Radial processes on $\mathsf{RCD}^*(K,N)$ spaces

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Abstract

In this paper, we show a stochastic expression of radial processes of Brownian motions on $\mathsf{RCD}^*(K,N)$-spaces. The expression holds under the law for all starting point provided the reference point is sufficiently regular. We further prove that the regularity condition is satisfied for almost every reference point on $\mathsf{RCD}^*(K,N)$-space. Our results extend the comparison theorems over Alexandrov spaces proved by [30, 37].

1 Introduction

In this paper, we show a stochastic expression of radial processes of Brownian motions on $\mathsf{RCD}^*(K,N)$-spaces and its applications. Let (M,g) be a complete smooth Riemannian manifold and $d(x,y) = d_g(x,y)$, $x,y \in M$ is its Riemannian distance. We put $r_p(x) := d(x,p)$ for $x,p \in M$ and call r_p a *radial function*. Let us consider the Brownian motion $\mathbf{X} = (X_t, \mathbb{P}_x)$ on M starting from $x \in M$, which is a diffusion processes associated with the Laplace-Bertrami operator Δ_M. Then $r_p(X_t)$ is called the *radial process* of $\mathbf{X}$. It is proved in [26, Theorem 1.1] that the radial process on M has the following expression:

$$r_p(X_t) - r_p(X_0) = \sqrt{2}B_t + \int_0^t \Delta_M r_p(X_s)\,\mathrm{d}s - L_t, \qquad t < \zeta. \tag{1.1}$$

Here B_t is a one-dimensional standard Brownian motion, L_t is a non-decreasing process which increases only at $X_t \in C(p)$, where $C(p)$ is the cut-locus of (M,g) with respect to p, and ζ is the life time of $\mathbf{X}$. Moreover, $\Delta_M r_p = 0$ is assumed at the point on which r_p is not differentiable.

The radial process of a Brownian motion on a Riemannian manifold has played an important role in many applications to differential geometry. As an application of the expression of radial process, under the lower bound of Ricci curvature (resp. upper bound of sectional curvature), we can compare the radial process to that of the space form of constant curvature, consequently we obtain a comparison of the heat kernel

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to that of the space form of constant curvature. From the heat kernel comparison, S. Y. Cheng's eigenvalue comparison theorem follows under the lower bound of Ricci curvature. Moreover, the stochastic expression of radial process yields a stochastic proof of S. Y. Cheng's Liouville theorem for sublinear growth harmonic functions.

The stochastic expression of radial processes like (1.1) has been also expected on singular metric measure spaces with a lower curvature bound, such as Alexandrov spaces or more generally for $\mathsf{RCD}^*(K,N)$-spaces. Due to the technical reason, this was not accomplished yet, because the cut-locus may be dense even on an Alexandrov space. However, we can consider another (but similar) type of stochastic expression of radial processes based on the Laplacian comparison theorem holding for $\mathsf{RCD}^*(K,N)$-spaces, in particular, we prove the semimartingale property of radial processes on $\mathsf{RCD}^*(K,N)$-spaces.

Laplacian comparison theorem on Alexandrov spaces was done by von Renesse [37] under an additional condition. He proved a heat kernel comparison and comparison of radial processes of weaker type, because the semimartingale property of radial processes was not proved in [37]. The Laplacian comparison theorem, which asserts $\Delta r_p \leq (N-1)\cot_\kappa \circ r_p$ in a distributional sense (see (2.5) below), for Alexandrov spaces was done by [30] without assuming the additional assumption as in [37], however a comparison of radial processes of weak type is only announced, because of the lack of semimartingale property of radial processes. Laplacian comparison theorem on $\mathsf{RCD}(K,N)$-space was proved by Gigli [19, Theorem 5.14] based on the technique of optimal mass transport theory. Since $\mathsf{RCD}^*(K,N)$-space coincides with $\mathsf{RCD}(K,N)$-space by Cavalletti-Milman [9], Laplacian comparison theorem holds for $\mathsf{RCD}^*(K,N)$-space (see [19, Remark 5.16] also).

Hereafter we take an $\mathsf{RCD}^*(K,N)$-space $(X,d,\mathfrak{m})$, which is a geodesic metric measure space having a notion of lower Ricci bound by $K \in \mathbb{R}$ together with a notion of upper bound of dimensions by $N \in [1,\infty[$ (see Subsection 2.1 for the precise definition). Based on the Laplacian comparison theorem on $\mathsf{RCD}^*(K,N)$-space, we first prove the following stochastic expression of the radial process:

$$r_p(X_t) - r_p(X_0) = \sqrt{2}B_t + (N-1)\int_0^t \cot_\kappa \circ r_p(X_s)\,\mathrm{d}s - A_t \tag{1.2}$$

holds until X_t hits p under the law $\mathbb{P}_x$ for all quasi-every starting point $x \in X \setminus \{p\}$. Here A_t is a positive continuous additive functional. The expression (1.2) is different from (1.1), because we do not use the notion of cut-locus. Under a condition (R1) (see Definition 3.5 below) to the reference point p, we can strengthen the statement so that (1.2) holds for all time $t \in [0,+\infty[$ under the law $\mathbb{P}_x$ for quasi-every starting point $x \in X$. These statements are very weak in applying, since it has to neglect some exceptional set among all possible starting points. We further refine the statement so that (1.2) holds until X_t hits p under the law $\mathbb{P}_x$ for all $x \in X \setminus \{p\}$, and it holds for $t \in [0,+\infty[$ under the law $\mathbb{P}_x$ for all $x \in X$ provided p verifies a stronger condition (R2) than (R1) (see Definition 4.5 for (R2)). These refinements will be done by the global upper Gaussian estimate for the heat kernel of $\mathsf{RCD}^*(K,N)$-space established in [24]. From (1.2), we prove a comparison of radial processes, comparison of heat kernels, S.Y. Cheng's eigenvalue comparison theorem and S.Y. Cheng's Liouville theorem for sublinear growth harmonic functions. Note that the comparison of radial processes is not a weak type, because the radial process is a semimartingale in view of (1.2) provided p verifies the condition ($\mathsf{R2}$).

Let (X, d) be an N-dimensional Alexandrov space with $\mathrm{curv}(X) \geq \kappa$ and $N \geq 2$, which is a metric space having a notion of lower sectional curvature bound by $\kappa \in \mathbb{R}$ (see [29, 30] and the references therein for the definition of Alexandrov spaces). Then, for the N-dimensional Hausdorff measure $\mathfrak{m} = \mathcal{H}^N$, $(X, d, \mathfrak{m})$ is an $\mathsf{RCD}^*(K, N)$-space with $K = (N-1)\kappa$ as proved by [35] (see [44] and [1] also). Moreover, $(X, d, \mathfrak{m})$ satisfies the Bishop inequality (3.2). In particular, the condition ($\mathsf{R2}$) is satisfied for any reference point $p \in X$ for Alexandrov space $(X, d, \mathfrak{m})$ (see Lemma 4.6 and a comment after it). This means that (1.2) holds for all $t \in [0, +\infty[$ under $\mathbb{P}_x$ for all points $x, p \in X$ for Alexandrov space (X, d).

