}_{1,p} \\ \|\alpha\|_{L^2_1(dm)} \le 1}} \langle \tilde{w}_t - w_t, \alpha \rangle \\ &\le C \left(2L_1(\tilde{w}_t - w_t)\right)^{1/2} \\ &\le C \left(2L(\tilde{w} - w)t)\right)^{1/2}. \end{aligned} \tag{3.6}$$

Last inequality follows from Jensen's inequality since L_1 is convex. By Lemma 3.14, we can take a sequence $\{w^n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ so that $\lim_{n\to\infty} L(w^n - w) = \lim_{n\to\infty} L(w^n) - L(w) = 0$ holds. Then, combining with (3.6), w^n converges to w in $\mathscr{C}_p$. Thus we complete the proof of Theorem 2.6.

3.3 Proof of Theorem 2.8

3.3.1 Existence of the large deviation for $\bar{Y}^\lambda$

Our goal in this section is to prove the existence of the large deviation principle for $\bar{Y}^\lambda$ in Theorem 2.8. First we prove the large deviation for the law of current-valued random variables $\{\bar{Y}^\lambda_1\}_{\lambda>0}$ in $\mathscr{D}_{1,-p}$.

Proposition 3.15 *The law of $\{\bar{Y}^\lambda_1\}_{\lambda>0}$ satisfies the large deviation principle as $\lambda \to \infty$ in $\mathscr{D}_{1,-p}$ under the measure $\mathbb{P}_x$ uniformly in $x \in M$ with speed λ.*

For the proof of Proposition 3.15, we prepare the following lemma.

Lemma 3.16 *Let $B_r(\zeta)$ be a ball in $\mathscr{D}_{1,-p}$ centered at ζ with radius r. There is a constant $C_2 > 0$ so that for all $r > 0$, $\zeta \in \mathscr{D}_{1,-p}$, $x, y \in M$, $\varepsilon > 0$ and $\lambda > 1$,*

$$\mathbb{P}_x\left[\bar{Y}_1^\lambda \in B_r(\zeta)\right] \le C_2 \mathbb{P}_y\left[\bar{Y}_1^\lambda \in B_{r+\varepsilon}(\zeta)\right] + (1 + C_2)C_1 \exp\left(-\frac{\lambda^2\varepsilon^2}{4C_1}\right).$$

Proof. We have

$$\mathbb{P}_x\left[\bar{Y}_1^\lambda \in B_r(\zeta)\right] \le \mathbb{P}_x\left[\frac{1}{\lambda}(Y_\lambda - Y_1) \in B_{r+\varepsilon/2}(\zeta)\right] + \mathbb{P}_x\left[\frac{1}{\lambda}\|Y_1\|_{-p} \ge \frac{\varepsilon}{2}\right].$$

By Lemma 3.2, the following estimate holds:

$$\mathbb{P}_x\left[\frac{1}{\lambda}\|Y_1\|_{-p} \ge \frac{\varepsilon}{2}\right] \le C_1 \exp\left(-\frac{\lambda^2\varepsilon^2}{4C_1}\right). \tag{3.7}$$

Note that the transition density $p_t(x, y)$ of z_t is positive and continuous in x and y for each fixed $t > 0$ (for example, see [17]). We set $C_2 := \sup_{x,y\in M} p_1(x, y)/\inf_{x',y'\in M} p_1(x', y')$. Then, the Markov property of $\{z_t\}_{t\ge 0}$ implies

$$\begin{aligned}
\mathbb{P}_x\left[\frac{1}{\lambda}(Y_\lambda - Y_1) \in B_{r+\varepsilon/2}(\zeta)\right] &= \mathbb{E}_x\left[\mathbb{P}_{z_1}\left[\frac{1}{\lambda}Y_{\lambda-1} \in B_{r+\varepsilon/2}(\zeta)\right]\right] \\
&\le C_2\mathbb{E}_y\left[\mathbb{P}_{z_1}\left[\frac{1}{\lambda}Y_{\lambda-1} \in B_{r+\varepsilon/2}(\zeta)\right]\right] \\
&= C_2\mathbb{P}_y\left[\frac{1}{\lambda}(Y_\lambda - Y_1) \in B_{r+\varepsilon/2}(\zeta)\right] \\
&\le C_2\left(\mathbb{P}_y\left[\bar{Y}_1^\lambda \in B_{r+\varepsilon}(\zeta)\right] + \mathbb{P}_y\left[\frac{1}{\lambda}\|Y_1\|_{-p} \ge \frac{\varepsilon}{2}\right]\right).
\end{aligned}$$

Thus, the estimate (3.7) implies the conclusion. $\langle$**q.e.d.**$\rangle$

Proof of Proposition 3.15. By virtue of Lemma 3.2, the family $\{\bar{Y}_1^\lambda\}_{\lambda>0}$ is exponentially tight uniformly in initial points $x \in M$. Indeed, as a compact set, we can take a ball in $\mathscr{D}_{1,-p_0}$ for $d < p_0 < p$ centered at origin with sufficiently large radius. Thus we need only to prove the weak large deviation. By the existence theorem of the large deviation (Theorem 4.1.11 of [4], cf. Remark 3.3), it suffices to show the following:

$$\begin{aligned}
&\sup_{r>0}\left\{-\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\left(\sup_{x\in M}\mathbb{P}_x\left[\bar{Y}_1^\lambda \in B_r(\zeta)\right]\right)\right\} \\
&\qquad = \sup_{r>0}\left\{-\liminf_{\lambda\to\infty}\frac{1}{\lambda}\log\left(\inf_{x\in M}\mathbb{P}_x\left[\bar{Y}_1^\lambda \in B_r(\zeta)\right]\right)\right\} \quad \text{for all } \zeta \in \mathscr{D}_{1,-p}.
\end{aligned}$$

By using Lemma 3.2, we can prove that, for any $T > 0$,

$$\lim_{R\to\infty}\sup_{x\in M}\sup_{0\le t\le T}\mathbb{P}_x\left[\|Y_t\|_{-p} \ge R\right] = 0.$$

Then, an argument similar to the proof of Lemma 4.2.3 and Lemma 4.2.7 of [5] implies the existence of the limit

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\left(\inf_{x\in M}\mathbb{P}_x\left[\bar{Y}_1^\lambda\in B_r(\zeta)\right]\right).$$

By Lemma 3.16, we obtain

$$\begin{aligned}\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\left(\sup_{x\in M}\mathbb{P}_x\left[\bar{Y}_1^\lambda\in B_r(\zeta)\right]\right) &\le \limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\left(\inf_{y\in M}\mathbb{P}_y\left[\bar{Y}_1^\lambda\in B_{r+\varepsilon}(\zeta)\right]\right)\\ &= \liminf_{\lambda\to\infty}\frac{1}{\lambda}\log\left(\inf_{y\in M}\mathbb{P}_y\left[\bar{Y}_1^\lambda\in B_{r+\varepsilon}(\zeta)\right]\right)\end{aligned}$$

for all $\varepsilon>0$. Hence it yields Proposition 3.15. ⟨**q.e.d.**⟩

Remark 3.17 Let us define a functional $\Lambda_1^*:\mathscr{D}_{1,-p}\to[0,\infty]$ by

$$\Lambda_1^*(\omega)=\sup_{r>0}\left\{-\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\left(\inf_{x\in M}\mathbb{P}_x\left[\bar{Y}_1^\lambda\in B_r(\omega)\right]\right)\right\}.$$

Then Λ_1^* becomes the rate function which governs the large deviation for $\{\bar{Y}_1^\lambda\}_{\lambda>0}$. By the Markov property of $\{z_t\}_{t\ge0}$, Λ_1^* is convex and lower semi-continuous (cf. Lemma 4.1.7 of [5]). Moreover, Λ_1^* is good since $\{\bar{Y}_1^\lambda\}_{\lambda>0}$ is exponentially tight.

In the rest of this section, we extend the large deviation estimate for $\{\bar{Y}_1^\lambda\}_{\lambda>0}$ to that for $\{\bar{Y}^\lambda\}_{\lambda>0}$.

Proposition 3.18 *The law of $\{\bar{Y}^\lambda\}_{\lambda>0}$ satisfies the large deviation principle as $\lambda\to\infty$ in $\mathscr{C}_p$ under $\mathbb{P}_x$ uniformly in $x\in M$ with speed λ.*

Proof. By Proposition 3.5, the law of $\{\bar{Y}^\lambda\}_{\lambda>0}$ is exponentially tight in $\mathscr{C}_p$ uniformly in $x\in M$. In particular, goodness of the rate function follows once we prove the existence of the large deviation. Let $\phi_n:(\mathscr{D}_{1,-p})^{n2^n}\to\mathscr{C}_p$ be the mapping to piecewise linear functions of dyadic partitions. More precisely,

$$\phi_n(w_1,\dots,w_{n2^n})_t:=\sum_{k=0}^{n2^n-1}\left(1\wedge(2^nt-k)\vee 0\right)(w_{k+1}-w_k).$$

Let $\pi_n:\mathscr{C}_p\to(\mathscr{D}_{1,-p})^{n2^n}$ be an evaluation map given by $\pi_n(w)=\{w_{k2^{-n}}\}_{k=1}^{n2^n}$. Let ρ be a distance on $\mathscr{C}_p$ defined by

$$\rho(w,\eta):=\sum_{k=1}^{\infty}\frac{1}{2^k}\big(\sup_{0\le t\le k}\|w_t-\eta_t\|_{-p}\wedge 1\big). \tag{3.8}$$

Then, by virtue of the approximation theorem of large deviation (Theorem 4.2.16 of [4], cf. Remark 3.3) and the uniform exponential tightness, it suffices to show the uniform large deviation for $\{\phi_n\circ\pi_n(Y^\lambda)\}_{\lambda>0}$ and for each $\varepsilon>0$

$$\lim_{n\to\infty}\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\left(\sup_{x\in M}\mathbb{P}_x[\rho(\bar{Y}^\lambda,\phi_n\circ\pi_n(\bar{Y}^\lambda))>\varepsilon]\right)=-\infty. \tag{3.9}$$

Given a partition $0 = t_0 < t_1 < \cdots < t_n$, a family of $(\mathscr{D}_{1,-p})^n$-valued random variables $\{\bar{Y}^\lambda_{t_1}, \dots, \bar{Y}^\lambda_{t_n}\}_{\lambda>0}$ satisfies the large deviation principle under $\mathbb{P}_x$, uniformly in $x \in M$. It is a consequence of the Markov property of $\{z_t\}_{t\geq 0}$. Since ϕ_n is continuous, the contraction principle (Theorem 4.2.1 of [4], cf. Remark 3.3) implies that $\{\phi_n \circ \pi_n(\bar{Y}^\lambda)\}_{\lambda>0}$ satisfies the large deviation principle uniformly in $x \in M$.

Let us turn to the proof of (3.9). We take $N \in \mathbb{N}$ such that $\sum_{k=N+1}^\infty 2^{-k} < \varepsilon/2$. Then

$$\begin{aligned}\{\rho(w, \phi_n \circ \pi_n(w)) > \varepsilon\} &\subset \left\{\sum_{k=1}^N \frac{1}{2^k} \sup_{0\leq t\leq k} \|w_t - (\phi_n \circ \pi_n(w))_t\|_{-p} > \frac{\varepsilon}{2}\right\} \\ &\subset \bigcup_{k=1}^N \left\{\sup_{0\leq t\leq k} \|w_t - (\phi_n \circ \pi_n(w))_t\|_{-p} > \frac{2^{k-1}\varepsilon}{N}\right\}.\end{aligned}$$

In addition, for $n > N$ and $k = 1, \dots, N$,

$$\begin{aligned}&\sup_{0\leq t\leq k} \|w_t - (\phi_n \circ \pi_n(w))_t\|_{-p} \\ &\quad= \max_{1\leq j\leq k2^n} \sup_{0\leq t<2^{-n}} \|(w_{t+(j-1)2^{-n}} - w_{(j-1)2^{-n}}) - 2^n t(w_{j2^{-n}} - w_{(j-1)2^{-n}})\|_{-p} \\ &\quad= \max_{1\leq j\leq k2^n} \sup_{0\leq t<2^{-n}} \|2^n t(w_{t+(j-1)2^{-n}} - w_{j2^{-n}}) + (1-2^n t)(w_{t+(j-1)2^{-n}} - w_{(j-1)2^{-n}})\|_{-p} \\ &\quad\leq \max_{1\leq j\leq k2^n} \sup_{s,t\in[(j-1)2^{-n},\, j2^{-n})} \|w_t - w_s\|_{-p} \\ &\quad\leq 2 \max_{1\leq j\leq k2^n} \sup_{t\in[(j-1)2^{-n},\, j2^{-n})} \|w_t - w_{(j-1)2^{-n}}\|_{-p}.\end{aligned}$$

Thus, with the aid of Lemma 3.2, we obtain

$$\begin{aligned}&\sup_{x\in M} \mathbb{P}_x\big[\rho\big(\bar{Y}^\lambda, \phi_n \circ \pi_n(\bar{Y}^\lambda)\big) > \varepsilon\big] \\ &\qquad\leq \sum_{k=1}^N \sup_{x\in M} \mathbb{P}_x\left[\sup_{0\leq t\leq k} \big\|\bar{Y}^\lambda_t - (\phi_n \circ \pi_n(\bar{Y}^\lambda))_t\big\|_{-p} > \frac{2^{k-1}\varepsilon}{N}\right] \\ &\qquad\leq \sum_{k=1}^N \sum_{j=1}^{k2^n} \sup_{x\in M} \mathbb{P}_x\left[\sup_{(j-1)2^{-n}\leq t\leq j2^{-n}} \left\|\bar{Y}^\lambda_t - \bar{Y}^\lambda_{(j-1)2^{-n}}\right\|_{-p} > \frac{2^{k-2}\varepsilon}{N}\right] \\ &\qquad= \sum_{k=1}^N k2^n \sup_{x\in M} \mathbb{P}_x\left[\sup_{0\leq t\leq 2^{-n}} \|\bar{Y}^\lambda_t\|_{-p} > \frac{2^{k-2}\varepsilon}{N}\right] \\ &\qquad\leq C_1 \sum_{k=1}^N k2^n \exp\left(-\frac{4^{k-2}\varepsilon^2}{C_1 N^2}\lambda 2^n\right).\end{aligned}$$

Consequently, for each $n > N$,

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log\left(\sup_{x\in M} \mathbb{P}_x\left[\rho\left(\bar{Y}^\lambda, \phi_n \circ \pi_n\left(\bar{Y}^\lambda\right)\right) > \varepsilon\right]\right) \leq -\frac{\varepsilon^2}{4C_1 N^2} 2^n$$

and hence the desired estimate (3.9) follows. $\langle$**q.e.d.**$\rangle$

3.3.2 Extension of a variational formula

Let us define $\Lambda_1 : \mathscr{D}_{1,p} \to \mathbb{R}$ by

$$\Lambda_1(\alpha) := \lim_{t\to\infty} \frac{1}{t} \log \left(\sup_{x\in M} \mathbb{P}_x \left[\exp\left(Y_t(\alpha)\right)\right] \right). \tag{3.10}$$

Then we have

$$\Lambda_1^*(\omega) = \sup_{\alpha\in\mathscr{D}_{1,p}} \left(\langle \omega, \alpha\rangle - \Lambda_1(\alpha)\right). \tag{3.11}$$

This is merely a consequence of Theorem 2.2.21 in [5] (cf. Remark 3.17).