The constitution of this paper as follows. In the next section we introduce our framework and state our main result in a simplified form. Most of notations we require in our argument are summarized there. In Section 3, we will prove a preliminary version of theorem. We first give a deeper study of the Laplacian comparison theorem, which will be used in the subsequent section, and prove an expression of the radial process from quasi-every starting points. The key idea is to realize the Laplacian to the radial function as a Radon measure. By checking the smoothness of the measure, we can consider the additive functional corresponding to it. Then we can obtain a more detailed description of the Fukushima decomposition for the radial process. In order to refine the preliminary result, we will provide some estimates of resolvent operators in Section 4. Based on it, we will show that the Radon measure describing the Laplacian comparison theorem is a smooth measure in the strict sense. The global Gaussian heat kernel upper bound plays a prominent role there. The proof of our main theorem will be finished in Section 5. We also discuss some related results. In Section 6, we will show applications of our main theorem. As mentioned above, we prove comparison theorems and a Liouville type theorem. Note that, both in Sections 3 and 5, we state our theorem in two different ways: a weaker expression (up to the hitting time to the reference point) and a stronger result (beyond the hitting time) under an additional assumption. In Section 7, we will discuss when we can obtain our expression of the radial process beyond the hitting time to the reference point. We consider two situations: The case that the Brownian motion does not hit the reference point and the case that the assumption of our main theorem for a stronger result is satisfied. In both cases, we require a detailed analysis based on very recent results on a local structure of RCD spaces. The reason why we arranged this section at this position is on this fact. Indeed, the argument looks somewhat different from the ones in other sections. In our paper, we try to cover general situations among RCD spaces as much as possible. On the other hand, the case $N = 1$, where N is the upper dimension bound of the space, is somewhat exceptional from the viewpoint of the Laplacian comparison theorem. To deal with this case, we classify such spaces in appendix by refining the previous result [27].

2 Framework and Main Results

In this section, we will introduce our framework and state the main result. In the next subsection we define the RCD spaces. There are already many papers on RCD spaces and we try to make our description minimal. In Subsection 2.2, we state our main results together with a small observation. In Subsection 2.3, we prepare some properties of RCD spaces which will be used in the sequel.

2.1 Framework

Let $(X, d, \mathfrak{m})$ be a metric measure space, i.e., (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a σ-finite Borel measure on X. Suppose that $\mathfrak{m}(B_r(x)) \in]0, \infty[$ for any metric ball $B_r(x)$ of radius $r > 0$ centered at $x \in X$. In particular, $\operatorname{supp} \mathfrak{m} = X$. Suppose also that d is a geodesic distance, i.e., for any $x_0, x_1 \in X$, there exists $\gamma : [0, 1] \to X$ such that $\gamma(i) = x_i$ $(i = 0, 1)$ and $d(\gamma(s), \gamma(t)) = |s - t| d(x_0, x_1)$. We call such γ a minimal geodesic joining x_0 and x_1.

To define RCD spaces, we introduce the Cheeger's energy functional. Let $\mathcal{C}^{\mathrm{Lip}}(X)$ be the set of Lipschitz functions on X. Let $\mathsf{Ch} : L^2(X; \mathfrak{m}) \to [0, \infty]$ be given by

$$\begin{aligned} \mathsf{Ch}(f) &:= \frac{1}{2} \inf \left\{ \varliminf_{n \to \infty} \int_X |Df_n|^2 \, d\mathfrak{m} \;\middle|\; f_n \in \mathcal{C}^{\mathrm{Lip}}(X) \cap L^2(X; \mathfrak{m}), f_n \to f \text{ in } L^2(X; \mathfrak{m}) \right\}, \\ \mathcal{D}(\mathsf{Ch}) &:= \{ f \in L^2(X; \mathfrak{m}) \mid \mathsf{Ch}(f) < \infty \}, \end{aligned}$$

where $|Dg| : X \to [0, \infty]$ is local Lipschitz constant of $g : X \to \mathbb{R}$ defined by

$$|Dg|(x) := \varlimsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

For $f \in L^2(X; \mathfrak{m})$ with $\mathsf{Ch}(f) < \infty$, we have $|Df|_w \in L^2(X; \mathfrak{m})$ such that

$$\mathsf{Ch}(f) = \frac{1}{2} \int_X |Df|_w^2 \, d\mathfrak{m}. \tag{2.1}$$

We call $|Df|_w$ minimal weak upper gradient of f. To state the precise definition of $|Df|_w$, we need some notions in optimal transport, and it is done in Subsection 2.3 below. We call $(X, d, \mathfrak{m})$ to be *infinitesimally Hilbertian*, if Ch satisfies the parallelogram law. Note that minimal weak upper gradients also satisfies the parallelogram law if $(X, d, \mathfrak{m})$ is infinitesimally Hilbertian (see [4]). It means that there exists a bilinear form $\langle D\cdot, D\cdot \rangle : \mathcal{D}(\mathsf{Ch}) \times \mathcal{D}(\mathsf{Ch}) \to L^1(X; \mathfrak{m})$ such that $\langle Df, Df \rangle = |Df|_w^2$. We denote the (non-positive definite) selfadjoint operator associated with $2\mathsf{Ch}$ by Δ. Throughout this paper, we will assume $K \in \mathbb{R}$ and $N \in [1, \infty[$.

Definition 2.1 *We call that $(X, d, \mathfrak{m})$ is $\mathsf{RCD}^*(K, N)$ space if it satisfies the following conditions:*

(i) *$(X, d, \mathfrak{m})$ is infinitesimally Hilbertian.*

(ii) *There exists $x_0 \in X$ and a constant $c, C > 0$ such that $V_r(x_0) \le C e^{cr^2}$.*

(iii) *If $f \in \mathcal{D}(\mathsf{Ch})$ satisfies $|Df|_w \le 1$ $\mathfrak{m}$-a.e., then f has a 1-Lipschitz representative.*

(iv) *For any $f \in \mathcal{D}(\Delta)$ with $\Delta f \in \mathcal{D}(\mathsf{Ch})$ and $g \in \mathcal{D}(\Delta) \cap L^\infty(X;\mathfrak{m})$ with $g \ge 0$ and $\Delta g \in L^\infty(X;\mathfrak{m})$,*

$$\frac{1}{2}\int_X |Df|^2 \Delta g \, \mathrm{d}\mathfrak{m} - \int_X \langle Df, D\Delta f\rangle g \, \mathrm{d}\mathfrak{m} \ge K \int_X |Df|_w^2 g \, \mathrm{d}\mathfrak{m} + \frac{1}{N}\int_X (\Delta f)^2 g \, \mathrm{d}\mathfrak{m}.$$

The condition (iv) of Definition 2.1 is a weak form of the Bochner inequality and it is well known that it holds on Riemannian manifolds with $\mathrm{Ric} \ge K$ and $\dim \le N$. The corresponding characterization by Bakry-Émery Ricci tensor also exists for weighted Riemannian manifolds. See [7, 14] and references therein for more details, related results and other equivalent conditions. For $N < \infty$, this sort of condition is first introduced in [19] as $\mathsf{RCD}(K,N)$ space. It is an infinitesimally Hilbertian space satisfying $\mathsf{CD}(K,N)$ condition introduced in [42] (see [31] also). As mentioned in the introduction, recently, F. Cavalletti and E. Milman [9] show that $\mathsf{RCD}(K,N)$ condition is indeed equivalent to $\mathsf{RCD}^*(K,N)$ condition.

When $N = 1$, we may (and will) assume $K = 0$ because $\mathsf{RCD}^*(K,1)$ space $(X,d,\mathfrak{m})$ is isomorphic to $\mathbb{R}$, $[0,+\infty[$, $\mathbb{S}^1(r) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = r^2\}$ for some $r > 0$, or $[0,\ell]$ for some $\ell > 0$ and $\mathfrak{m}$ is the one-dimensional Hausdorff measure (see Proposition A.1 below; This is a refinement of [27]).