Lemma 3.19 *For $\alpha \in \mathscr{D}_{1,p}$, $\Lambda_1(\alpha)$ is equal to the principal eigenvalue of the operator $\Delta/2 + b + \check{\alpha} + |\alpha|^2/2$, where $\check{\alpha}$ is a vector field corresponding to α.*

Proof. Let us define a semigroup T_t^α by $T_t^\alpha f(x) = \mathbb{E}_x[\exp(Y_t(\alpha)) f(z_t)]$ for $f \in C(M)$. Then we can extend T_t^α to L^r-semigroup for all $1 \le r \le \infty$. By (3.10), we have

$$\Lambda_1(\alpha) := \lim_{t\to\infty} \frac{1}{t} \log \left(\sup_{x\in M} \mathbb{P}_x \left[\exp\left(Y_t(\alpha) - \frac{1}{2}\langle Y(\alpha)\rangle_t + \frac{1}{2}\langle Y(\alpha)\rangle_t \right)\right]\right).$$

Since the quadratic variation $\langle Y(\alpha)\rangle_t$ is given by (2.2), the Girsanov formula and the Feynman-Kac formula imply that T_t^α is the semigroup generated by $\Delta/2 + b + \check{\alpha} + |\alpha|^2/2$. Now we have

$$\Lambda(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log \|T_t^\alpha 1\|_{L^\infty} = \lim_{t\to\infty} \frac{1}{t} \log \|T_t^\alpha\|_{L^\infty\to L^\infty}.$$

Then, by virtue of the ultracontractivity of the semigroup corresponding to $\Delta/2$, we have $\|T_1^\alpha\|_{L^2\to L^\infty} < \infty$. Thus we obtain

$$\lim_{t\to\infty} \frac{1}{t} \log \|T_t^\alpha\|_{L^\infty\to L^\infty} = \lim_{t\to\infty} \frac{1}{t} \log \|T_t^\alpha\|_{L^2\to L^2}. \tag{3.12}$$

The spectral mapping theorem implies that the right-hand side of (3.12) is equal to the principal eigenvalue of $\Delta/2 + b + \check{\alpha} + |\alpha|^2/2$. ⟨**q.e.d.**⟩

Given a smooth vector field β and a smooth function V on M, let λ^* be a principal eigenvalue of the operator $\Delta/2 + \beta + V$ on $L^2(dv)$. Then the following variational formula for λ^* is well-known (see [12], for example).

$$\begin{aligned} -\lambda^* &= \min_{f\in\mathscr{W}_0} \left\{ \int_M \left(\frac{1}{2}(df, df) - f\beta f + \frac{1}{2}|\hat{\beta}|^2 f^2 - V f^2 \right) dv - \frac{1}{2}\sigma^2(\hat{\beta}, f) \right\} \\ &= \min_{f\in\mathscr{W}_0} \left\{ \int_M \left(\frac{1}{2}(df, df) + \frac{1}{2}(|\hat{\beta}|^2 - \delta\hat{\beta}) f^2 - V f^2 \right) dv - \frac{1}{2}\sigma^2(\hat{\beta}, f) \right\}, \end{aligned} \tag{3.13}$$

where $\hat{\beta}$ is the 1-form corresponding to β and

$$\begin{aligned} \mathscr{W}_0 &:= \left\{ f \in C^2(M)\ ;\ f > 0,\ \int_M f^2\, dv = 1 \right\}, \\ \sigma^2(\alpha, f) &:= \inf_{U\in C^1(M)} \int_M |\alpha - dU|^2 f^2\, dv, \quad \alpha \in \mathscr{D}_{1,\infty}. \end{aligned}$$

We want to use the formula (3.13) for the calculation of the rate function Λ_1^*. For this purpose, we extend it to the following form.

Proposition 3.20

$$-\lambda_1^* = \inf_{\mu\in\mathscr{M}_1}\left[\frac{1}{2}\mathscr{E}(\mu)+\int_M\left\{\frac{1}{2}(|\hat\beta|^2-\delta\hat\beta)-V\right\}d\mu-\frac{1}{2}\sigma^2(\hat\beta,\mu)\right], \tag{3.14}$$

where

$$\mathscr{E}(\mu) := \begin{cases}\mathscr{E}(f) & \text{if } \mu \ll v \text{ and } d\mu = f^2 dv \text{ with } f\in\mathscr{W},\\ \infty & \text{otherwise,}\end{cases}$$

$$\sigma^2(\alpha,\mu) := \inf_{U\in C^\infty(M)}\int_M|\alpha-dU|^2d\mu$$

for $\mu\in\mathscr{M}_1$ *and* $\alpha\in\mathscr{D}_{1,\infty}$.

We prepare some notations for the proof. Set $E=\{du\ ;\ u\in C^\infty(M)\}$. For $f\in\mathscr{W}$, we denote $L^2_1(f^2dv)$-closure of E by $\bar E^f$. The orthogonal projection to $\bar E^f$ on $L^2_1(f^2dv)$ is denoted by P_f.

Proof of Proposition 3.20. It is almost the same argument as the original proof of the variational formula (3.13). Most part of the proof is devoted to extend the range of minimum from $\mathscr{W}_0$ to $\mathscr{W}$.

The Krein-Rutman theorem allows us to take a unique, $L^2(dv)$-normalized, strictly positive eigenfunction u_0 corresponding to the principal eigenvalue λ^*. Note that u_0 is smooth by the hypoellipticity. Set $\psi=-\log u_0$. Then, with a bit of calculation, we obtain

$$\frac{1}{2}\Delta\psi-\frac{1}{2}(d\psi,d\psi)+(\hat\beta,d\psi)-V=-\lambda^*.$$

It is equivalent to the following:

$$\frac{1}{2}\Delta\psi+\min_\alpha\left\{(d\psi,\alpha)+\frac{1}{2}(\hat\beta-\alpha,\hat\beta-\alpha)-V\right\}=-\lambda^*, \tag{3.15}$$

where the range of the minimum above can be taken over any class of measurable 1-forms containing $\hat\beta-d\psi=\hat\beta+u_0^{-1}du_0$, which attains the minimum. Let $\mathscr{W}'$ be the totality of $f\in\mathscr{W}$ with $f>0$ a.e. For each $f\in\mathscr{W}'$ we set $\alpha_f=(1-P_f)\hat\beta+f^{-1}df$. We will adopt the family $\{\alpha_f\ ;\ f\in\mathscr{W}'\}$ as the range of the minimum in (3.15), which is possible once we show that there is $f_0\in\mathscr{W}'$ so that $\alpha_{f_0}=\hat\beta-d\psi$.

Consider the following differential equation:

$$\mathscr{L}^*h=\frac{1}{2}\Delta h-(\hat\beta-\frac{du_0}{u_0},dh)+\left(\delta\hat\beta-2(\hat\beta,\frac{du_0}{u_0})\right)h=0. \tag{3.16}$$

Note that $\mathscr{L}^*$ is the adjoint operator of $\mathscr{L}:=\Delta/2+\beta-\mathrm{grad}\psi$ on $L^2(u_0^2dv)$. Since the principal eigenvalue of $\mathscr{L}$ is 0, the Krein-Rutman theorem asserts that the principal eigenvalue corresponding to $\mathscr{L}^*$ is also 0. Thus (3.16) has the $L^2(u_0^2dv)$-normalized, strictly positive smooth solution. We give $W_0 \in C^\infty(M)$ and $f_0 \in \mathscr{W}'$ by the relations $W_0 = (\log h)/2$ and $f_0 = u_0 \exp W_0$. Then we have

$$\int_M (\hat{\beta} - dW_0, du) f_0^2 dv = \int_M \left(\hat{\beta} - \frac{dh}{2h}, du\right) h\, u_0^2 dv = \int_M h \mathscr{L} u\, u_0^2 dv = 0.$$

Accordingly, $dW_0 = P_f \hat{\beta}$ holds and therefore we obtain

$$\hat{\beta} - d\psi = \hat{\beta} + u_0^{-1} du_0 = (1 - P_f)\hat{\beta} + f_0^{-1} df_0$$

and hence $\alpha_{f_0} = \hat{\beta} - d\psi$.

By (3.15), for each $f \in \mathscr{W}'$ we have

$$\frac{1}{2}\Delta\psi + (d\psi, \alpha_f) + \frac{1}{2}(\hat{\beta} - \alpha_f, \hat{\beta} - \alpha_f) - V \geq -\lambda^*.$$

Multiplying f^2 and integrating over M we obtain

$$\frac{1}{2}\int_M \Delta\psi f^2\, dv + \int_M (d\psi, \alpha_f) f^2\, dv + \int_M \left\{\frac{1}{2}(\hat{\beta} - \alpha_f, \hat{\beta} - \alpha_f) - V\right\} f^2\, dv \geq -\lambda^*.$$

On the left-hand side,

$$\begin{aligned} \frac{1}{2}\int_M \Delta\psi f^2\, dv + \int_M (d\psi, \alpha_f) f^2\, dv &= \frac{1}{2}\int_M \Delta\psi f^2\, dv + \int_M (d\psi, (1 - P_f)\hat{\beta} + \frac{df}{f}) f^2\, dv \\ &= \int_M (d\psi, (1 - P_f)\hat{\beta}) f^2\, dv = 0. \end{aligned}$$

The second equality follows from the Green formula

$$\int_M \Delta\psi f^2\, dv = -2\int_M (d\psi, df) f\, dv,$$

which is proved by approximating f by smooth functions. Then we have

$$-\lambda^* \leq \int_M \left\{\frac{1}{2}(\hat{\beta} - \alpha_f, \hat{\beta} - \alpha_f) - V\right\} f^2\, dv. \tag{3.17}$$

Since the equality holds in (3.17) when $f = f_0$, we obtain

$$-\lambda^* = \inf_{f \in \mathscr{W}'} \int_M \left\{\frac{1}{2}(\hat{\beta} - \alpha_f, \hat{\beta} - \alpha_f) - V\right\} f^2\, dv.$$

On the right-hand side, we have

$$\begin{aligned} &\frac{1}{2}\int_M \left\{(\hat{\beta} - \alpha_f, \hat{\beta} - \alpha_f) - V\right\} f^2\, dv \\ &\qquad = \int_M \left\{\frac{1}{2}(df, df) - (P_f\hat{\beta}, df) f + \frac{1}{2}|P_f\hat{\beta}|^2 f^2 - V f^2\right\} dv. \end{aligned} \tag{3.18}$$

Here we have

$$\int_M |P_f\hat\beta|^2 f^2\,dv = \int_M |\hat\beta|^2 f^2\,dv - \sigma^2(\hat\beta, f)$$

since $\sigma^2(\hat\beta, f) = \|(1-P_f)\hat\beta\|^2_{L^2(f^2dv)}$. On the other hand, the lemma below (Lemma 3.21) asserts

$$\int_M (P_f\hat\beta, df)f\,dv = \int_M (\hat\beta, df)f\,dv = \frac{1}{2}\int_M \delta\hat\beta f^2\,dv.$$

Thus, we conclude that

$$\begin{aligned} -\lambda^* &= \inf_{f\in\mathscr{W}'}\left[\frac{1}{2}\mathscr{E}(f) + \int_M \left\{\frac{1}{2}\left(|\hat\beta|^2 - \delta\hat\beta\right) - V\right\} f^2 dv - \frac{1}{2}\sigma^2(\hat\beta, f)\right] \\ &= \inf_{\substack{\mu\in\mathscr{M}_1 \\ d\mu = f^2dv \\ f\in\mathscr{W}'}}\left[\frac{1}{2}\mathscr{E}(\mu) + \int_M \left\{\frac{1}{2}\left(|\hat\beta|^2 - \delta\hat\beta\right) - V\right\} d\mu - \frac{1}{2}\sigma^2(\hat\beta, \mu)\right]. \qquad (3.19)\end{aligned}$$

Now we call the term in the infimum of (3.19) by $\Psi(\mu)$. We claim that $\Psi(\mu)$ is convex. To prove it, all we need to show is the convexity of $-\sigma^2(\hat\beta, \mu)$ since the convexity of $\mathscr{E}(\mu)$ is well-known. For each $U \in C^1(M)$ fixed, the functional $\int_M |\alpha - dU|^2 d\mu$ is linear with respect to μ. By taking infimum over U, we conclude that $\sigma^2(\alpha, \cdot)$ is concave. Thus we prove the convexity of $\Psi(\mu)$.

In order to replace the range of infimum in (3.19) to the whole $\mathscr{M}_1$, first we extend $\mathscr{W}'$ to $\mathscr{W}$. For each $f \in \mathscr{W}$, we set $f_\varepsilon = \{(1-\varepsilon)f^2 + \varepsilon\}^{1/2} \in \mathscr{W}'$. Take $\mu \in \mathscr{M}_1$ and $\mu_\varepsilon \in \mathscr{M}_1$ by $d\mu = f^2dv$ and $d\mu_\varepsilon = f_\varepsilon^2 dv$. Then we obtain

$$-\lambda^* \le \Psi(\mu_\varepsilon) \le (1-\varepsilon)\Psi(\mu) + \varepsilon\Psi(v).$$

Letting $\varepsilon \to 0$, $\Psi(\mu) \ge -\lambda^*$ follows and hence we can extend the range of infimum from $\mathscr{W}'$ to $\mathscr{W}$. Extension to the whole $\mathscr{M}_1$ needs only the fact that $\Psi(\mu) = \infty$ unless $d\mu = f^2dv$ for some $f \in \mathscr{W}$. $\langle$**q.e.d.**$\rangle$

Lastly, we prove a lemma used in the proof of Proposition 3.20.

Lemma 3.21 *For* $f \in \mathscr{W}$,

$$\int_M (\alpha, df)f\,dv = 0, \quad \alpha \in \left(\bar E^f\right)^\perp.$$

Proof. The method here is essentially due to that in the proof of Theorem 6.3.19 of [5]. For the proof, we require the following variational formula for the Dirichlet form $\mathscr{E}$:

$$\mathscr{E}(f) = \sup\left\{-\int_M \frac{\Delta u}{u} f^2\,dv\ ;\ u \in C^\infty(M),\ u \ge 1\right\}. \qquad (3.20)$$

By virtue of (3.20), we can take a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty(M)$ with $u_n \ge 1$ so that $-\int_M u_n^{-1}\Delta u_n f^2\,dv$ tends to $\mathscr{E}(f)$ as n goes to ∞. Then, by the definition of α and the Schwarz inequality,

$$\begin{aligned}\left|\int_M (\alpha, df)f\,dv\right| &= \left|\int_M \left(\alpha, \frac{du_n}{u_n}\right) f^2\,dv - \int_M (\alpha, df)f\,dv\right| \\ &\le \left\{\int_M |\alpha|^2 f^2\,dv\right\}^{1/2} \left\{\int_M \left|f\frac{du_n}{u_n} - df\right|^2 dv\right\}^{1/2}.\end{aligned}$$

Here we have

$$\begin{aligned}\limsup_{n\to\infty}\int_M\left|f\frac{du_n}{u_n}-df\right|^2 dv &= \limsup_{n\to\infty}\left\{\mathscr{E}(f)-\int_M\delta\left(\frac{du_n}{u_n}\right)f^2\,dv+\int_M\frac{|du_n|^2}{u_n^2}f^2\,dv\right\}\\ &= \limsup_{n\to\infty}\left(\mathscr{E}(f)+\int_M\frac{\Delta u_n}{u_n}f^2\,dv\right)=0.\end{aligned}$$

$\langle$**q.e.d.**$\rangle$

3.3.3 Rate function I

In this section we are going to complete the proof of Theorem 2.8 by giving a representation of the rate function. Our first goal is in the following.

Proposition 3.22 $\Lambda_1^*=I_1$.

Recall that Λ_1^* appears in (3.11) and I_1 is given by (2.9). First we show that we may restrict the range of supremum in (3.11) to $\mathscr{D}_{1,\infty}$.

Lemma 3.23 *The functional* $\Lambda_1:\mathscr{D}_{1,p}\to\mathbb{R}$ *is continuous.*

Proof. By Lemma 3.19 we can identify $\Lambda_1(\alpha)$ with the principal eigenvalue of the differential operator $\Delta/2+b+\check{\alpha}+|\alpha|^2/2$. The Krein-Rutman theorem implies that it is always a simple eigenvalue. Thus we can apply the perturbation theory to prove the continuity of Λ_1.

Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a family of elements in $\mathscr{D}_{1,p}$ which converges to $\alpha_\infty\in\mathscr{D}_{1,p}$ as n tends to infinity. We consider associated differential operators $L_n:=\Delta/2+b+\check{\alpha}_n+|\alpha_n|^2/2$ and $L_\infty:=\Delta/2+b+\check{\alpha}_\infty+|\alpha_\infty|^2/2$. The perturbation theory asserts the continuity $\lim_{n\to\infty}\Lambda_1(\alpha_n)=\Lambda_1(\alpha_\infty)$ once we prove the following claims (3.21) and (3.22) (see [20] Chapter IV§3.5):

$$\lim_{n\to\infty}\sup_{\substack{u\in\mathrm{Dom}(L_n)\\ \|u\|_{L^2(dv)}+\|L_nu\|_{L^2(dv)}=1}}\inf_{w\in\mathrm{Dom}(L_\infty)}\left(\|u-w\|_{L^2(dv)}+\|L_nu-L_\infty w\|_{L^2(dv)}\right)=0, \tag{3.21}$$

$$\lim_{n\to\infty}\sup_{\substack{w\in\mathrm{Dom}(L_\infty)\\ \|w\|_{L^2(dv)}+\|L_\infty w\|_{L^2(dv)}=1}}\inf_{u\in\mathrm{Dom}(L_n)}\left(\|u-w\|_{L^2(dv)}+\|L_nu-L_\infty w\|_{L^2(dv)}\right)=0. \tag{3.22}$$

We give a proof only for (3.21) since we can prove (3.22) similarly. Note that $\mathrm{Dom}\,(L_n)=\mathrm{Dom}\,(L_\infty)=\mathrm{Dom}(\Delta)$ holds. We give a constant $C_3:=\max\{\|\hat{b}\|_p,\sup_{n\in\mathbb{N}\cup\{\infty\}}\|\alpha_n\|_p\}$. Recall that there is a constant $C>0$ so that, for each $\alpha\in\mathscr{D}_{1,p}$, $\sup_{x\in M}|\alpha|(x)\le C\|\alpha\|_p$ holds by the Sobolev embedding theorem. For $u\in\mathrm{Dom}\,(L_n)$ with $\|u\|_{L^2(dv)}+\|L_nu\|_{L^2(dv)}=1$, we have

$$\begin{aligned}&\inf_{w\in\mathrm{Dom}(L_\infty)}\left(\|u-w\|_{L^2(dv)}+\|L_nu-L_\infty w\|_{L^2(dv)}\right)\\ &\qquad\le\|L_nu-L_\infty u\|_{L^2(dv)}\\ &\qquad=\left\|(\alpha_n-\alpha_\infty,du)+\frac{1}{2}\left(|\alpha_n|^2-|\alpha_\infty|^2\right)u\right\|_{L^2(dv)}\\ &\qquad\le C\left(\|\alpha_n-\alpha_\infty\|_p\|du\|_{L^2_1(dv)}+C_3\|\alpha_n-\alpha_\infty\|_p\|u\|_{L^2(dv)}\right).\end{aligned}$$

Now we estimate the term $\|du\|_{L^2_1(dv)}$. We have

$$\|du\|^2_{L^2_1(dv)} = (\Delta u, u)_{L^2(dv)} = (L_n u, u)_{L^2(dv)} - \int_M (\hat{b} + \alpha_n, du) u \, dv - \frac{1}{2} \int_M |\alpha_n|^2 u^2 \, dv.$$

On each term of the right-hand side, for each $\varepsilon > 0$, we have

$$\Big| (L_n u, u)_{L^2(dv)} \Big| \leq \|L_n u\|_{L^2(dv)} \|u\|_{L^2(dv)} \leq \left(\frac{\|L_n u\|_{L^2(dv)} + \|u\|_{L^2(dv)}}{2} \right)^2 = \frac{1}{4},$$

$$\Big| \int_M (\hat{b} + \alpha_n, du) u \, dv \Big| \leq CC_3 \|du\|_{L^2_1(dv)} \|u\|_{L^2(dv)} \leq CC_3 \varepsilon \|du\|^2_{L^2_1(dv)} + CC_3 \frac{1}{4\varepsilon} \|u\|^2_{L^2(dv)},$$

$$\Big| \int_M |\alpha_n|^2 u^2 \, dv \Big| \leq C_3^2 \|u\|^2_{L^2(dv)} \leq C_3^2.$$

Combining all these estimates with sufficiently small $\varepsilon > 0$ so that $CC_3 \varepsilon < 1$, we conclude (3.21). $\langle$**q.e.d.**$\rangle$

As a consequence of Lemma 3.23, we obtain

$$\Lambda_1^*(\omega) = \sup_{\alpha \in \mathscr{D}_{1,p}} (\langle \omega, \alpha \rangle - \Lambda_1(\alpha)) = \sup_{\alpha \in \mathscr{D}_{1,\infty}} (\langle \omega, \alpha \rangle - \Lambda_1(\alpha))$$

since $\mathscr{D}_{1,\infty}$ is dense in $\mathscr{D}_{1,p}$. By virtue of Proposition 3.20, we obtain the following representation of Λ_1^*:

$$\begin{aligned}
\Lambda_1^*(\omega) &= \sup_{\alpha \in \mathscr{D}_{1,\infty}} \inf_{\mu \in \mathscr{M}_1} \bigg[\langle \omega, \alpha \rangle + \frac{1}{2} \mathscr{E}(\mu) \\
&\qquad + \int_M \left\{ \frac{1}{2} \left(|\hat{b} + \alpha|^2 - \delta(\hat{b} + \alpha) \right) - \frac{1}{2} |\alpha|^2 \right\} d\mu - \frac{1}{2} \sigma^2(\hat{b} + \alpha, \mu) \bigg] \\
&= \sup_{\alpha \in \mathscr{D}_{1,\infty}} \inf_{\mu \in \mathscr{M}_1} \bigg[-\frac{1}{2} \sigma^2(\alpha, \mu) + \langle \omega, \alpha \rangle + \int_M \left((\hat{b}, \alpha) - \frac{1}{2} \delta \alpha \right) d\mu \\
&\qquad + \frac{1}{2} \mathscr{E}(\mu) - \langle \omega, \hat{b} \rangle - \frac{1}{2} \int_M |\hat{b}|^2 d\mu \bigg]. \qquad (3.23)
\end{aligned}$$

We want to exchange the order of supremum and infimum by using the minimax theorem. For this purpose, we define the functional Φ by

$$\Phi(\alpha, \mu) := -\frac{1}{2} \sigma^2(\alpha, \mu) + \langle \omega, \alpha \rangle + \int_M \left((\hat{b}, \alpha) - \frac{1}{2} \delta \alpha \right) d\mu + \frac{1}{2} \mathscr{E}(\mu) - \langle \omega, \hat{b} \rangle - \frac{1}{2} \int_M |\hat{b}|^2 d\mu$$

and verify some properties.

Lemma 3.24 (i)*For each $\mu \in \mathscr{M}_1$ fixed, $\Phi(\cdot, \mu)$ is concave and continuous.* (ii) *For each $\alpha \in \mathscr{D}_{1,\infty}$ fixed, $\Phi(\alpha, \cdot)$ is convex and lower semi-continuous.*

Proof. (i) We may suppose $\mathscr{E}(\mu) < \infty$ and denote $d\mu = f^2 dv$ with $f \in \mathscr{W}$. Then it suffices to show the continuity and the concavity of $-\sigma^2(\alpha, \mu)$ since other functionals which appear in the definition of Φ are all linear and bounded. Recall that we can rewrite $\sigma^2(\alpha, \mu)$ by $\|(1 - P_f)\alpha\|^2_{L^2_1(d\mu)}$. This expression implies the continuity of $\sigma^2(\cdot, \mu)$. The concavity follows from that of the functional $\alpha \mapsto -\int_M |\alpha - dU|^2 d\mu$ by taking infimum on $U \in C^\infty(M)$.

(ii) It is well-known that $\mathscr{E}(\mu)$ is convex and lower semi-continuous with respect to the weak topology on $\mathscr{M}_1$. Thus it suffices to prove the lower semi-continuity and convexity of $-\sigma^2(\alpha, \cdot)$. Note that for each $U \in C^1(M)$ fixed, the functional $\int_M |\alpha - dU|^2 d\mu$ is continuous with respect to μ. By taking infimum over U, we conclude that $\sigma(\alpha, \cdot)$ is upper semi-continuous. We can prove the convexity as well. $\langle$**q.e.d.**$\rangle$

Proof of Proposition 3.22. Take $\omega \in \mathscr{D}_{1,-p}$ with $\Lambda_1^*(\omega) < \infty$. First we show $\omega \in \mathscr{H}$. Since $\mathscr{M}_1$ and $\mathscr{D}_{1,\infty}$ are convex and $\mathscr{M}_1$ is compact, we can apply the Sion minimax theorem [25] to (3.23) by virtue of Lemma 3.24. Then we obtain

$$\begin{aligned}\Lambda_1^*(\omega) = \inf_{\mu \in \mathscr{M}_1} \sup_{\alpha \in \mathscr{D}_{1,\infty}} \Bigg[-\frac{1}{2}\sigma^2(\alpha, \mu) + \langle \omega, \alpha \rangle + \int_M \left((\hat{b}, \alpha) - \frac{1}{2}\delta\alpha \right) d\mu \\ + \frac{1}{2}\mathscr{E}(\mu) - \langle \omega, \hat{b} \rangle - \frac{1}{2}\int_M |\hat{b}|^2 d\mu \Bigg].\end{aligned}$$

If the 1-form α is exact, namely $\alpha = du$ for some $u \in C^\infty(M)$, then clearly $\sigma(\alpha, \mu) = 0$ holds for all $\mu \in \mathscr{M}_1$. In this case, we have

$$\Phi(du, \mu) = \langle \omega, du \rangle + \int_M \left(\frac{1}{2}\Delta u + bu \right) d\mu + \frac{1}{2}\mathscr{E}(\mu) - \langle \omega, \hat{b} \rangle - \frac{1}{2}\int_M |\hat{b}|^2 d\mu.$$

If $\omega \notin \tilde{\Omega}$, then for each $\mu \in \mathscr{M}_1$ there exists $u^\mu \in C^\infty(M)$ so that

$$\langle \omega, du^\mu \rangle + \int_M \left(\frac{1}{2}\Delta + b \right) u^\mu \, d\mu \neq 0.$$

Thus we can conclude that $\sup_{\alpha \in \mathscr{D}_{1,\infty}} \Phi(\alpha, \mu) = +\infty$ for each μ by taking $\alpha = R du^\mu$ with $R \in \mathbb{R}$ and letting $|R| \to \infty$. Since we assume $\Lambda_1^*(\omega) < \infty$, ω must be in $\tilde{\Omega}$. Hence there exists $\mu \in \mathscr{M}_1$ so that

$$\langle \omega, du \rangle + \int_M \left(\frac{1}{2}\Delta + b \right) u \, d\mu = 0 \tag{3.24}$$

holds for all $u \in C^\infty(M)$. Note that there is at most one $\mu \in \mathscr{M}_1$ which satisfies (3.24). Indeed, if both μ and ν satisfy the relation (3.24) then we have

$$\int_M \left(\frac{1}{2}\Delta u + bu \right) d\mu = \int_M \left(\frac{1}{2}\Delta u + bu \right) d\nu$$

for all $u \in C^\infty(M)$. Given arbitrary $\phi \in C^\infty(M)$, there exists $u \in C^\infty(M)$ which satisfies the equation

$$\left(\frac{1}{2}\Delta + b \right) u = \phi - \int_M \phi \, dm \tag{3.25}$$

(see [13]). Recall that m is the normalized invariant measure of $\Delta/2 + b$. Thus we obtain $\int_M \phi \, d\mu = \int_M \phi \, d\nu$ for all $\phi \in C^\infty(M)$ and therefore $\mu = \nu$. We denote such μ by μ^ω. Thus we have $\sup_{\alpha \in \mathscr{D}_{1,\infty}} \Phi(\alpha, \nu) = +\infty$ for $\nu \neq \mu^\omega$. Therefore we must have $\sup_{\alpha \in \mathscr{D}_{1,\infty}} \Phi(\alpha, \mu^\omega) < \infty$ in order to satisfy $\Lambda_1^*(\omega) < \infty$. Hence we require $\omega \in \Omega$. Set $f = \chi(\omega)$. Then we obtain

$$\begin{aligned} \Lambda_1^*(\omega) &= \inf_{\mu \in \mathscr{M}_1} \sup_{\alpha \in \mathscr{D}_{1,\infty}} \Phi(\alpha, \mu) \\ &= \sup_{\alpha \in \mathscr{D}_{1,\infty}} \left\{ -\frac{1}{2} \|(1 - P_f)\alpha\|^2_{L^2_1(f^2 dv)} + \langle \omega, \alpha \rangle + \int_M (\hat{b}, \alpha) f^2 \, dv - \int_M (\alpha, df) f \, dv \right. \\ &\qquad \left. + \frac{1}{2}\mathscr{E}(f) - \langle \omega, \hat{b} \rangle - \frac{1}{2} \int_M |\hat{b}|^2 f^2 \, dv \right\}. \end{aligned} \tag{3.26}$$

Except $\langle \omega, \alpha \rangle$, the term appearing in $\Phi(\alpha, \mu)$ which depends on α is bounded on the set $\{\alpha \in \mathscr{D}_{1,\infty} \ ; \ \|\alpha\|_{L^2_1(f^2 dv)} \leq 1\}$. Thus $\Lambda_1^*(\omega) = \infty$ if ω is not a bounded functional on $L^2_1(f^2 dv)$. Hence we conclude $\omega \in \mathscr{H}$.