Let $\mathcal{E} := 2\mathsf{Ch}$ and $\mathcal{F} := \mathcal{D}(\mathsf{Ch})$. Then $(\mathcal{E},\mathcal{F})$ is strongly local regular Dirichlet form. Indeed, quasi-regularity is shown in [2] when $N = \infty$ and the regularity follows from the facts that Lipschitz functions in $L^2(X;\mathfrak{m})$ is dense in $\mathcal{F}$ with respect to the Sobolev norm $\|\cdot\|_{W^{1,2}}$ (see [5]) and that (X,d) is locally compact by the Bishop-Gromov inequality in the next subsection. The condition (ii) of Definition 2.1 ensures that $(\mathcal{E},\mathcal{F})$ is conservative (see [3, Theorem 4.20]). We bring several concepts and notations in the theory of Dirichlet form from [17].

2.2 Main Results

For $p \in X$, let the radial function $r_p : X \to \mathbb{R}$ be given by $r_p(x) := d(p,x)$. Let

$$\kappa := \begin{cases} \dfrac{K}{N-1} & (N > 1), \\ 0 & (N = 1), \end{cases}$$

$\mathfrak{s}_\kappa(t) := \sin(\sqrt{\kappa}t)/\sqrt{\kappa}$ and $\cot_\kappa(t) := (\log \mathfrak{s}_k(t))' = \mathfrak{s}_k'(t)/\mathfrak{s}_k(t)$. We interpret $\mathfrak{s}_0(t)$ as $\lim_{\kappa\to 0}\mathfrak{s}_\kappa(t)(= t)$.

Let $\mathbf{X} = (\Omega, \mathcal{M}, (X_t)_{t\ge 0}, (\mathbb{P}_x)_{x\in X})$ with a filtration $(\mathscr{F}_t)_{t\ge 0}$ be the diffusion process canonically associated with $(\mathcal{E},\mathcal{F})$. Let σ_p be the first hitting time of $\mathbf{X}$ to $\{p\}$. Our main theorem is a stochastic expression of the radial process $r_p(X_t)$. A simplified version can be stated as follows:

Theorem 2.2 (Stochastic expression of radial process (simplified version))

(i) *For all $x \in X \setminus \{p\}$, there exists a positive continuous additive functional A_t in the strict sense and a one-dimensional standard Brownian motion B such that*

$$r_p(X_t) - r_p(X_0) = \sqrt{2}B_t + (N-1)\int_0^t \cot_\kappa \circ r_p(X_s)\,\mathrm{d}s - A_t \tag{2.2}$$

holds for all $t \in [0, \sigma_p[$ $\mathbb{P}_x$-a.s.

(ii) *Suppose* (R2) *in Definition 4.5 below holds. Then* (2.2) *holds for all $t \in [0, +\infty[$ $\mathbb{P}_x$-a.s. for all $x \in X$. In particular, $r_p(X_t)$ is a semimartingale.*

Readers may wonder how restrictive the condition (R2) is. Indeed, it is not extremely restrictive since $\mathfrak{m}$-a.e. $p \in X$ verifies (R2) when the space is *not essentially one-dimensional* (see Proposition 7.2 below).

A version of Theorem 2.2 for $f(r_p(X_t))$ with $f \in C^2(\mathbb{R})$ (Theorem 3.11) admitting only q.e. starting points is shown in Section 3. As a corollary of it, we extend the Laplacian comparison theorem (Corollary 3.13). By using it, we will show the full version (that is, for $f(r_p(X_t))$) of Theorem 2.2 (Theorem 5.3). Note that Corollary 3.13 completely extends the one proved in [37, Theorem I] on a class of Alexandrov spaces.

2.3 Basic properties

As mentioned after Definition 2.1 above, RCD condition extends the class of spaces from Riemannian manifolds with a lower Ricci curvature bound. Indeed, many geometric or analytic properties on Riemannian manifolds are extended to $\mathsf{RCD}^*(K, N)$ spaces. We would like to review some of them which will be required in this paper. First we see geometric properties for diameter and volume growth:

- (Bonnet-Myers theorem) When $K > 0$,

$$\mathrm{diam}(X) \le \frac{\pi}{\sqrt{\kappa}}, \tag{2.3}$$

  where $\mathrm{diam}(X) := \sup_{x,y\in X} d(x, y)$.

- (Bishop-Gromov inequality) For $0 < r < R < \pi/\sqrt{\kappa_+}$ and $x \in X$,

$$\frac{V_R(x)}{V_r(x)} \le \frac{\bar{V}_R}{\bar{V}_r}, \tag{2.4}$$

  where $\bar{V}_r := \displaystyle\int_0^r \mathfrak{s}_\kappa^{N-1}(u)\,\mathrm{d}u$. Note that $\bar{V}_r$ under $N \in \mathbb{N}$ differs from the volume of a metric ball or radius r on a space form up to a multiplicative constant.

These are indeed a consequence of the measure contraction property $\mathsf{MCP}(K, N)$ in [34, 42] which is weaker than $\mathsf{RCD}^*(K, N)$ condition (see [10, 14]). Note that the Bishop-Gromov inequality implies that $\mathfrak{m}$ satisfies the volume doubling property locally uniformly, that is, for each $R > 0$, there exists $C_D > 0$ such that $\mathfrak{m}(B_{2r}(x)) \le C_D\mathfrak{m}(B_r(x))$ for any $x \in X$ and $r \le R$. As an important consequence of this property,

(X, d) is locally compact. Thus $\mathsf{RCD}^*(K, N)$ space is compact if $K > 0$ by the Bonnet-Myers theorem.

We turn to analytic properties. The most important one in relation with our result is the following Laplacian comparison theorem: When $N > 1$, we have $r_p \in \mathcal{F}_{\mathrm{loc}} \cap \mathcal{C}(X)$ and

$$\mathcal{E}(r_p, v) \geq -(N-1) \int_X \cot_\kappa \circ r_p \, v \, \mathrm{d}\mathfrak{m} \tag{2.5}$$

holds for any $v \in \mathcal{C}_c^{\mathrm{Lip}}(X \setminus \{p\})_+$ (see [19, Corollary 5.15]). We will extend this in Proposition 3.7 below. Next we mention some properties of minimal weak upper gradient. It is immediate from the definition that

$$|Df|_w \leq |Df| \tag{2.6}$$

holds for $f \in \mathcal{C}^{\mathrm{Lip}}(X) \cap L^2(X; \mathfrak{m})$. For r_p, $|Dr_p| \leq 1$ obviously holds and in addition

$$|Dr_p|_w = 1 \quad \mathfrak{m}\text{-a.e.} \tag{2.7}$$

(see [19, The proof of Corollary 5.15]). Let $(P_t)_{t \geq 0}$ be the semigroup of contractions on $L^2(X; \mathfrak{m})$ generated by Δ, which can be extended to contraction operators on $L^q(X; \mathfrak{m})$ for $1 \leq q \leq \infty$. As in Bakry-Émery theory based on the Bochner inequality, P_t satisfies the following L^1-gradient estimate:

$$|DP_t f| \leq \mathrm{e}^{-Kt} P_t(|Df|_w) \quad \text{for } f \in \mathcal{F} \cap L^\infty(X; \mathfrak{m}) \tag{2.8}$$

(see [39, Corollary 4.3]). It should be remarked that a local Lipschitz constant appears in the left hand side. P_t can be expressed as integral operator (see [2, Section 7.1]; see [4] also), and its density (or the heat kernel density) $\mathfrak{p}_t(x, y)$ satisfies the following Gaussian heat kernel upper bound: There exists $C_1, C_2, C_3 > 0$ such that, for $t > 0$ and $x, y \in X$,

$$\mathfrak{p}_t(x, y) \leq \frac{C_1}{V_{\sqrt{t}}(x)} \exp\left(-\frac{d(x, y)^2}{C_2 t} + C_3 t \right) \tag{2.9}$$

(see [24]). In particular, our diffusion process $\mathbf{X}$ satisfies the absolute continuity condition in [17, Section 4.2].