Let us define $\xi_f := \lim_{\varepsilon \downarrow 0} (f + \varepsilon)^{-1} df \in L^2_1(f^2 dv)$. Then, for $\omega \in \mathscr{H}$, we have $\hat{\omega} + \hat{b} - \xi_f \in (\bar{E}^f)^\perp$ where $\hat{\omega} \in L^2_1(f^2 dv)$ is determined by (2.10). It is a consequence of (2.8) and (2.10). Thus we have

$$\langle \omega, \alpha \rangle + \int_M (\hat{b}, \alpha) f^2 \, dv - \int_M (\alpha, df) f \, dv = \int_M (\hat{\omega} + \hat{b} - \xi_f, (1 - P_f)\alpha) f^2 \, dv.$$

By virtue of (2.10) and (3.26), we obtain

$$\begin{aligned} \Lambda_1^*(\omega) = -\frac{1}{2} \int_M |(1 - P_f)\alpha - \hat{\omega} - \hat{b} + \xi_f|^2 f^2 \, dv + \frac{1}{2} \int_M |\hat{\omega} + \hat{b} - \xi_f|^2 f^2 \, dv \\ + \frac{1}{2}\mathscr{E}(f) - \int_M (\hat{\omega}, \hat{b}) f^2 \, dv - \frac{1}{2} \int_M |\hat{b}|^2 f^2 \, dv. \end{aligned}$$

Since $\{(1 - P_f)\alpha\}_{\alpha \in \mathscr{D}_{1,\infty}}$ is dense in $\left(\bar{E}^f\right)^\perp$, we claim

$$\Lambda_1^*(\omega) = \frac{1}{2} \int_M |\hat{\omega}|^2 f^2 \, dv + \frac{1}{2}\mathscr{E}(f) + \frac{1}{2} \int_M |\xi_f|^2 f^2 \, dv - \int_M (\hat{\omega} + \hat{b}, \xi_f) f^2 \, dv. \tag{3.27}$$

By virtue of Lemma 3.21, we have

$$\begin{aligned} \int_M (\hat{\omega} + \hat{b}, \xi_f) f^2 \, dv &= \lim_{\varepsilon \downarrow 0} \int_M \left(\hat{\omega} + \hat{b}, \frac{df}{f + \varepsilon} \right) f^2 \, dv \\ &= \int_M (\hat{\omega} + \hat{b}, df) f \, dv = \int_M (\xi_f, df) f \, dv = \int_M |\xi_f|^2 f^2 \, dv \end{aligned} \tag{3.28}$$

and

$$\int_M |\xi_f|^2 \, f^2 \, dv = \lim_{\varepsilon \downarrow 0} \int_M \left| \frac{df}{f + \varepsilon} \right|^2 f^2 \, dv = \lim_{\varepsilon \downarrow 0} \mathscr{E}(\phi_\varepsilon(f)) = \mathscr{E}(f), \tag{3.29}$$

where $\phi_\varepsilon(x) = x - \varepsilon \log[(x+\varepsilon)/\varepsilon]$ for $\varepsilon > 0$. Here the last equality in (3.29) follows from the fact that $\lim_{\varepsilon\to 0} \mathscr{E}(\phi_\varepsilon(f) - f) = 0$. It is proved in the same way as Theorem 1.4.2 (v) of [9]. Substituting (3.28) and (3.29) into (3.27), we obtain $\Lambda_1^*(\omega) = I_1(\omega)$.

Now all we need to show is that Λ_1^* coincides with I_1 on $\mathscr{H}$ because I_1 is finite on $\mathscr{H}$. This assertion follows from the fact that the observation to obtain (3.27) from (3.23) is all valid for $\omega \in \mathscr{H}$ without assuming $\Lambda_1^*(\omega) < \infty$. $\langle$**q.e.d.**$\rangle$

Remark 3.25 The rate function $I_1(\omega)$ attains its minimum 0 only when $\omega = 0$. Indeed, by Definition 2.7, $I_1(\omega) = 0$ if and only if $\omega \in \mathscr{H}$ and $\hat{\omega} = 0$. In addition, (2.10) and (2.8) assert that $\omega \in \mathscr{H}$ and $\hat{\omega} = 0$ is equivalent to $\omega = 0$.

To complete the proof of Theorem 2.8, we prepare the space $(\mathscr{D}_{1,-p})^{[0,\infty)}$, which is the space of maps from $[0,\infty)$ to $\mathscr{D}_{1,-p}$ with pointwise convergence topology. We may regard $\{\bar{Y}^\lambda\}_{\lambda>0}$ as $(\mathscr{D}_{1,-p})^{[0,\infty)}$-valued random variables. As we remarked in the proof of Proposition 3.18, for each partition $0 = t_0 < t_1 < \cdots < t_n$, $\{(\bar{Y}^\lambda_{t_1}, \ldots, \bar{Y}^\lambda_{t_n})\}_{\lambda>0}$ satisfies the large deviation. We can easily show that the corresponding rate function $I_{t_1,\ldots,t_n}$ is described as follows:

$$I_{t_1,\ldots,t_n}(w_1, \ldots, w_n) = \sum_{\ell=1}^{n} (t_\ell - t_{\ell-1}) I_1\left(\frac{w_\ell - w_{\ell-1}}{t_\ell - t_{\ell-1}}\right),$$

where $w_0 = 0$. Thus, in the same way as in Lemma 5.1.6 of [4], we can prove that $\{\bar{Y}^\lambda\}_{\lambda>0}$ satisfies the large deviation in $(\mathscr{D}_{1,-p})^{[0,\infty)}$. Moreover, the rate function coincides with I on $\mathscr{C}_p$ and attains infinity on $(\mathscr{D}_{1,-p})^{[0,\infty)} \setminus \mathscr{C}_p$. On the other hand, the canonical injection from $\mathscr{C}_p$ to $(\mathscr{D}_{1,-p})^{[0,\infty)}$ is clearly continuous. Thus, the contraction principle and the uniqueness of rate function result that the rate function which governs the large deviation in $\mathscr{C}_p$ coincides with I. This is just what we wanted to prove.

4 Applications

In this section we assume $p > d + 1$.

4.1 Large deviation for X and A

First we will establish large deviations for X and A. Recall that X is a current-valued process determined by the stochastic line integrals themselves and A its bounded variation part.

Define the scaled processes $\{\tilde{X}^\lambda\}_{\lambda>0}$, $\{\tilde{A}^\lambda\}_{\lambda>0}$, $\{\bar{X}^\lambda\}_{\lambda>0}$ and $\{\bar{A}^\lambda\}_{\lambda>0}$ by the following:

$$\begin{aligned}
\tilde{X}^\lambda_t &:= \frac{1}{g(\lambda)} X^\lambda_t, \\
\tilde{A}^\lambda_t &:= \frac{1}{g(\lambda)} A^\lambda_t, \\
\bar{X}^\lambda_t &:= \frac{1}{\lambda}\left(X_{\lambda t} - \lambda t\, e\right) = \frac{1}{\lambda} X_{\lambda t} - t\, e = \frac{1}{\sqrt{\lambda}} X^\lambda_t, \\
\bar{A}^\lambda_t &:= \frac{1}{\lambda}\left(A_{\lambda t} - \lambda t\, e\right) = \frac{1}{\lambda} A_{\lambda t} - t\, e = \frac{1}{\sqrt{\lambda}} A^\lambda_t,
\end{aligned}$$

where X^λ, A^λ and e are given in section 2.2. Let functionals L_{X1}, L_{A1}, I_{X1} and I_{A1} on $\mathscr{D}_{1,-p}$ be given by

$$\begin{aligned} L_{X1}(\omega) &:= \inf_{(1-Q^*)\eta=\omega} L_1(\eta), \\ L_{A1}(\omega) &:= \inf_{-Q^*\eta=\omega} L_1(\eta), \qquad (4.1) \\ I_{X1}(\omega) &:= \inf_{(1-Q^*)\eta=\omega} I_1(\eta), \\ I_{A1}(\omega) &:= \inf_{-Q^*\eta=\omega} I_1(\eta). \qquad (4.2) \end{aligned}$$

Recall that Q^* is defined in section 2.2. By using them we define rate functions L_X, L_A, I_X and I_A as follows:

$$L_X(w) := \begin{cases} \int_0^\infty L_{X1}(\dot{w}_t)dt & \text{if } w_t = \int_0^t \dot{w}_s ds \text{ with } \dot{w}_s \in \mathscr{D}_{1,-p} \text{ for a.e.} s, \\ \infty & \text{otherwise,} \end{cases}$$

$$L_A(w) := \begin{cases} \int_0^\infty L_{A1}(\dot{w}_t)dt & \text{if } w_t = \int_0^t \dot{w}_s ds \text{ with } \dot{w}_s \in \mathscr{D}_{1,-p} \text{ for a.e.} s, \\ \infty & \text{otherwise,} \end{cases}$$

$$I_X(w) := \begin{cases} \int_0^\infty I_{X1}(\dot{w}_t)dt & \text{if } w_t = \int_0^t \dot{w}_s ds \text{ with } \dot{w}_s \in \mathscr{D}_{1,-p} \text{ for a.e.} s, \\ \infty & \text{otherwise,} \end{cases}$$

$$I_A(w) := \begin{cases} \int_0^\infty I_{A1}(\dot{w}_t)dt & \text{if } w_t = \int_0^t \dot{w}_s ds \text{ with } \dot{w}_s \in \mathscr{D}_{1,-p} \text{ for a.e.} s, \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 4.1 (i) *Suppose $g(\lambda) = o(\sqrt{\lambda})$. Then, the law of $\{\tilde{X}^\lambda\}_{\lambda>0}$ (resp. $\{\tilde{A}^\lambda\}_{\lambda>0}$) satisfies the large deviation principle as $\lambda \to \infty$ in $\mathscr{C}_p$ under $\mathbb{P}_x$ uniformly in $x \in M$ with speed $g(\lambda)^2$ and the rate function L_X (resp. L_A).*

(ii) *The law of $\{\bar{X}^\lambda\}_{\lambda>0}$ (resp. $\{\bar{A}^\lambda\}_{\lambda>0}$) satisfies the large deviation principle as $\lambda \to \infty$ in $\mathscr{C}_p$ under $\mathbb{P}_x$ uniformly in $x \in M$ with speed λ and the rate function I_X (resp. I_A).*

Proof. Note that we have $Q^* e = 0$ and $(1 - Q^*)\,\tilde{A}^\lambda_t = 0$ a.e. by combining the definition of Q and e with (2.3). Then, by (2.1),

$$\begin{aligned} \tilde{X}^\lambda_t(\alpha) &= \tilde{X}^\lambda_t(\alpha - Q\alpha) + \tilde{X}^\lambda_t(Q\alpha) \\ &= \tilde{Y}^\lambda_t(\alpha - Q\alpha) + \frac{1}{g(\lambda)\sqrt{\lambda}}\left(u_\alpha(z_{\lambda t}) - u_\alpha(z_0)\right) \\ &= \left\langle (1 - Q^*)\tilde{Y}^\lambda_t, \alpha \right\rangle + \frac{1}{g(\lambda)\sqrt{\lambda}}\left(u_\alpha(z_{\lambda t}) - u_\alpha(z_0)\right). \qquad (4.3) \end{aligned}$$

By the contraction principle, $\{(1 - Q^*)\tilde{Y}^\lambda\}_{\lambda>0}$ satisfies the large deviation principle with rate function L_X uniformly in $x \in M$. Note that $\sup_{x\in M} |u_\alpha(x)| \le C \|\alpha\|_p$ holds. This fact comes from the hypoellipticity of $\Delta/2+b$ and the Sobolev embedding theorem. Thus, by (4.3), we can easily show that, for each $\varepsilon > 0$,

$$\limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log\left(\sup_{x\in M} \mathbb{P}_x\left[\rho(\tilde{X}^\lambda, (1-Q^*)\tilde{Y}^\lambda) > \varepsilon\right]\right) = -\infty,$$

where ρ is a metric on $\mathscr{C}_p$ defined by (3.8). The equality above asserts that $\tilde{X}^\lambda$ and $(1-Q^*)\tilde{Y}^\lambda$ are exponentially equivalent uniformly in $x \in M$ (see Theorem 4.2.13 of [4]; cf. Remark 3.3). Thus $\{\tilde{X}^\lambda\}_{\lambda>0}$ satisfies the large deviation with the rate function L_X.

As for $\tilde{A}^\lambda$, we have

$$\tilde{A}^\lambda_t(\alpha) = \tilde{A}^\lambda_t(Q\alpha) = -\left\langle Q^*\tilde{Y}^\lambda_t, \alpha\right\rangle + \frac{1}{\lambda}\left(u_\alpha(z_{\lambda t}) - u_\alpha(z_0)\right).$$

Thus, in the same way, we conclude that $\tilde{A}^\lambda$ satisfies the large deviation with the rate function L_A, which governs the large deviation for $\{-Q^*\tilde{Y}^\lambda\}_{\lambda>0}$. The second assertion also follows by the same argument. $\langle$**q.e.d.**$\rangle$

Remark 4.2 Rate functions L_{X1}, L_{A1}, I_{X1} and I_{A1} attain their minimum only at 0. It is a consequence of the fact that Q^* is a continuous linear operator and the goodness of L_1 or I_1.

In the case of $\tilde{A}^\lambda$, we can obtain more explicit form of L_{A1} or I_{A1} than (4.1) or (4.2). First we deal with I_{A1}.

Proposition 4.3

$$I_{A1}(\omega) = \begin{cases} \frac{1}{2}\mathscr{E}(f) - \int_M (\hat{b}, df) f\, dv + \frac{1}{2}\int_M |P_f\hat{b}|^2 f^2\, dv. & \textit{if } -\omega \in \mathscr{H}_A \textit{ and } \chi(-\omega) = f\ , \\ \infty & \textit{otherwise,} \end{cases}$$

where $\mathscr{H}_A := \mathrm{Range}(Q^*) \cap \Omega$.

Proof. Take $\omega_0 \in \mathscr{D}_{1,-p}$ with $I_{A1}(\omega_0) < \infty$. Then there is $\eta \in \mathscr{H}$ with $-Q^*\eta = \omega_0$. For $u \in C^\infty(M)$,

$$\langle \eta, du\rangle = \langle \eta, Q(du)\rangle = \langle Q^*\eta, du\rangle = -\langle \omega_0, du\rangle\,.$$

Thus $-\omega_0 \in \Omega$ and $\chi(\eta) = \chi(-\omega_0)$ holds. In particular, $-\omega_0 \in \mathscr{H}_A$ follows. Set $f = \chi(-\omega_0)$. Now we have

$$I_1(\eta) = \frac{1}{2}\int_M |\hat{\eta}|^2 f^2 dv = \frac{1}{2}\int_M |(1-P_f)\hat{\eta}|^2 f^2 dv + \frac{1}{2}\int_M |P_f\hat{\eta}|^2 f^2 dv. \tag{4.4}$$

Let $\eta_0 \in \mathscr{D}_{1,-p} \cap \mathscr{H}$ be determined by

$$\langle \eta_0, \alpha\rangle = -\int_M \left(P_f\hat{b} - \xi_f, \alpha\right) f^2 dv. \tag{4.5}$$

Recall that ξ_f is given by $\xi_f = \lim_{\varepsilon\downarrow 0}(f+\varepsilon)^{-1}df \in L^2_1(f^2dv)$. Note that, by Lemma 3.21, $\xi_f \in \bar{E}^f$ holds. Then

$$-\langle \eta_0, Q\alpha\rangle = \int_M \left(\hat{b} - \xi_f, Q\alpha\right) f^2 dv = \langle \omega_0, Q\alpha\rangle = \langle \omega_0, \alpha\rangle$$

holds. This means $-Q^*\eta_0 = \omega_0$. In addition, since we have $\widehat{\eta_0} = \xi_f - P_f\hat{b}$,

$$\int_M \left|(1-P_f)\widehat{\eta_0}\right|^2 f^2\, dv = 0.$$

On the other hand, for each $\eta \in \mathscr{H}$ with $-Q^*\eta = \omega_0$,

$$\begin{aligned}
\int_M |P_f\hat{\eta}|^2 f^2\, dv &= \sup_{\substack{\alpha\in\mathscr{D}_{1,\infty} \\ \|\alpha\|_{L^2_1(f^2dv)}=1}} \left|\int_M (P_f\hat{\eta},\alpha) f^2\, dv\right|^2 = \sup_{\substack{\alpha\in\mathrm{Range}(P_f) \\ \|\alpha\|_{L^2_1(f^2dv)}=1}} \left|\int_M (\hat{\eta},\alpha) f^2\, dv\right|^2 \\
&= \sup_{\substack{\alpha\in E \\ \|\alpha\|_{L^2_1(f^2dv)}=1}} |\langle \eta,\alpha\rangle|^2 = \sup_{\substack{\alpha\in E \\ \|\alpha\|_{L^2_1(f^2dv)}=1}} |\langle \omega_0,\alpha\rangle|^2 \\
&= \int_M |P_f\hat{b} - \xi_f|^2 f^2\, dv.
\end{aligned}$$

The last equality follows from the relation (2.8). Therefore, the second term in (4.4) is independent of the choice of η. Hence we obtain

$$I_{A1}(\omega_0) = \frac{1}{2}\int_M |P_f\hat{b} - \xi_f|^2 f^2\, dv = \frac{1}{2}\mathscr{E}(f) - \int_M (\hat{b}, df) f\, dv + \frac{1}{2}\int_M |P_f\hat{b}|^2 f^2\, dv \tag{4.6}$$

with the aid of (3.29).

To complete the proof, we need to show that $I_{A1}(\omega) < \infty$ holds if $-\omega \in \mathscr{H}_A$. For $f = \chi(-\omega)$ we define η_0 by (4.5). Then, as we proved, $\eta_0 \in \mathscr{H}$ and $-Q^*\eta_0 = \omega$. Hence we obtain $I_{A1}(\omega) < \infty$ by the definition of I_{A1}. ⟨**q.e.d.**⟩

Remark 4.4 Note that $-\omega \in \mathscr{H}_A$ holds if and only if ω is given by

$$\langle \omega, \alpha\rangle = \int_M \left((\hat{b},\alpha) - \frac{1}{2}\delta\alpha\right) f^2 dv - e(\alpha) \tag{4.7}$$

for some $f \in \mathscr{W}$. Indeed, when ω is given by (4.7), clearly $-\omega \in \mathscr{H}_A$. Conversely, when $\omega \in \mathscr{H}_A$, by virtue of (2.5), we have

$$\langle \omega, \alpha\rangle = \langle \omega, du_\alpha\rangle = \int_M \left(\frac{1}{2}\Delta + b\right) u_\alpha f^2 dv = \int_M \left((\hat{b},\alpha) - \frac{1}{2}\delta\alpha\right) f^2 dv - e(\alpha).$$

Proposition 4.3 implies $-\omega \in \mathscr{H}_A$ if and only if $I_{A1}(\omega) < \infty$. Theorem 4.1 yields $\mathscr{H} \cap \mathrm{Range}(Q^*) \subset \mathscr{H}_A$ since $I_{A1}(\omega) < \infty$ when $-\omega \in \mathscr{H} \cap \mathrm{Range}(Q^*)$. However, in order to obtain $\omega \in \mathscr{H} \cap \mathrm{Range}(Q^*)$ for given $\omega \in \mathscr{H}_A$, we require that $|e(\alpha)| \le C\|\alpha\|_{L^2_1(f^2dv)}$ holds with $\chi(\omega) = f$ for some constant C by (4.7). Hence $\mathscr{H} \cap \mathrm{Range}(Q^*) = \mathscr{H}_A$ does not hold in general.

Next we discuss L_{A1}. Let us denote the Radon-Nikodym density of m with respect to v by φ^2. Recall that we can take φ to be smooth and strictly positive.

We say $\omega \in \mathscr{H}'_A$ if and only if

$$\langle \omega, \alpha \rangle = \int_M \left((\hat{b}, \alpha) - \frac{1}{2}\delta\alpha \right) F dm$$

for some $F \in \mathrm{Dom}(\mathscr{E})$ with $\int_M F dv = 0$. We define a map $\tilde{\chi} : \mathscr{H}'_A \to \mathrm{Dom}(\mathscr{E})$ by $\tilde{\chi}(\omega) = F$.

Proposition 4.5

$$L_{A1}(\omega) = \begin{cases} \frac{1}{8}\int_M |dF|^2 dm - \frac{1}{2}\int_M \left(\hat{b} - \frac{d\varphi}{\varphi}\, ,\, dF \right) F\, dm & \\ \qquad\qquad + \frac{1}{2}\int_M \left| P_\varphi \left[F \left(\hat{b} - \frac{d\varphi}{\varphi} \right) \right] \right|^2 dm & \textit{if } \omega \in \mathscr{H}'_A,\ \tilde{\chi}(\omega) = F, \\ \infty & \textit{otherwise.} \end{cases}$$

Moreover, $\mathscr{H}'_A = \mathscr{H}' \cap \mathrm{Range}(Q^*)$ *holds.*

Proof. Take $\omega \in \mathscr{D}_{1,-p}$ with $L_{A1}(\omega) < \infty$. Then there exists $\eta \in \mathscr{H}'$ with $-Q^*\eta = \omega$. For $\alpha \in \mathscr{D}_{1,p}$, we have

$$\langle \omega, \alpha \rangle = \langle Q^* \omega, \alpha \rangle = \langle \omega, Q\alpha \rangle = - \langle \eta, Q\alpha \rangle = - \int_M (\check{\eta}, Q\alpha) dm.$$

Accordingly,

$$|\langle \omega, \alpha \rangle| \le C \left\| Q\alpha \right\|_{L^2_1(dm)} \le C \left\| \alpha \right\|_{L^2_1(dm)}$$

holds for some constant C. The last inequality follows from the fact that we can extend Q to a continuous operator on $L^2_1(dm)$. Thus $\omega \in \mathscr{H}'$ follows. Hence, by virtue of (4.1), $L_{A1}(\omega) < \infty$ occurs if and only if $\omega \in \mathscr{H}' \cap \mathrm{Range}(Q^*)$.

The space of all signed measures μ on M with $\mu(M) = 0$ is denoted by $\mathscr{M}_0$. Also let us define a map $\iota : \mathscr{M}_0 \to \mathscr{D}_{1,-p}$ as follows:

$$\langle \iota(\mu), \alpha \rangle = \int_M \left((\hat{b}, \alpha) - \frac{1}{2}\delta\alpha \right) d\mu. \tag{4.8}$$

By definition, each $\tilde{A}^\lambda$ take its values in $\iota(\mathscr{M}_0)$. Note that L_{A1} is given by the following Legendre transform:

$$\begin{aligned} L_{A1}(\omega) &= \sup_{\alpha \in \mathscr{D}_{1,p}} \left(\langle \omega, \alpha \rangle - \Lambda_{A1}(\alpha) \right), \\ \Lambda_{A1}(\alpha) &:= \lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left(\mathbb{E}_x \left[\exp \left(g(\lambda)^2 \tilde{A}^\lambda_1(\alpha) \right) \right] \right). \end{aligned}$$

If $\omega \notin \overline{\iota(\mathscr{M}_0)}^{\mathscr{D}_{1,-p}}$, then there is $\alpha \in \mathscr{D}_{1,p}$ which annihilates on $\overline{\iota(\mathscr{M}_0)}^{\mathscr{D}_{1,-p}}$ and $\langle \omega, \alpha \rangle = 1$. Then for $R > 0$ we have $\langle \omega, R\alpha \rangle - \Lambda_A(R\alpha) = R$ and therefore $L_{A1}(\omega) = \infty$ holds. Thus we may assume $\omega \in \overline{\iota(\mathscr{M}_0)}^{\mathscr{D}_{1,-p}}$.

Accordingly, we can take a sequence $\{F_n\}_{n\in\mathbb{N}} \subset C^\infty(M)$ with $\int_M F_n\, dm = 0$ so that the corresponding $\omega_n \in \mathscr{D}_{1,-p}$ defined by

$$\langle \omega_n, \alpha\rangle = \int_M \left((\hat{b}, \alpha) - \frac{1}{2}\delta\alpha \right) F_n \varphi^2 dv$$

converges to ω in $\mathscr{D}_{1,-p}$. Let $H^r(M)$ be the r-th order $L^2(dv)$-Sobolev space of functions on M with norm $\|\cdot\|_r$ for each $r \in \mathbb{R}$. Then,

$$\begin{aligned}
\left\|\varphi^2(F_n - F_k)\right\|_{-p+1} &= \sup_{\substack{\|h\|_{p-1}\le 1 \\ \int_M h dm = 0}} \left| \left\langle \varphi^2(F_n - F_k), h \right\rangle_{H^{p-1}(M)} \right| \\
&\le \sup_{\substack{u\in C^\infty(M) \\ \|du\|_p \le C}} \left| \left\langle \varphi^2(F_n - F_k)\,,\, \left(\frac{1}{2}\Delta + b\right) u \right\rangle_{H^{p-1}(M)} \right| \\
&\le \sup_{\|du\|_p\le C} |\langle \omega_n - \omega_k, du\rangle| \le C \left\|\omega_n - \omega_k\right\|_{-p}
\end{aligned}$$

for some constant C. Since the multiplication of φ^{-2} is continuous on $H^{-p+1}(M)$, $\{F_n\}_{n\in\mathbb{N}}$ forms a Cauchy sequence in $H^{-p+1}(M)$. We denote the limit by F. Then

$$\langle \omega, \alpha\rangle = \left\langle \varphi^2 F\,,\, (\hat{b}, \alpha) - \frac{1}{2}\delta\alpha \right\rangle_{H^{p-1}(M)}$$

holds. Since $\omega \in \mathscr{H}'$, $(\hat{b} - d/2)(\varphi^2 F) \in L^2_1(dv)$ holds. Here, the multiplication of $\hat{b}$ and the exterior derivative d are operated in the sense of distribution. Thus, by using the Gårding inequality iteratively, we find that F is in $H^1(M) = \mathrm{Dom}(\mathscr{E})$ and therefore

$$\langle \omega, \alpha\rangle = \int_M \left(F\left(\hat{b} - \frac{d\varphi}{\varphi}\right) - \frac{1}{2}dF\,,\, \alpha \right) dm = \int_M \left((\hat{b}, \alpha) - \frac{1}{2}\delta\alpha \right) F\, dm.$$

For $\eta \in \mathscr{H}'$ with $-Q^*\eta = \omega$, we have

$$L_1(\eta) = \frac{1}{2}\int_M |(1 - P_\varphi)\check{\eta}|^2\, dm + \frac{1}{2}\int_M |P_\varphi\check{\eta}|^2\, dm.$$

Then

$$\frac{1}{2}\int_M |P_\varphi\check{\eta}|^2\, dm = \frac{1}{2}\int_M \left| P_\varphi\left(F\left(\hat{b} - \frac{d\varphi}{\varphi}\right) - \frac{1}{2}dF \right) \right|^2 dm.$$

This fact comes from the same argument as in the proof of Proposition 4.3. On the other hand, we can take $\eta_0 \in \mathscr{H}'$ with $\check{\eta}_0 = -P_\varphi[F(\hat{b} - \varphi^{-1}d\varphi) - dF/2]$. Then, we can easily verify that $-Q^*\eta_0 = \omega$ and $(1 - P_\varphi)\check{\eta}_0 = 0$. Therefore, we obtain

$$\begin{aligned}
L_{A1}(\omega) &= \frac{1}{2}\int_M \left| P_\varphi\left[F\left(\hat{b} - \frac{d\varphi}{\varphi}\right) - \frac{1}{2}dF \right] \right|^2 dm \\
&= \frac{1}{8}\int_M |dF|^2 dm - \frac{1}{2}\int_M \left(\hat{b} - \frac{d\varphi}{\varphi}\,,\, dF\right) F\, dm + \frac{1}{2}\int_M \left| P_\varphi\left[F\left(\hat{b} - \frac{d\varphi}{\varphi}\right) \right] \right|^2 dm.
\end{aligned}$$

$\langle$**q.e.d.**$\rangle$

4.2 Comparison of rate functions

In this section, we compare the rate function I, I_X and I_A with those corresponding to the moderate deviations. As we have seen in Definition 2.7, the rate function I which governs the sample path large deviation in $\mathscr{C}_p$ is expressed by using the rate function I_1. Here, Proposition 3.15 and Proposition 3.22 ensure that I_1 governs the large deviation for 1-dimensional distribution evaluated at time 1. We remark that the same observation is still true for other sample path large deviations we proved. Thus, to investigate the difference of rate functions, it is sufficient to concentrate on the rate functions I_1, L_1 and so on.