Before closing this subsection, we introduce some notions in optimal transport. Note that these are used only in Section 6. The L^2-Wasserstein distance $W_2(\mu_0, \mu_1)$ between two probability measures μ_0, μ_1 on X is given by

$$W_2(\mu_0, \mu_1)^2 := \inf \left\{ \int_{X \times X} d(x, y)^2 \, \mathrm{d}\pi(x, y) \;\middle|\; \pi \in \Pi(\mu_0, \mu_1) \right\},$$

where $\Pi(\mu_0, \mu_1)$ is the set of couplings of μ_0 and μ_1. That is, $\pi \in \Pi(\mu_0, \mu_1)$ means that π is a probability measure on $X \times X$ and that $\pi(A \times X) = \mu_0(A)$, $\pi(X \times B) = \mu_1(B)$ for $A, B \in \mathscr{B}(X)$. In duality with Bakry-Émery's L^2-gradient estimate we have the following estimate in W_2 of heat flows:

$$W_2(\mu_0 P_t, \mu_1 P_t) \leq \mathrm{e}^{-Kt} W_2(\mu_0, \mu_1), \tag{2.10}$$

where $\mu_i P_t$ is a dual action of P_t to a probability measure μ_i on X $(i = 0, 1)$ (see [2, (7.2)]). For $t \in [0,1]$, let $e_t : \mathcal{C}([0,1] \to X) \to X$ be the evaluation map given by $e_t(\gamma) := \gamma_t$. We say that $\boldsymbol{\pi} \in \mathscr{P}(\mathcal{C}([0,1] \to X))$ be a 2-test plan if $\boldsymbol{\pi}$ is concentrated on 2-absolutely continuous curves $\mathrm{AC}^2((0,1) \to X)$, there exists $C_{\boldsymbol{\pi}} > 0$ such that $(e_t)_\sharp \pi \le C_{\boldsymbol{\pi}} \mathfrak{m}$ for all $t \in [0,1]$, and

$$\int \left(\int_0^1 |\dot{\gamma}_s|^2 \, \mathrm{d}s \right) \boldsymbol{\pi}(\mathrm{d}\gamma) < \infty$$

(see [19, Definition 2.4]). We call that a Borel measurable function $f : X \to \mathbb{R}$ belongs to 2-Sobolev class if there exists $G \in L^2(X; \mathfrak{m})$ such that

$$\int |f(\gamma_1) - f(\gamma_0)| \, \boldsymbol{\pi}(\mathrm{d}\gamma) \le \int \int_0^1 G(\gamma_s) |\dot{\gamma}_s| \, \mathrm{d}s \, \boldsymbol{\pi}(\mathrm{d}\gamma)$$

for any 2-test plan. We call G (2-)weak upper gradient. It is known that there exists a minimal G in $\mathfrak{m}$-a.e. sense for each f in 2-Sobolev class (see [3, Section 5]). We call such G minimal weak upper gradient and denote it by $|Df|_w$. It is also known that $f \in \mathcal{D}(\mathsf{Ch})$ belongs to the 2-Sobolev class and we have (2.1) (see [3, Section 6]). Note that $|Df|_w^2$ or $\langle Df, Dg \rangle$ enjoys the Leibniz rule and the chain rule. See [3, 4, 19] for more details and other properties of $|Df|_w$.

3 Laplacian comparison and its applications

In this section, we will extend the Laplacian comparison theorem in order to apply it to our problem. Proposition 3.9 and Corollary 3.13 are main assertions in this direction. On the way, we also show a preliminary version of our main theorem (Theorem 3.11).

We begin with the following auxiliary lemma. We give a proof for completeness, but it is well-known for experts.

Lemma 3.1 $\mathfrak{m}$ *has no atoms.*

Proof. Let $p \in X$. Since (X, d) is a geodesic space, we have $q_n \in X$ with $d(p, q_n) = 1/n$ for each sufficiently large $n \in \mathbb{N}$. By the Bishop-Gromov inequality (2.4),

$$V_{(n-1)/n^2}(q) + \mathfrak{m}(\{p\}) \le V_{(n+1)/n^2}(q) \le \frac{\bar{V}_{(n+1)/n^2} V_{n-1/n^2}(q)}{\bar{V}_{(n-1)/n^2}}.$$

Thus we have

$$\mathfrak{m}(\{p\}) \le \left(\frac{\bar{V}_{(n+1)/n^2}}{\bar{V}_{(n-1)/n^2}} - 1 \right) V_{(n-1)/n^2}(q_n) \le \left(\frac{\bar{V}_{(n+1)/n^2}}{\bar{V}_{(n-1)/n^2}} - 1 \right) V_2(p).$$

Since $\lim_{r \to 0} \bar{V}_r / r^N = 1$ and $V_2(p) < \infty$, the conclusion follows by letting $n \to \infty$ in the last inequality. $\square$

The next lemmas are required for studying Laplacian comparison theorems.

Lemma 3.2 *Fix* $q \in [1, +\infty[$ *and suppose* $\cot_\kappa \circ r_p \in L^q_{\rm loc}(X; \mathfrak{m})$.

(i) $\lim_{n\to\infty} n^q \mathfrak{m}(B_{\frac{1}{n}}(p)) = 0$ *holds. In particular, we have* $N > q$.

(ii) *If* $q \geq 2$*, then* $\{p\}$ *is polar.*

Proof. (i): Since $\cot_\kappa(t) \sim 1/t$ as $t \to 0$, $\cot_\kappa \circ r_p \in L^q_{\rm loc}(X; \mathfrak{m})$ implies $1/r_p \in L^q_{\rm loc}(X; \mathfrak{m})$. Hence

$$\mathfrak{m}(B_{\frac{1}{n}}(p)) \leq \frac{1}{n^q} \int_{B_{\frac{1}{n}}(p)} \frac{{\rm d}\mathfrak{m}}{r_p^q}$$

Applying this with Lebesgue's dominated convergence theorem with Lemma 3.1 in mind, we see

$$n^q \mathfrak{m}(B_{\frac{1}{n}}(p)) \leq \int_{B_{\frac{1}{n}}(p)} \frac{{\rm d}\mathfrak{m}}{r_p^q} \to 0 \quad \text{as} \quad n \to \infty.$$

Applying the Bishop-Gromov inequality (2.4), we see

$$\mathfrak{m}(B_{\frac{1}{n}}(p)) \geq \bar{V}_{\frac{1}{n}} \frac{\mathfrak{m}(B_1(p))}{\bar{V}_1} \sim \frac{1}{n^N} \frac{\mathfrak{m}(B_1(p))}{\bar{V}_1} \quad \text{as} \quad n \to \infty.$$

This yields $N > q$.
(ii): For the polarity of $\{p\}$, it suffices to prove ${\rm Cap}(\{p\}) = 0$ by [17, Theorems 4.1.2 and 4.2.4]. From [17, Lemma 2.2.7(ii)], we have

$$\begin{aligned}
{\rm Cap}(\{p\}) &= \inf\{\mathcal{E}_1(f,f) \mid f \in \mathcal{C}_c^{\rm Lip}(X),\ f \geq 1 \text{ on } p\} \\
&\leq \mathcal{E}_1((1-nr_p)_+, (1-nr_p)_+) \\
&= \int_{B_{\frac{1}{n}}(p)} n^2 |Dr_p|^2 \,{\rm d}\mathfrak{m} + \int_{B_{\frac{1}{n}}(p)} (1-nr_p)_+^2 \,{\rm d}\mathfrak{m} \\
&\leq (n^2+1)\mathfrak{m}(B_{\frac{1}{n}}(p)) \leq (n^q+1)\mathfrak{m}(B_{\frac{1}{n}}(p)) \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence the conclusion holds. □

Lemma 3.3 $\cot_\kappa \circ r_p \in L^1_{\rm loc}(X \setminus \{p\}; \mathfrak{m})$. *In particular,* $r_p \cot_\kappa \circ r_p \in L^1_{\rm loc}(X; \mathfrak{m})$.

Proof. When $N = 1$, we have $\cot_\kappa \circ r_p = r_p^{-1}$ and hence the assertion obviously holds. Thus we consider the case $N > 1$. The latter assertion, the local integrability of $r_p \cot_\kappa \circ r_p$, comes from the fact that $t|\cot_\kappa t| \leq 2$ for sufficiently small $t > 0$. If $\kappa \leq 0$, there is nothing to prove, because $\cot_\kappa \circ r_p$ is bounded on any compact subset of $X \setminus \{p\}$. Thus we consider only the case $\kappa > 0$. Recall the Bonnet-Myers' diameter bound (2.3). If $\sup_{x\in X} d(p,x) < \pi/\sqrt{\kappa}$, again the assertion obviously holds. Let us suppose that there is $p' \in X$ with $d(p,p') = \pi/\sqrt{\kappa}$. Then the problem is reduced to show

$$\int_{B_{\pi/(2\sqrt{\kappa})}(p')} \frac{{\rm d}\mathfrak{m}}{\sin(r_p\sqrt{\kappa})} < \infty \tag{3.1}$$

and this is indeed shown in the proof of [19, Lemma 5.11]. □

Remark 3.4 *The assertion of Lemma 3.3 remains true even if* $(X, d, \mathfrak{m})$ *satisfies merely* $\mathsf{MCP}(K, N)$*-condition instead of* $\mathsf{RCD}(K, N)$ *condition.*

We now introduce a condition which ensures the Laplacian comparison of r_p on the whole X instead of $X \setminus \{p\}$. In other words, it ensures a stochastic expression of $r_p(X_t)$ beyond σ_p,

Definition 3.5 *We say that* $p \in X$ *verifies the condition* (R1) *if* $\cot_\kappa \circ r_p \in L^1_{\mathrm{loc}}(X; \mathfrak{m})$.

We will give a sufficient condition for (R1) in terms of volume growth exponent.

Lemma 3.6 *Fix* $q \in [1, +\infty[$. *Suppose that there exist* $C > 0$ *and* $\varepsilon, \delta > 0$ *such that* $V_r(p) \le Cr^{\varepsilon+q}$ *for all* $r \in]0, \delta[$. *Then* $\frac{1}{r_p} \in L^q_{\mathrm{loc}}(X; \mathfrak{m})$. *In particular,* (R1) *holds if this assumption is verified for some* q.

Note that the assumption in Lemma 3.6 holds if the Bishop inequality

$$\mathfrak{m}(B_r(p)) \le \varpi_N \bar{V}_r \quad (r \in [0, \pi/\sqrt{\kappa_+}[) \tag{3.2}$$

holds, where $\varpi_N := \pi^{N/2}/\Gamma(1 + N/2)$. In particular, for $N \in \mathbb{N}$, it holds on N-dimensional Alexandrov spaces (X, d) with $\mathfrak{m} = \mathcal{H}^N$ (see [43, (1.3)]), or more generally, on any N-dimensional Alexandrov space (X, d) with $N \ge 2$ equipped with the weighted measure $\mathfrak{m} = \mathrm{e}^{-V}\mathcal{H}^N$ on X, where $V : X \to \mathbb{R}$ is lower semi-continuous. Note here that there is no need to require the RCD-condition for the weighted Alexandrov space $(X, d, \mathfrak{m})$ in showing Lemma 3.6.

Proof. The second assertion follows from Lemma 3.3. For the first one,

$$\begin{aligned}\int_{B_\delta(p)} \frac{\mathrm{d}\mathfrak{m}}{r_p^q} &= \sum_{n=1}^\infty \int_{\{\delta/2^n \le r_p < \delta/2^{n-1}\}} \frac{\mathrm{d}\mathfrak{m}}{r_p^q} \le \sum_{n=1}^\infty \left(\frac{2^n}{\delta}\right)^q V_{\frac{\delta}{2^{n-1}}}(p) \\ &\le C \sum_{n=1}^\infty \left(\frac{2^n}{\delta}\right)^q \left(\frac{\delta}{2^{n-1}}\right)^{q+\varepsilon} = C2^q(2\delta)^\varepsilon \sum_{n=1}^\infty \left(\frac{1}{2^\varepsilon}\right)^n < \infty\end{aligned}$$

and thus the conclusion holds. $\square$

Proposition 3.7 (Laplacian comparison I) *We have* $r_p \in \mathcal{F}_{\mathrm{loc}} \cap \mathcal{C}(X)$ *and* (2.5) *holds for any* $v \in \mathcal{C}_c^{\mathrm{Lip}}(X \setminus \{p\})_+$. *If* (R1) *holds, then* (2.5) *holds for any* $v \in \mathcal{C}_c^{\mathrm{Lip}}(X)_+$.

Proof. As mentioned in Subsection 2.3, the first assertion is already proved when $N > 1$. The case $N = 1$ can be shown easily since $(X, d, \mathfrak{m})$ is very much specified (see Proposition A.1 below). We now prove the latter assertion. The condition $\cot_\kappa \circ r_p \in L^1_{\mathrm{loc}}(X; \mathfrak{m})$ excludes the case $N = 1$ by Lemma 3.2. Note that the both sides of (2.5) are always meaningful for $v \in \mathcal{C}_c^{\mathrm{Lip}}(X)$ under (R1). Take $v \in \mathcal{C}_c^{\mathrm{Lip}}(X)_+$. Set $v_n := v(1 - 1 \wedge (2 - 2nr_p)_+)$. Then $v_n \in \mathcal{C}_c^{\mathrm{Lip}}(X \setminus \{p\})$. Applying Lemma 3.2 together with Lemma 3.3, we have

$$\begin{aligned}|\mathcal{E}(r_p, v - v_n)| &\le \int_X |D(1 \wedge (2 - 2nr_p)_+)v| \,\mathrm{d}\mathfrak{m} \\ &\le \int_X |D(1 \wedge (2 - 2nr_p)_+)| \cdot |v| \,\mathrm{d}\mathfrak{m} + \int_X |Dv|(1 \wedge (2 - 2nr_p)_+) \,\mathrm{d}\mathfrak{m} \\ &= 2\|v\|_\infty n\mathfrak{m}(B_{\frac{1}{n}}(p)) + \|Dv\|_\infty \mathfrak{m}(B_{\frac{1}{n}}(p)) \to 0 \text{ as } n \to \infty\end{aligned}$$

and

$$
\begin{aligned}
|(\cot_\kappa\circ r_p, v-v_n)_{\mathfrak{m}}| &\le \int_X |\cot_\kappa\circ r_p|(1\wedge(2-2nr_p)_+)|v|\,\mathrm{d}\mathfrak{m}\\
&\le \|v\|_\infty \int_{B_{\frac{1}{n}}(p)} |\cot_\kappa\circ r_p|\,\mathrm{d}\mathfrak{m}\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Clearly, $v_n\ge 0$ under $v\ge 0$. Therefore, we obtain