First we compare the explicit form of I_{A1} with that of L_{A1}. By Remark 4.4, $-\omega \in \mathscr{H}_A$ if and only if

$$\langle \omega, \alpha \rangle = \int_M \left((\hat{b}, \alpha) - \frac{1}{2}\delta\alpha \right) \left(\frac{f^2}{\varphi^2} - 1 \right) dm$$

for $f = \chi(-\omega)$. We can express the rate function $I_{A1}(\omega)$ by using $\tilde{f} := f/\varphi$ as follows:

$$I_{A1}(\omega) = \frac{1}{2} \int_M |d\tilde{f}|^2 dm - \int_M \left(\hat{b} - \frac{d\varphi}{\varphi}, d\tilde{f} \right) \tilde{f}\, dm + \frac{1}{2} \int_M \left| P_f \left[\hat{b} - \frac{d\varphi}{\varphi} \right] \right|^2 \tilde{f}^2 dm. \qquad (4.9)$$

This expression is similar to that of L_{A1}. But two functions $\tilde{f}$ and F are quite different. Indeed, $\int_M \tilde{f} dm = 1$ holds while $\int_M F dm = 0$.

Given $f \in \mathscr{W}$ with $f^2/\varphi^2 - 1 \in \mathrm{Dom}(\mathscr{E})$, we can take $\omega \in \mathscr{D}_{1,-p}$ through (4.7). In this case, $-\omega \in \mathscr{H}'_A \cap \mathscr{H}_A$ holds. But, as we will see in Example 4.8, there is no domination between $I_{A1}(\omega)$ and $L_{A1}(\omega)$ in general even when $b = 0$.

Next, we investigate some relations between rate functions.

Proposition 4.6 *Take $\omega \in \mathscr{D}_{1,-p}$.*

(i) *When $Q^*\omega = 0$, $\omega \in \mathscr{H}$ is equivalent to $\omega \in \mathscr{H}'$ and $I_1(\omega) = L_1(\omega)$.*

(ii) *Suppose that $\hat{b}$ is an exact 1-form.*

  (a) *If $L_{X1}(\omega) < \infty$ then $\omega \in \mathscr{H}' \cap \mathrm{Range}(1 - Q^*)$ and $L_{X1}(\omega) = L_1(\omega)$. Moreover, $\{\omega \in \mathscr{D}_{1,-p} \,;\, L_{X1}(\omega) < \infty\} = \mathscr{H}' \cap \mathrm{Range}(1 - Q^*)$ holds.*

  (b) *The domain $\mathscr{H}'_A$ is equal to $\mathscr{H}' \cap \mathrm{Range}(Q^*)$. Moreover, $L_{A1}(\omega) = L_1(-\omega)$ holds when $-\omega \in \mathscr{H}'_A$.*

  (c) *The domain $\mathscr{H}_A$ is equal to $\mathscr{H} \cap \mathrm{Range}(Q^*)$. Moreover, $I_{A1}(\omega) = I_1(-\omega)$ holds when $-\omega \in \mathscr{H}_A$.*

  (d) *$I_{X1} \le L_{X1}$.*

Proof. (i) Since $\omega \in \mathrm{Ker}(Q^*)$, (2.8) implies that $\omega \in \Omega$ and $\chi(\omega) = \varphi$. Hence, the existence of $\hat{\omega} \in L^2_1(dm)$ is equivalent to that of $\check{\omega} \in L^2_1(dm)$ and $\check{\omega} = \hat{\omega}$ holds when either of them exists.

(ii-a) We can easily verify that, when $\hat{b}$ is an exact 1-form, the ranges of Q and $(1-Q)$ are orthogonal in $L^2_1(dm)$ each other. Thus, the extension $\bar{Q}$ of Q to the continuous operator on $L^2_1(dm)$ becomes an orthogonal projection. Then, for $\eta \in \mathscr{H}'$, $Q^*\eta \in \mathscr{H}'$ and

$$\langle Q^*\eta, \alpha\rangle = \int_M (\bar{Q}\check{\omega}, \alpha) dm$$

holds. For $\omega \in \mathrm{Range}(1-Q^*) \cap \mathscr{H}'$ and $\eta \in \mathscr{H}'$ with $(1-Q^*)\eta = \omega$, we have

$$\int_M |\check{\eta}|^2 dm = \int_M |\check{\omega} + \bar{Q}\check{\eta}|^2 dm = \int_M |\check{\omega}|^2 dm + \int_M |\bar{Q}\check{\eta}|^2 dm$$

since $\check{\omega} \in \mathrm{Range}(1-\bar{Q})$. Hence we obtain

$$\inf_{(1-Q^*)\eta=\omega} L_1(\eta) = L_1(\omega)$$

and the conclusion follows.

(ii-b) This is proved in the same way as (ii-a). Note that, as we proved in Proposition 4.5, the former assertion holds without the assumption on $\hat{b}$.

(ii-c) By virtue of Remark 4.4, it suffices to show $\mathscr{H}_A \subset \mathscr{H} \cap \mathrm{Range}(Q^*)$ and $I_{A1}(\omega) = I_1(-\omega)$ for $\omega \in \mathscr{H}_A$. Take $\omega \in \mathscr{H}_A$. Since $\hat{b} = \varphi^{-1}d\varphi$, we obtain $e = 0$. Thus we can rewrite (4.7) as follows:

$$\langle \omega, \alpha\rangle = \int_M \left(\hat{b} - \xi_f, \alpha\right) f^2 dv.$$

In addition, $\hat{b} - \xi_f \in L^2_1(f^2 dv)$ and $P_f\hat{b} = \hat{b}$ hold. Thus $\omega \in \mathscr{H}$ holds and a direct calculation yields $I_1(-\omega) = I_{A1}(\omega)$.

(ii-d) Take ω with $L_{X1}(\omega) < \infty$. Then (ii-a) and (i) imply $L_{X1}(\omega) = I_1(\omega)$. Thus the conclusion comes from the definition of I_{X1} in Theorem 4.1. $\langle$**q.e.d.**$\rangle$

Remark 4.7 As we will see in Example 4.9, in the case of I_{X1}, such a relation as (ii-a)-(ii-c) of Proposition 4.6 fails in general even when $b = 0$. This example also provides the case that $I_{X1} = L_{X1}$ does not hold.

Example 4.8 Let us consider the case that M is equal to the unit circle $S^1 \simeq [0,1]/\{0 \sim 1\}$ with flat metric. We assume $b = 0$. Given $f \in \mathscr{W}$, we define $\omega \in \mathscr{H}$ by

$$\langle \omega, \alpha\rangle = -\int_M (\alpha, df) f \, dv. \tag{4.10}$$

Then

$$L_1(\omega) = L_{A1}(-\omega) = \frac{1}{2}\int_M |df|^2 f^2 \, dv$$

obviously holds. As for $I_1(\omega)$, we can easily check that $\omega \in \mathscr{H}$, $\hat{\omega} = \xi_f$ and $\chi(\omega) = f$. Thus we obtain

$$I_1(\omega) = I_{A1}(-\omega) = \frac{1}{2}\mathscr{E}(f).$$

(i) Let $f_1 : [0,1] \to \mathbb{R}$ with $f_1(0) = f_1(1)$ be given by $f_1(x) = 4\sqrt{5}(x - 1/2)^2$. In the case of $f = f_1$ we have

$$\begin{aligned} L_1(\omega) &= \frac{1}{2}\int_{S^1} |df_1|^2 f_1^2 dv = \frac{200}{7}, \\ I_1(\omega) &= \frac{1}{2}\int_{S^1} |df_1|^2 dv = \frac{40}{3}, \end{aligned}$$

and therefore $L_1(\omega) > I_1(\omega)$.

(ii) Next we take $f = f_2$ by $f_2(x) = \sqrt{3}(1 - |2x - 1|)$. Then, for ω given by (4.10), we have

$$\begin{aligned} L_1(\omega) &= \frac{1}{2}\int_{S^1} |df_2|^2 f_2^2 dv = 6, \\ I_1(\omega) &= \frac{1}{2}\int_{S^1} |df_2|^2 dv = 6, \end{aligned}$$

and therefore $L_1(\omega) = I_1(\omega)$. More generally, if df in (4.10) has a constant length, then $L_1(\omega) = I_1(\omega)$ holds on an arbitrary compact manifold M when $b = 0$.

(iii) Let $f_3 = \sqrt{2/3}(1 + \cos(2\pi x))$. Then, for ω corresponding to f_3,

$$\begin{aligned} L_1(\omega) &= \frac{1}{2}\int_{S^1} |df_3|^2 f_3^2 dv = \frac{5\pi^2}{9}, \\ I_1(\omega) &= \frac{1}{2}\int_{S^1} |df_3|^2 dv = \frac{2\pi^2}{3}, \end{aligned}$$

and therefore $L_1(\omega) < I_1(\omega)$.

Example 4.9 Let M be a 2-dimensional torus $S^1 \times S^1$ with flat metric. Assume $b = 0$. Take a function $h : [0,1] \to \mathbb{R}$ which has its support in $(1/3, 2/3)$ and $\int_0^1 |h(x)|^2 dx = 52$. Consider a current ω given as follows:

$$\langle \omega, \alpha \rangle = \int_{S^1\times S^1} (\beta, \alpha)\, dv$$

where $\beta = h(y)dx$. Then obviously $L_{X1}(\omega) = L_1(\omega) = I_1(\omega) = 26$ holds. On the other hand, for $f \in \mathscr{W}$ with $f > 0$ on $S^1 \times [1/3, 2/3]$ and $1/f \in L^2(S^1 \times [1/3, 2/3], dv)$, we define $\eta \in \mathscr{H}$ by

$$\langle \eta, \alpha \rangle = \int_{S^1\times S^1} (\beta, \alpha)\, dv + \int_{S^1\times S^1} (\alpha, df)\, f dv.$$

Then $(1 - Q^*)\eta = \omega$, $\chi(\eta) = f$ and $\hat{\eta} = f^{-2}\beta + \xi_f$ holds. Thus

$$I_{X1}(\omega) \le I_1(\eta) = \frac{1}{2}\int_{S^1\times S^1} \frac{|\beta|^2}{f^2} dv + \frac{1}{2}\mathscr{E}(f).$$

Let $f = f(x, y) = f_2(y)$. Here f_2 is the same one as Example 4.8 (ii). Then

$$I_1(\eta) \le \frac{1}{2}\left(\mathscr{E}(f) + \int_{1/3}^{2/3} \frac{|h(y)|^2}{3(1 - |2y - 1|)^2} dy\right) \le \frac{51}{2} < 26 = I_1(\omega)$$

and therefore $I_{X1}(\omega) < I_1(\omega)$, namely, $I_{X1}(\omega) < L_{X1}(\omega)$.

As we have seen in this subsection, the essential difference between $I_1(\omega)$ and $L_1(\omega)$ comes from the action of ω on exact 1-forms. It is closely related to the asymptotic behavior of the bounded variation part A, which is determined by occupation time integrals. Intuitively, in the case of moderate deviation, the empirical law converges to the normalized invariant measure faster than the rate of decay. Thus the invariant measure always appears in the rate function L_1. In other words, we can observe no information about the position of the process in the rate function since it is homogenized. However, in the case of $g(\lambda) = \sqrt{\lambda}$, we can find the information of the trajectory of the process affecting the value of the rate function I_1. It is *not* homogenized. These observation gives us an intuitive reason why such phenomena as mentioned in Remark 4.7 occur only on I_{X1} since I_{X1} is given by $I_{X1}(\omega) = \inf_{(1-Q^*)\eta=\omega} I_1(\eta)$.

4.3 Empirical laws

Here we give a remark on the connection with the large deviation for empirical law. Recall that $\mathscr{M}_0$ is the totality of all signed measures μ on M with $\mu(M) = 0$ and ι is determined by (4.8). Consider $\mathscr{M}_0$-valued stochastic processes $\{\Xi^\lambda\}_{\lambda>0}$ given by

$$\Xi_t^\lambda = \frac{1}{g(\lambda)\sqrt{\lambda}} \left(\int_0^{\lambda t} \delta_{z_s} ds - \lambda t\, m \right).$$

We use the same symbol ι to denote an extended operator from $C([0,\infty) \to \mathscr{M}_0)$ to $\mathscr{C}_p$. Then we have $\iota(\Xi^\lambda) = \tilde{A}^\lambda$. In particular, when $g(\lambda) = \sqrt{\lambda}$, we have $\iota(\Xi^\lambda) = \bar{A}^\lambda$. Hence we can consider the large deviation or the moderate deviation for occupation measure processes $\{\Xi^\lambda\}_{\lambda>0}$ by regarding them as current-valued processes through the embedding ι.

We should point out that the large deviation for $\bar{A}_1^\lambda$ is a generalization of the large deviation for mean empirical laws in $\mathscr{M}_1$ under the weak topology. To see it, we define the map $S : \mathscr{M}_1 \to \mathscr{M}_0$ by $S(\mu) = \mu - m$. Then we have $\bar{A}_1^\lambda = \iota \circ S(\lambda^{-1} \int_0^\lambda \delta_{z_s} ds)$. In other words, $\bar{A}_1^\lambda$ is the image of the mean empirical law of the diffusion $\{z_t\}_{t\geq 0}$ by the map $\iota \circ S$. If we can apply the inverse contraction principle, the large deviation for empirical law follows as a consequence of our result. For this purpose, we need to verify the assumptions of the inverse contraction principle listed in the following (cf. Remark 3.3):

(i) $\iota \circ S$ is injective.

(ii) $\{I_{A1} < \infty\} \subset \mathrm{Range}(\iota \circ S)$.

(iii) $\iota \circ S$ is continuous.

(iv) $\{\lambda^{-1} \int_0^\lambda \delta_{z_s} ds\}_{\lambda>0}$ is exponentially tight uniformly in $x \in M$.

First we prove the injectivity of $\iota \circ S$. It suffices to show the injectivity of ι. Indeed, suppose that there are $\mu, \nu \in \mathscr{M}_0$ so that $\iota(\mu) = \iota(\nu)$ holds. For each $\phi \in C^\infty(M)$, take a solution u of (3.25) and substitute $\alpha = du$ in (4.8). Then we obtain $\int_M \phi\, d\mu = \int_M \phi\, d\nu$ and therefore $\mu = \nu$. The assumption (ii) follows from Remark 4.4. For (iii) and (iv), we need a careful treatment since these statements depend on topologies we consider on

$\mathscr{D}_{1,-p}$ or $\mathscr{M}_1$. When we consider the weak topology on $\mathscr{M}_1$, (iv) automatically follows since $\mathscr{M}_1$ is compact. However, ι is not continuous under the norm topology on $\mathscr{D}_{1,-p}$. Thus, in order to obtain the continuity, we need to consider the weak topology on $\mathscr{D}_{1,-p}$. Fortunately, the large deviation is preserved when the topology is weakened.

Thus we conclude the following. It is a special case of the Donsker-Varadhan law [7].

Corollary 4.10 *Suppose that $\mathscr{M}_1$ is equipped with a weak topology. Then the mean empirical law of the diffusion $\{z_t\}_{t\geq 0}$ satisfies the large deviation principle in $\mathscr{M}_1$ under $\mathbb{P}_x$ uniformly in $x \in M$ with the rate function given by $I_{A1} \circ \iota \circ S$.*

We can easily check that $I_{A1} \circ \iota \circ S$ coincides with the rate function which was developed in Chapter 6.3 of [5]. They considered diffusion processes whose generator has a Hörmander form satisfying the Hörmander condition. Their results are more general than Corollary 4.10 in this respect.

Remark 4.11 The reason why we weaken the topology is to obtain the exponential tightness for empirical laws. Indeed, under the norm topology on $\mathscr{D}_{1,-p}$, ι is continuous even when we consider the total variation distance on $\mathscr{M}_0$ or $\mathscr{M}_1$.

4.4 The law of the iterated logarithm

As an application of the sample path moderate deviation estimate, we prove the law of the iterated logarithm in our framework.

Theorem 4.12 *Let $g(\lambda) = \sqrt{\log\log\lambda}$.*

(i) *For $p > d$, the family $\{\tilde{Y}^\lambda\}_{\lambda>0}$ is almost surely relatively compact in $\mathscr{C}_p$ and the limit set*

$$\left\{ w \in \mathscr{C}_p \ ; \ \text{there exists } \{\lambda_n\}_{n\in\mathbb{N}} \text{ with } \lambda_n \to \infty \text{ such that } \lim_{n\to\infty} \tilde{Y}^{\lambda_n} = w \right\}$$

almost surely coincides with $\mathscr{K}$ given by $\mathscr{K} := \{w \in \mathscr{C}_p \ ; \ L(w) \leq 1\}$.

(ii) *For $p > d+1$, the family $\{\tilde{X}^\lambda\}_{\lambda>0}$ is almost surely relatively compact in $\mathscr{C}_p$ and the limit set almost surely coincides with $\mathscr{K}_X := \{w \in \mathscr{C}_p \ ; \ L_X(w) \leq 1\}$.*

(iii) *For $p > d+1$, the family $\{\tilde{A}^\lambda\}_{\lambda>0}$ is almost surely relatively compact in $\mathscr{C}_p$ and the limit set almost surely coincides with $\mathscr{K}_A := \{w \in \mathscr{C}_p \ ; \ L_A(w) \leq 1\}$.*

We shall give only the outline of the proof of the first assertion here. Other assertions are similarly proved. Taking it into consideration that L is a good rate function, we can prove the following proposition in the same way as in [3].

Proposition 4.13 *For every $\varepsilon > 0$ and $T > 0$, there exists a positive real number λ_0 almost surely such that for any $\lambda > \lambda_0$, we have $\inf_{w\in\mathscr{K}} \sup_{0\leq t\leq T} \|\tilde{Y}^\lambda_t - w_t\|_{-p} \leq \varepsilon$.*

Thus, the limit set of $\tilde{Y}^\lambda$ is contained in $\mathscr{K}$. It is enough to prove the following proposition for the inverse inclusion.

Proposition 4.14 *Take an arbitrary $w \in \mathscr{K}$ with $L(w) < 1$. Then, for any $\varepsilon > 0$, there exists $c > 1$ such that*

$$\mathbb{P}\left[\limsup_{n\to\infty}\left\{\sup_{0\le t\le T}\|\tilde{Y}_t^{c^n} - w_t\|_{-p} \le \varepsilon\right\}\right] = 1.$$

For the proof, we use the uniformity of the large deviation and the fact that Y is an additive functional of $\{z_t\}_{t\ge 0}$ instead of the Markov property used in [3].

Proof of Proposition 4.14. By a version of the Borel-Cantelli lemma, it suffices to show that $\sum_{n=1}^{\infty}\mathbb{P}[\sup_{0\le t\le T}\|\tilde{Y}_t^{c^n} - w_t\|_{-p} \le \varepsilon | \mathscr{F}_{c^{n-1}T}] = \infty$, where $\mathscr{F}_t = \sigma\{z_s\ ;\ 0 \le s \le t\}$. We have

$$\begin{aligned}
&\left\{\sup_{0\le t\le T}\|\tilde{Y}_t^{c^n} - w_t\|_{-p} \le \varepsilon\right\}\\
&\qquad = \left\{\sup_{0\le t\le c^{-1}T}\|\tilde{Y}_t^{c^n} - w_t\|_{-p} \le \varepsilon\right\} \cap \left\{\sup_{c^{-1}T\le t\le T}\|\tilde{Y}_t^{c^n} - w_t\|_{-p} \le \varepsilon\right\}\\
&\qquad \supset \left\{\sup_{0\le t\le c^{-1}T}\|\tilde{Y}_t^{c^n} - w_t\|_{-p} \le \frac{\varepsilon}{2}\right\}\\
&\qquad\qquad\qquad \cap \left\{\sup_{c^{-1}T\le t\le T}\|(\tilde{Y}_t^{c^n} - \tilde{Y}_{c^{-1}T}^{c^n}) - (w_t - w_{c^{-1}T})\|_{-p} \le \frac{\varepsilon}{2}\right\}\\
&\qquad = \mathscr{A}_1^{(n)} \cap \mathscr{A}_2^{(n)}.
\end{aligned}$$

Then, the Markov property of z_t implies

$$\mathbb{P}\left[\sup_{0\le t\le T}\|\tilde{Y}_t^{c^n} - w_t\|_{-p} \le \frac{\varepsilon}{2} | \mathscr{F}_{c^{n-1}T}\right] \ge 1_{\mathscr{A}_1^{(n)}}\mathbb{P}_{z_{c^{n-1}T}}\left[\tilde{Y}^{c^n} \in E_\varepsilon\right],$$

where

$$E_\varepsilon = \left\{\eta \in \mathscr{C}_p\ ;\ \sup_{0\le t\le (1-c^{-1})T}\|\eta_t - (w_{t+c^{-1}T} - w_{c^{-1}T})\|_{-p} \le \frac{\varepsilon}{2}\right\}.$$

There is a constant C_3 so that

$$\|w_t\|_{-p}^2 \le C_3\,\|\check{w}_t\|_{L_1^2(dm)}^2 \le 2C_3\left\{\int_0^t L_1(\dot{w}_s)^{1/2}ds\right\}^2 \le 2C_3 t L(w) \le 2C_3 t,$$

and Proposition 4.13 implies that for sufficiently large n we have

$$\begin{aligned}
\sup_{0\le t\le c^{-1}T}\|\tilde{Y}_t^{c^n}\|_{-p} &\le \frac{\sqrt{c^{n-1}}g(c^{n-1})}{\sqrt{c^n}g(c^n)}\sup_{0\le t\le T}\|\tilde{Y}_t^{c^{n-1}}\|_{-p}\\
&\le \frac{1}{\sqrt{c}}\left(\sup_{\substack{\varphi\in\mathscr{K}\\0\le t\le T}}\|\varphi_t\|_{-p} + 1\right) \le \frac{1}{\sqrt{c}}(2C_3T + 1).
\end{aligned}$$

Thus, if we fix $c > 0$ large enough, the event $\mathscr{A}_1^{(n)}$ occurs for all sufficiently large n.

On the other hand, there is $\psi \in E_\varepsilon$ so that $L(\psi) < 1$. Indeed, we can take ψ as follows:

$$\psi_t = \begin{cases} w_{t+c^{-1}T} - w_{c^{-1}T} & t \in [0, (1-c^{-1})T), \\ w_T - w_{c^{-1}T} & t \in [(1-c^{-1})T, \infty). \end{cases}$$

Hence Theorem 2.6 implies that there exists $0 < \beta < 1$ so that

$$\mathbb{P}_{z_{c^{n-1}T}}[E_\varepsilon] \geq \exp\left\{-\beta g(c^n)^2\right\} \tag{4.11}$$

holds for sufficiently large n. Since the right-hand side of (4.11) is not summable, the conclusion follows. $\langle$**q.e.d.**$\rangle$

As an application, Theorem 4.12 refines Theorem 2.2 as follows.

Corollary 4.15

$$\lim_{\lambda\to\infty} \frac{1}{h(\lambda)}(X_{\lambda t} - \lambda t e) = \frac{1}{h(\lambda)}(A_{\lambda t} - \lambda t e) = \frac{1}{h(\lambda)} Y_{\lambda t} = 0$$

$\mathbb{P}_x$-almost surely in $\mathscr{C}_p$ for all $x \in M$ if $\lim_{\lambda\to\infty} h(\lambda)/\sqrt{\lambda \log\log \lambda} = \infty$ *holds. In particular,*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^a}(X_{\lambda t} - \lambda t e) = \frac{1}{\lambda^a}(A_{\lambda t} - \lambda t e) = \frac{1}{\lambda^a} Y_{\lambda t} = 0$$

$\mathbb{P}_x$-almost surely in $\mathscr{C}_p$ for all $x \in M$ if and only if $a > 1/2$.

4.5 Long time asymptotics of the Brownian motion on Abelian covering manifolds

Let N be a noncompact Riemannian covering manifold of M with its covering transformation group Γ being Abelian. In this section, we apply our theorems to the study of long time asymptotics of the Brownian motion on N.

For this purpose, we give some preparations following [21]. Let π be the canonical projection from N to M. Take $x_0 \in N$ and let $x_1 = \pi(x_0)$. We denote the fundamental group of M with the base point x_1 by $\pi_1(M, x_1)$ and that of N with the base point x_0 by $\pi_1(N, x_0)$. For simplicity, we abbreviate the base point and write $\pi_1(M, x_1) = \pi_1(M)$ and $\pi_1(N, x_0) = \pi_1(N)$. In this framework, there is a surjection ρ from $\pi_1(M)$ to $\Gamma \simeq \pi_1(M)/\pi_*(\pi_1(N))$. Since Γ is Abelian, it induces the surjective mapping $\bar{\rho}$ from the first homology group $H_1(M;\mathbb{Z}) \simeq \pi_1(M)/[\pi_1(M), \pi_1(M)]$ to Γ. Here $[\pi_1(M), \pi_1(M)]$ is the commutator subgroup of $\pi_1(M)$. Moreover, we obtain the extended map $\bar{\rho}^{\otimes\mathbb{R}}$: $H_1(M;\mathbb{R}) \to \Gamma \otimes \mathbb{R}$ by taking tensor product with $\mathbb{R}$. For simplicity, we denote the extension $\bar{\rho}^{\otimes\mathbb{R}}$ by the same symbol $\bar{\rho}$. Then we obtain the adjoint injective map $\bar{\rho}^T$: $\mathrm{Hom}(\Gamma, \mathbb{R}) \to H^1(M;\mathbb{R})$. Note that the first cohomology group $H^1(M;\mathbb{R})$ is identified with the totality of harmonic 1-forms $\mathbf{H}_1(M)$ (see [26], for example). Since $\mathbf{H}_1(M)$ is regarded as a subspace of $\mathscr{D}_{1,p}$, we can pull-back the norm on $\mathrm{Hom}(\Gamma, \mathbb{R})$. Note that the induced topology on $\mathrm{Hom}(\Gamma, \mathbb{R})$ coincides with what comes from $L_1^2(dv)$ since $\|\alpha\|_p =$

$\|\alpha\|_{L^2_1(dv)}$ holds for each $\alpha \in \mathbf{H}_1(M)$. Then, it determines the dual norm on $\Gamma \otimes \mathbb{R}$ that makes $\Gamma \otimes \mathbb{R}$ a normed space. We define the map $\varphi : N \to \Gamma \otimes \mathbb{R}$ by

$$ {}_{\Gamma\otimes\mathbb{R}}\langle \varphi(x), \alpha \rangle_{\mathrm{Hom}(\Gamma,\mathbb{R})} = \int_c \pi^*(\bar{\rho}^T(\alpha)) \tag{4.12} $$

for each $\alpha \in \mathrm{Hom}(\Gamma, \mathbb{R})$. Here, $\pi^*(\bar{\rho}^T(\alpha))$ is the pull-back of $\bar{\rho}^T(\alpha)$ by π and c is a piecewise smooth path from x_0 to x. Note that the line integral in (4.12) is independent of the choice of c.