$$
\mathcal{E}(r_p,v)=\lim_{n\to\infty}\mathcal{E}(r_p,v_n)\ge -(N-1)\lim_{n\to\infty}(\cot_\kappa\circ r_p, v_n)_{\mathfrak{m}}\ge -(N-1)(\cot_\kappa\circ r_p, v)_{\mathfrak{m}}
$$

for $v\in\mathcal{C}_c^{\mathrm{Lip}}(X)_+$. □

Remark 3.8 *The proof of the first assertion of Proposition 3.7 including the case $N=1$ can be done alternatively along the method of the proof of [30, Proposition 3.1] (see [19, Remark 5.16] also) based on the fact that Cheeger's Lipschitz differentiability theorem remains valid for $\mathsf{RCD}^*(K,N)$-space for $K\in\mathbb{R}$ and $N\in[1,+\infty[$. Cheeger's Lipschitz differentiability theorem over $\mathsf{RCD}^*(K,N)$-space for $K\in\mathbb{R}$ and $N\in[1,+\infty[$ follows from the volume doubling condition for $\mathfrak{m}$ and a local weak $(1,1)$-Poincaré inequality. Note here that a local weak $(1,1)$-Poincaré inequality for $\mathsf{CD}(K,\infty)$-space was proved by Rajala [36, Theorem 1.2] and $\mathsf{RCD}^*(K,N)$-space for $K\in\mathbb{R}$ and $N\in[1,+\infty[$ is an $\mathsf{RCD}(K,\infty)$-space by [14, Theorem 7].*

Let $S_0(\mathbf{X})$ or $S_0(\mathbf{X}_{X\setminus\{p\}})$ be the family of positive smooth measures of finite energy integrals associated to $(\mathcal{E},\mathcal{F})$ or the part $(\mathcal{E}_{X\setminus\{p\}},\mathcal{F}_{X\setminus\{p\}})$ of $(\mathcal{E},\mathcal{F})$ on $X\setminus\{p\}$ (see [17, Section 2.2])) respectively, i.e., $\nu\in S_0(\mathbf{X})$ (resp. $\nu\in S_0(\mathbf{X}_{X\setminus\{p\}})$) if and only if there exists $C>0$ such that

$$
\int_X |v|\,\mathrm{d}\nu\le C\sqrt{\mathcal{E}_1(v,v)}\quad\text{for}\quad v\in\mathcal{F}\cap\mathcal{C}_c(X)\quad(\text{resp. } v\in\mathcal{F}_{X\setminus\{p\}}\cap\mathcal{C}_c(X\setminus\{p\}).
$$

For $\nu\in S_0(\mathbf{X})$ (resp. $\nu\in S_0(\mathbf{X}_{X\setminus\{p\}})$) with $\alpha>0$, there exists $U_\alpha\nu\in\mathcal{F}$ (resp. $U_\alpha\nu\in\mathcal{F}_{X\setminus\{p\}}$) such that $\mathcal{E}_\alpha(U_\alpha\nu,v)=\langle\nu,v\rangle$ for any $v\in\mathcal{F}\cap\mathcal{C}_c(X)$ (resp. $v\in\mathcal{F}_{X\setminus\{p\}}\cap\mathcal{C}_c(X\setminus\{p\})$, which is called the *$\alpha$-potential* of ν with respect to $(\mathcal{E},\mathcal{F})$ (resp. $(\mathcal{E}_{X\setminus\{p\}},\mathcal{F}_{X\setminus\{p\}})$). Let $S(\mathbf{X})$ or $S(\mathbf{X}_{X\setminus\{p\}})$ be the family of positive smooth measures (see [17, Section 2.2]) associated to $(\mathcal{E},\mathcal{F})$ or the part $(\mathcal{E}_{X\setminus\{p\}},\mathcal{F}_{X\setminus\{p\}})$ of $(\mathcal{E},\mathcal{F})$ on $X\setminus\{p\}$ (see [17, Section 4.4])) respectively. It is known that for any $\mu\in S(\mathbf{X})$ there exists a positive continuous additive functional A_t^μ (PCAF in short) admitting exceptional set such that

$$
\langle f\mu, v\rangle=\lim_{t\to 0}\frac{1}{t}\mathbb{E}_{v\mathfrak{m}}\left[\int_0^t f(X_s)\mathrm{d}A_s^\mu\right]\qquad\text{for any }\ \gamma\text{-excessive function } v\ (\gamma\ge 0)\tag{3.3}
$$

(see [17, Theorems 5.1.1–5.1.4]). The relation $\mu\in S(\mathbf{X})\leftrightarrow(A_t^\mu)_{t\ge 0}$ characterized by (3.3) is called the *Revuz correspondence*. By definition, if for any relatively compact open subset G satisfying $\overline{G}\subset X$ (resp. $\overline{G}\subset X\setminus\{p\}$), $\mathbf{1}_G\nu\in S_0(\mathbf{X})$ (resp. $\nu\in S_0(\mathbf{X}_{X\setminus\{p\}})$) holds, then $\nu\in S(\mathbf{X})$ (resp. $\nu\in S(\mathbf{X}_{X\setminus\{p\}})$).

Proposition 3.9 (A realization of Δr_p) (i) *There exists a positive Radon measure ν on $X \setminus \{p\}$ such that*

$$\mathcal{E}(r_p, v) + (N-1)\int_X \cot_\kappa \circ r_p \, v \, \mathrm{d}\mathfrak{m} = \int_X v \, \mathrm{d}\nu \tag{3.4}$$

holds for $v \in \mathcal{C}_c^{\mathrm{Lip}}(X \setminus \{p\})$. If (R1) *holds, then ν can be regarded as a positive Radon measure on X such that* (3.4) *holds for $v \in \mathcal{C}_c^{\mathrm{Lip}}(X)$ and $\nu(\{p\}) = 0$.*

(ii) *The positive Radon measure ν specified in* (i) *belongs to $S(\mathbf{X}_{X\setminus\{p\}})$. In addition, $|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S(\mathbf{X}_{X\setminus\{p\}})$, $r_p|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S(\mathbf{X})$ and $r_p\nu \in S(\mathbf{X})$. If* (R1) *holds, then $|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S(\mathbf{X})$ and $\nu \in S(\mathbf{X})$.*

Proof. (i): Set $L := \mathcal{C}_c^{\mathrm{Lip}}(X \setminus \{p\})$ and $I(v) := \mathcal{E}(r_p, v) + ((N-1)\cot_\kappa \circ r_p, v)_{\mathfrak{m}}$ for $v \in L$. Then L is a vector lattice satisfying that $u \in L$ implies $u \wedge 1 \in L$. Lemma 3.3 implies $I(v) < \infty$ for any $v \in L$. We claim that I is a Daniell integral. That is, for $u \in L$, $u \geq 0$ implies $I(u) \geq 0$, and for $u_n \in L$ with $u_n \geq u_{n+1} \geq 0$, $\lim_{n\to\infty} u_n = 0$ implies $\lim_{n\to\infty} I(u_n) = 0$. The former property is an immediate consequence of Proposition 3.7. For the latter one, take such a sequence $\{u_n\}_n$. Then $\{u_n\}_n$ uniformly converges to 0 as $n \to \infty$ and $\mathrm{supp}[u_n] \subset \mathrm{supp}[u_1] \subset X \setminus \{p\}$. Take $v \in \mathcal{C}_c^{\mathrm{Lip}}(X)$ such that $v = 1$ on $\mathrm{supp}[u_1]$. Then $0 \leq I(u_n) \leq \|u_n\|_\infty I(v) \to 0$ as $n \to \infty$. Thus the claim holds. It is known that the Daniell integral on L admits a measure ν on the Baire σ-field $\mathscr{B}_a(X \setminus \{p\})$ generated by L (see [13, Theorem 4.5.2], [11, 32, 38]), that is,

$$I(v) = \int_X v(x)\,\nu(\mathrm{d}x) \quad \text{for any } v \in L.$$

Since L is dense in $\mathcal{C}_c(X\setminus\{p\})$, the Baire σ-field $\mathscr{B}_a(X\setminus\{p\})$ coincides with $\mathscr{B}(X\setminus\{p\})$. Take any compact subset K of $X\setminus\{p\}$. Let $w \in L = \mathcal{C}_c^{\mathrm{Lip}}(X\setminus\{p\})_+$ such that $w = 1$ on K. Then $\mathbf{1}_K \leq w$ on $X \setminus \{p\}$ implies $\nu(K) \leq \int_X w(x)\,\nu(\mathrm{d}x) = I(w) < \infty$. Therefore the former assertion holds. Under $\cot_\kappa \circ r_p \in L^1_{\mathrm{loc}}(X;\mathfrak{m})$, the proof of the existence of a positive Radon measure ν on X satisfying (3.4) for $v \in \mathcal{C}_c^{\mathrm{Lip}}(X)$ is similar as above. By using Lemma 3.2, we have

$$\begin{aligned}
\nu(\{p\}) &\leq \int_{B_{\frac{1}{n}}(p)} (1 - nr_p)_+ \, \mathrm{d}\nu \\
&= \mathcal{E}(r_p, (1-nr_p)_+) + (N-1)(\cot_\kappa \circ r_p, (1-nr_p)_+)_{\mathfrak{m}} \\
&\leq n\int_{B_{\frac{1}{n}}(p)} \langle Dr_p, Dr_p\rangle \, \mathrm{d}\mathfrak{m} + (N-1)\int_{B_{\frac{1}{n}}(p)} \cot_\kappa \circ r_p \, \mathrm{d}\mathfrak{m} \\
&\leq n\mathfrak{m}(B_{\frac{1}{n}}(p)) + (N-1)\int_{B_{\frac{1}{n}}(p)} \cot_\kappa \circ r_p \, \mathrm{d}\mathfrak{m} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence $\nu(\{p\}) = 0$ follows.

(ii): Firstly, we prove that $\nu \in S(\mathbf{X}_{X\setminus\{p\}})$. We may assume $N > 1$, because the proof for the case $N = 1$ can be an easy modification of it. Note first that $|\cot_\kappa \circ r_p|\,\mathfrak{m}$ is a positive Radon measure on $X \setminus \{p\}$ under $N > 1$ by Lemma 3.3. Thus it suffices to show that ν charges no set of zero capacity. Let G be a relatively compact open set satisfying $\overline{G} \subset X \setminus \{p\}$. Then there exists a constant $C_G > 0$ such that

$$\int_X |v| \, \mathrm{d}\nu \le C_G \sqrt{\mathcal{E}(v,v) + \int_X v^2 |\cot_\kappa \circ r_p| \, \mathrm{d}\mathfrak{m}} \quad \text{for} \quad v \in \mathcal{C}_c^{\mathrm{Lip}}(G). \tag{3.5}$$

Indeed, since $|Dr_p| \le 1$ on X, we have

$$\begin{aligned} |\mathcal{E}(r_p, |v|)| &= \left| \int_X \langle Dr_p, D|v| \rangle \, \mathrm{d}\mathfrak{m} \right| \le \int_G |Dr_p| \cdot |D|v|| \, \mathrm{d}\mathfrak{m} \\ &\le \sqrt{\mathfrak{m}(G)} \sqrt{\int_X |D|v||^2 \, \mathrm{d}\mathfrak{m}} \le \sqrt{\mathfrak{m}(G)} \sqrt{\mathcal{E}(v,v)} \end{aligned}$$

and

$$(N-1)|(\cot_\kappa \circ r_p, |v|)_{\mathfrak{m}}| \le (N-1) \sqrt{\int_G |\cot_\kappa \circ r_p| \, \mathrm{d}\mathfrak{m}} \sqrt{\int_X v^2 |\cot_\kappa \circ r_p| \, \mathrm{d}\mathfrak{m}}.$$

Then the claim holds with $C_G := \sqrt{2} \left(\sqrt{\mathfrak{m}(G)} + (N-1) \sqrt{\int_G |\cot_\kappa \circ r_p| \, \mathrm{d}\mathfrak{m}} \right)$. (3.5) implies that ν charges no exceptional set with respect to the part space $(\check{\mathcal{E}}_G, \check{\mathcal{F}}_G)$ on G for the Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(X; |\cot_\kappa \circ r_p| \, \mathfrak{m})$ associated to the time changed process $(\check{\mathbf{X}}_{X \setminus \{p\}}, |\cot_\kappa \circ r_p| \, \mathfrak{m})$. Note here that polarity with respect to $\mathbf{X}_{X \setminus \{p\}}$ is equivalent to the polarity with respect to $(\check{\mathbf{X}}_{X \setminus \{p\}}, |\cot_\kappa \circ r_p| \, \mathfrak{m})$. Consequently, ν charges no exceptional set with respect to $\mathbf{X}_{X \setminus \{p\}}$ in view of [17, Theorem 4.4.3(ii)], which implies $\nu \in S(\mathbf{X}_{X \setminus \{p\}})$.

Secondly, we show $|\cot_\kappa \circ r_p| \, \mathfrak{m} \in S(\mathbf{X}_{X \setminus \{p\}})$ and $r_p |\cot_\kappa \circ r_p| \, \mathfrak{m} \in S(\mathbf{X})$. We consider only the latter one since the former one can be shown in the same way. Recall that $r_p \cot_\kappa \circ r_p \in L^1_{\mathrm{loc}}(X; \mathfrak{m})$ holds by Lemma 3.3. With the aid of this, we can show that $r_p |\cot_\kappa \circ r_p| \, \mathfrak{m}$ is a positive Radon measure charging no exceptional set in a similar manner as we did for $\nu \in S(\mathbf{X}_{X \setminus \{p\}})$. It implies $r_p |\cot_\kappa \circ r_p| \in S(\mathbf{X})$.

Thirdly, we prove $r_p \nu \in S(\mathbf{X})$. Let G be a relatively compact open set. For $v \in \mathcal{C}_c^{\mathrm{Lip}}(G)$,

$$\begin{aligned} |\mathcal{E}(r_p, |v| r_p)| &= \left| \int_X \langle Dr_p, D|v| r_p \rangle \, \mathrm{d}\mathfrak{m} \right| \le \int_G |D|v| r_p| \, \mathrm{d}\mathfrak{m} \\ &\le \sqrt{\int_G r_p^2 \, \mathrm{d}\mathfrak{m}} \sqrt{\int_X |D|v||^2 \, \mathrm{d}\mathfrak{m}} + \sqrt{\mathfrak{m}(G)} \cdot \sqrt{\int_G v^2 \, \mathrm{d}\mathfrak{m}} \\ &\le \sqrt{\int_G r_p^2 \, \mathrm{d}\mathfrak{m} + \mathfrak{m}(G)} \cdot \sqrt{\mathcal{E}_1(v,v)} \end{aligned}$$

and

$$(N-1)|(\cot_\kappa \circ r_p, |v| r_p)_{\mathfrak{m}}| \le (N-1) \sqrt{\int_G r_p |\cot_\kappa \circ r_p| \, \mathrm{d}\mathfrak{m}} \sqrt{\int_X v^2 r_p |\cot_\kappa \circ r_p| \, \mathrm{d}\mathfrak{m}}.$$

These inequalities imply

$$\int_X |v| r_p \,\mathrm{d}\nu \le C_G \sqrt{\mathcal{E}_1(v,v) + \int_X v^2 r_p |\cot_\kappa \circ r_p| \,\mathrm{d}\mathfrak{m}} \quad \text{for} \quad v \in \mathcal{C}_c^{\mathrm{Lip}}(G) \tag{3.6}$$

with $C_G := \sqrt{2}\left(\sqrt{\int_G r_p^2 \,\mathrm{d}\mathfrak{m} + \mathfrak{m}(G)} + (N-1)\sqrt{\int_G r_p |\cot_\kappa \circ r_p| \,\mathrm{d}\mathfrak{m}}\right)$. So we can conclude that the Radon measure $r_p\nu$ charges no exceptional set with respect to $\mathbf{X}$ by the same argument as above. This implies $r_p\nu \in S(\mathbf{X})$.