According to [21], through the spectral-geometric approach, we know the following precise long time asymptotics of the heat kernel $p(t,x,y)$ associated with $\Delta/2$ on N:

$$ \lim_{t\uparrow\infty} \left\{ (2\pi t)^{r/2} p(t,x,y) - C(N) \exp\{-d_\Gamma(x,y)^2/2t\} \right\} = 0 $$

uniformly in x and y. Here $r = \operatorname{rank}\Gamma$ and $C(N)$ is a constant determined explicitly in terms of Γ and the Riemannian metric. d_Γ is determined by $d_\Gamma(x,y) := |\varphi(x) - \varphi(y)|_{\Gamma\otimes\mathbb{R}}$. Roughly speaking, this asymptotic behavior indicates us that the heat kernel $p(t,x,y)$ approaches to the pull-back of the heat kernel on $\Gamma \otimes \mathbb{R}$ by φ. Thus, we expect that there is a connection between the long time asymptotics of the heat kernel and the asymptotics of $\varphi(B_t)$, where $(B_t, \bar{\mathbb{P}}_{x_0})$ is the Brownian motion on N starting at x_0. We would like to give a probabilistic approach to this problem and our main theorem gives some information about the asymptotics of $\varphi(B_t)$.

We remark that ${}_{\Gamma\otimes\mathbb{R}}\langle \varphi(B_t), \alpha \rangle_{\mathrm{Hom}(\Gamma,\mathbb{R})}$ coincides with the stochastic line integral of α along $\pi(B)$. Note that $\{\pi(B_t)\}_{t\ge 0}$ is a Brownian motion on M starting at x_1 under $\bar{\mathbb{P}}_{x_0}$. Fix $t > 0$. The approximation theorem of stochastic line integrals (Theorem 6.1 of [15]) guarantees the existence of the sequence of random paths $\{c^{(\ell)}\}_{\ell\in\mathbb{N}}$ on M which are all piecewise geodesics and $c^{(\ell)}_t = \pi(B_t)$ so that

$$ \lim_{\ell\to\infty} \bar{\mathbb{E}}_{x_0} \left[\sup_{0\le s\le t} \left| \int_{c^{(\ell)}[0,s]} \bar{\rho}^T(\alpha) - \int_{\pi(B)[0,s]} \bar{\rho}^T(\alpha) \right|^2 \right] = 0, $$

where $\int_{\pi(B)[0,t]} \bar{\rho}^T(\alpha)$ is the stochastic line integral of $\bar{\rho}^T(\alpha) \in \mathbf{H}_1(M)$ along $\pi(B)$. Now we have

$$ {}_{\Gamma\otimes\mathbb{R}}\langle \varphi(B_t), \alpha \rangle_{\mathrm{Hom}(\Gamma,\mathbb{R})} = \int_{c^{(\ell)}[0,t]} \alpha $$

for all ℓ by virtue of the independence of the choice of the path c in (4.12). Thus we obtain

$$ {}_{\Gamma\otimes\mathbb{R}}\langle \varphi(B_t), \alpha \rangle_{\mathrm{Hom}(\Gamma,\mathbb{R})} = \int_{\pi(B)[0,t]} \alpha \quad t \ge 0, \text{ a.s.} \tag{4.13} $$

Let us define $\tilde{\rho} : \mathscr{D}_{1,-p} \to \Gamma \otimes \mathbb{R}$ by

$$ {}_{\Gamma\otimes\mathbb{R}}\langle \tilde{\rho}(\omega), \alpha \rangle_{\mathrm{Hom}(\Gamma,\mathbb{R})} = \left\langle \omega, \bar{\rho}^T(\alpha) \right\rangle. $$

Then, for the current-valued process X_t associated with $\pi(B_t)$, $\varphi(B_t) = \tilde{\rho}(X_t)$ holds almost surely since the left hand side of (4.13) is clearly continuous with respect to α.

Moreover $\tilde{\rho}(X_t) = \tilde{\rho}(Y_t)$ holds by (2.1), the explicit form of the bounded variation part of X_t. Note that $\tilde{\rho}$ is continuous since so is $\bar{\rho}^T$ as a map from $\mathrm{Hom}(\Gamma, \mathbb{R})$ to $\mathscr{D}_{1,p}$.

Accordingly, we can apply our main theorem with the aid of the contraction principle and obtain the following estimate for $\varphi(B_t)$.

Corollary 4.16 *We set* $\tilde{V}_t^\lambda := g(\lambda)^{-1}\lambda^{-1/2}\varphi(B_{\lambda t})$*. Assume* $g(\lambda) = o(\sqrt{\lambda})$ *as* $\lambda \to \infty$*. Then, we have*

$$\begin{aligned}
\limsup_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \bar{\mathbb{P}}_{x_0}[\tilde{V}^\lambda \in \mathscr{A}] &\leq -\inf_{w\in\bar{\mathscr{A}}} \tilde{L}(w), \\
\liminf_{\lambda\to\infty} \frac{1}{g(\lambda)^2} \log \bar{\mathbb{P}}_{x_0}[\tilde{V}^\lambda \in \mathscr{A}] &\geq -\inf_{w\in\mathscr{A}^\circ} \tilde{L}(w)
\end{aligned}$$

for any Borel set $\mathscr{A} \subset C([0,\infty) \to \Gamma \otimes \mathbb{R})$*. Here*

$$\tilde{L}(w) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^\infty |\dot{w}_s|^2_{\Gamma\otimes\mathbb{R}} ds & \text{if } w_0 = 0 \text{ and } w \text{ is absolutely continuous,} \\ \infty & \text{otherwise.} \end{cases}$$

Proof. All we need to prove is $\inf_{\tilde{\rho}(\eta_t)=w_t} L(\eta) = \tilde{L}(w)$. It follows in the similar way as (ii-a) of Proposition 4.6. Indeed, let P be the orthonormal projection on $L^2_1(dv)$ to the range of $\bar{\rho}^T$. Then for $\omega \in \Gamma \otimes \mathbb{R}$ and $\eta \in L^2_1(dv)$ with $\tilde{\rho}(\eta) = \omega$, we have

$$\|\eta\|^2_{L^2_1(dv)} = \|P\eta\|^2_{L^2_1(dv)} + \|(1-P)\eta\|^2_{L^2_1(dv)}$$

and

$$\begin{aligned}
\|P\eta\|_{L^2_1(dv)} &= \sup_{\|\alpha\|_{L^2_1(dv)}=1} \left|(\eta, P\alpha)_{L^2_1(dv)}\right| = \sup_{|\gamma|_{\mathrm{Hom}(\Gamma,\mathbb{R})}=1} \left| {}_{\Gamma\otimes\mathbb{R}}\langle \omega, \gamma\rangle_{\mathrm{Hom}(\Gamma,\mathbb{R})}\right| \\
&= |\omega|_{\Gamma\otimes\mathbb{R}}\,.
\end{aligned}$$

Next we will construct an element $\eta_0 \in \mathscr{D}_{1,-p} \cap \mathscr{H}'$ with $(1-P)\check{\eta}_0 = 0$ and $\tilde{\rho}(\eta_0) = \omega$. Let $\hat{\omega} \in \mathrm{Hom}(\Gamma, \mathbb{R})$ be an element corresponding to $\omega \in \Gamma \otimes \mathbb{R}$. We define η_0 by the following relation:

$$\langle \eta_0, \alpha\rangle_p = \int_M (\bar{\rho}^T(\hat{\omega}), \alpha) dv, \quad \alpha \in \mathscr{D}_{1,p}.$$

Then, $(1-P)\check{\eta}_0 = (1-P)\bar{\rho}^T(\hat{\omega}) = 0$ clearly holds. In addition,

$$\begin{aligned}
{}_{\Gamma\otimes\mathbb{R}}\langle \tilde{\rho}(\eta_0), \gamma\rangle_{\mathrm{Hom}(\Gamma,\mathbb{R})} &= \int_M (\bar{\rho}^T(\hat{\omega}), \bar{\rho}^T(\gamma)) dv = (\hat{\omega}, \gamma)_{\mathrm{Hom}(\Gamma,\mathbb{R})} \\
&= {}_{\Gamma\otimes\mathbb{R}}\langle \omega, \gamma\rangle_{\mathrm{Hom}(\Gamma,\mathbb{R})}\,.
\end{aligned}$$

Thus the conclusion follows. $\langle$**q.e.d.**$\rangle$

Remark 4.17 In the same way as in the proof of the Corollary 4.16, we can prove the large deviation for $\bar{V}_t^\lambda := \lambda^{-1}\varphi(B_{\lambda t})$. But the rate function is more complicatedly described, which is given in [1].

In the same manner as section 4.4, we can prove the following law of the iterated logarithms.

Corollary 4.18 *Let $g(\lambda) = \sqrt{\log \log \lambda}$. Then the limit set of $\{\tilde{V}^\lambda\}_{\lambda>0}$ coincides with the compact set $\mathscr{K}_1$ given by*

$$\mathscr{K}_1 := \{w \in C([0,\infty) \to \Gamma \otimes \mathbb{R}) \; ; \; L_1(w) \leq 1\}.$$

Note that we have the following relation between $d_\Gamma(\cdot,\cdot)$ and the Riemannian distance $\mathrm{dist}(\cdot,\cdot)$: there are positive constants c_1, c_2 and c_3 such that for all $x, y \in N$

$$c_1 d_\Gamma(x,y) \leq \mathrm{dist}(x,y) \leq c_2 d_\Gamma(x,y) + c_3 \tag{4.14}$$

(see [21]). Now Corollary 4.18 gives us some information about the divergence order of the Brownian motion on N.

Corollary 4.19 *There is a nonrandom constant c with $c_1 \leq c \leq c_2$ so that*

$$\limsup_{t\to\infty} \frac{\mathrm{dist}(B_t, B_0)}{\sqrt{2t \log\log t}} = c$$

$\bar{\mathbb{P}}_{x_0}$*-almost surely.*

Proof. Note that the invariant σ-field of the Brownian motion on N is trivial. It is a consequence of [19]. This fact ensures the existence of the constant $c \in [0,\infty]$. By virtue of (4.14), the rest to be proved is

$$\limsup_{t\to\infty} \frac{d_\Gamma(B_t, B_0)}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \left| \frac{\varphi(B_t)}{\sqrt{2t\log\log t}} \right|_{\Gamma\otimes\mathbb{R}} = 1.$$

But Corollary 4.16 asserts that

$$\limsup_{t\to\infty} \left| \frac{\varphi(B_t)}{\sqrt{2t\log\log t}} \right|_{\Gamma\otimes\mathbb{R}} = \frac{1}{\sqrt{2}} \sup_{w\in\mathscr{K}_1} |w_1|_{\Gamma\otimes\mathbb{R}} \leq \sup_{w\in\mathscr{K}_1} \sqrt{L(w)} = 1.$$

The inequality above comes from the Schwarz inequality and we can easily show that the equality actually holds. $\langle$**q.e.d.**$\rangle$

References

[1] M. Avellaneda, *Large deviation estimates and the homological behavior of brownian motion on manifolds*, Ph.D. thesis, University of Minnesota, 1985.

[2] P. Baldi, *Large deviations and stochastic homogenization*, Ann. Mat. Pura Appl. **151** (1988), 161–177.

[3] ———, *Large deviations for diffusion processes with homogenization and applications*, Ann. Probab. **19** (1991), 509–524.

[4] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, second ed., Applications of Mathematics, 38, Springer-Verlag, New York, 1998.

[5] J. D. Deuschel and D. W. Stroock, *Large deviations*, Academic Press, Boston, 1989.

[6] J. Diestel and J. J. Jr. Uhl, *Vector measures*, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977.

[7] M. D. Donsker and S. R. S. Varadhan, *Asymptotic evaluation of certain markov process expectations for large time. I*, Comm. Pure Appl. Math. **28** (1975), 1–47.

[8] N. Dunford and J. T. Schwartz, *Linear operators. I. general theory*, Pure and Applied Mathematics, Vol. 7, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958.

[9] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric markov processes*, de Gruyter Studies in Mathematics, 19, Walter de Gruyter & Co., Berlin, 1994.

[10] M. P. Gaffney, *Asymptotic distributions associated with the laplacian for forms*, Comm. Pure. Appl. Math. **11** (1958), 535–545.

[11] J. Gärtner, *On large deviations from the invariant measure*, Theory Probab. Appl. **22** (1977), 24–39.

[12] C. J. Holland, *A minimum principle for the principal eigenvalue for second-order linear elliptic differential equations with natural boundary conditions*, Comm. Pure and Appl. Math. **31** (1978), no. 4, 509–519.

[13] L. Hörmander, *Linear partial differential operators*, Springer-Verlag, 1963.

[14] N. Ikeda, *Limit theorems for a class of random currents*, Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), Academic Press, Boston, MA, 1987, pp. 181–193.

[15] N. Ikeda and S. Manabe, *Integral of differential forms along the path of diffusion processes*, Publ. RIMS, Kyoto Univ. **15** (1979), 827–852.

[16] N. Ikeda and Y. Ochi, *Central limit theorems and random currents*, Lecture Notes in Contr. and Inform. Sciences **78** (1986), 195–205.

[17] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, second ed., North-Holland Mathematical Library, 24, North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1989.

[18] K. Itô, *Foundation of stochastic differential equations in infinite dimensional spaces*, CBMS-NSF, Regional Conference Series in Applied Mathematics 47, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.

[19] V. A. Kaĭmanovich, *Brownian motion and harmonic functions on covering manifolds. an entropy approach*, Soviet Math. Dokl. **33** (1986), no. 3, 812–816.

[20] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, 1980.

[21] M. Kotani and T. Sunada, *Albanese maps and off diagonal long time asymptotics for the heat kernel*, Comm. Math. Phys. **209** (2000), 633–670.

[22] S. Manabe, *Stochastic intersection number and homological behavior of diffusion processes on riemannian manifolds*, Osaka J. Math. **19** (1982), 429–454.

[23] ———, *Large deviation for a class of current-valued processes*, Osaka J. Math. **29** (1992), 89–102.

[24] Y. Ochi, *Limit theorems for a class of diffusion processes*, Stochastics **15** (1985), 251–269.

[25] M. Sion, *On general minimax theorems*, Pacific J. Math. **8** (1958), 171–176.

[26] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, 94, Springer-Verlag, New York-Berlin, 1983.