Finally we prove the last assertion. Under (R1), $|\cot_\kappa \circ r_p|\,\mathfrak{m}$ is a positive Radon measure on X. The assertion $|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S(\mathbf{X})$ follows from the fact that any Radon measure charging no exceptional set belongs to $S(\mathbf{X})$. By (i), we know $\nu(\{p\}) = 0$. Since ν is a positive Radon measure on X, ν is decomposed into the sum $\nu_0 + \mathbf{1}_{N_\nu}\nu$ of a positive Radon measure $\nu_0 \in S(\mathbf{X})$ and an exceptional set N_ν by [18, Lemma 2.1]. Since $\nu \in S(\mathbf{X}_{X\setminus\{p\}})$, we see $\nu(N_\nu \setminus \{p\}) = 0$. Combining this with $\nu(\{p\}) = 0$, we obtain $\nu = \nu_0 \in S(\mathbf{X})$. □

Remark 3.10 *Under $\kappa \le 0$ or $\mathrm{diam}(X) < \pi/\sqrt{\kappa^+}$, we can see from the proof of Proposition 3.9* (i) *that for any relatively compact open subset G satisfying $\overline{G} \subset X\setminus\{p\}$, $\mathbf{1}_G\nu \in S_0(\mathbf{X}_G)$ and $\mathbf{1}_G|\cot_\kappa \circ r_p|\mathfrak{m} \in S_0(\mathbf{X}_G)$ because of $\cot_\kappa \circ r_p \in L^\infty_{\mathrm{loc}}(X \setminus \{p\}; \mathfrak{m})$. If there exists a point $p' \in X$ with $d(p,p') = \pi/\sqrt{\kappa}$ under $\kappa > 0$, then $\mathbf{1}_G\nu \in S_0(\mathbf{X}_G)$ and $\mathbf{1}_G|\cot_\kappa \circ r_p|\mathfrak{m} \in S_0(\mathbf{X}_G)$ hold for any relatively compact open set G satisfying $\overline{G} \subset X \setminus \{p,p'\}$.*

We are now ready to prove a version of our main theorem corresponding to Theorem 2.2. Let A_t^ν be the PCAF associated to $\nu \in S(\mathbf{X}_{X\setminus\{p\}})$ (see [17, Chapter 5]). We use the same symbol A_t^ν for the PCAF associated with $\nu \in S(\mathbf{X})$.

Theorem 3.11 (Stochastic expression of radial process I)

(i) *There exists a martingale additive functional B in the strict sense behaving as a one-dimensional standard Brownian motion under each $\mathbb{P}_x$ and a continuous additive functional N^{r_p} in the strict sense locally of zero energy such that*

$$r_p(X_t) - r_p(X_0) = \sqrt{2}B_t + N_t^{r_p} \tag{3.7}$$

holds for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for every $x \in X$.

(ii) *For $f \in C^2(\mathbb{R})$,*

$$\begin{aligned} f(r_p(X_t)) - f(r_p(X_0)) &= \sqrt{2}\int_0^t f'(r_p(X_s))\,\mathrm{d}B_s + \int_0^t f''(r_p(X_s))\,\mathrm{d}s \\ &\quad + (N-1)\int_0^t f'(r_p(X_s))\cot_\kappa(r_p(X_s))\,\mathrm{d}s - \int_0^t f'(r_p(X_s))\,\mathrm{d}A_s^\nu \end{aligned} \tag{3.8}$$

holds for all $t \in [0,\sigma_p[$ $\mathbb{P}_x$-a.s. for q.e. $x \in X \setminus \{p\}$. In particular, $f(r_p)$ is a semimartingale up to σ_p. If (R1) *holds, then* (3.8) *holds for all $t \in [0,+\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in X$. In particular, $f(r_p)$ is a semimartingale.*

Proof. (i) Since $r_p \in \mathcal{C}^{\rm Lip}(X) \subset \mathcal{F}_{\rm loc} \cap \mathcal{C}(X)$, we have the Fukushima decomposition under $\mathbf{X}$ by [17, Theorem 5.5.1]:

$$r_p(X_t) - r_p(X_0) = M_t^{r_p} + N_t^{r_p} \quad t \in [0, +\infty[, \quad \mathbb{P}_x\text{-a.s.} \tag{3.9}$$

for q.e. $x \in X$. Here $M_t^{r_p}$ (resp. $N_t^{r_p}$) is a martingale additive functional locally of finite energy (resp. continuous additive functional locally of zero energy) and both of them are local additive functionals and the decomposition is unique up to the equivalence of local additive functionals with respect to $\mathbf{X}$ (cf. [17, Theorem 5.5.1]). According to (2.7), the energy measure $\mu_{\langle r_p \rangle} = 2|Dr_p|_w^2\, \mathfrak{m}$ with respect to $\mathbf{X}$ satisfies $\mu_{\langle r_p \rangle} = 2\mathfrak{m}$ on X. Since $\mathbf{X}$ is a conservative diffusion process and $\mathfrak{m}$ is a smooth measure in the strict sense (i.e. $\mathfrak{m} \in S_1(\mathbf{X})$, see Section 5 for $S_1(\mathbf{X})$), we can apply [15, Theorem 2.1] so that (3.9) holds for all $x \in X$, and $M_t^{r_p}$ and $N_t^{r_p}$ can be redefined as local additive functionals in the strict sense. The quadratic variational process $\langle M^{r_p} \rangle$ of M^{r_p} under $\mathbf{X}$ satisfies $\langle M^{r_p} \rangle_t = 2t$. This implies that there exists a one dimensional standard Brownian motion B_t under $\mathbb{P}_x$ for all $x \in X$ such that $M_t^{r_p} = \sqrt{2} B_t$, $t \in [0, +\infty[$ under $\mathbb{P}_x$ for all $x \in X$ in view of Lévy's Theorem.

(ii) We first prove (2.2) with $A_t = A_t^{\nu}$ holds for $t \in [0, \sigma_p[$ $\mathbb{P}_x$-a.s. for q.e. $x \in X \setminus \{p\}$. Recall that $|\cot_\kappa \circ r_p|\, \mathfrak{m} \in S(\mathbf{X}_{X\setminus\{p\}})$ by Proposition 3.9 (ii). Next we set $\mu := \mu_1 - \mu_2$ with $\mu_1 := \nu + (N-1)\cot_\kappa^- \circ r_p\, \mathfrak{m} \in S(\mathbf{X}_{X\setminus\{p\}})$ and $\mu_2 := (N-1)\cot_\kappa^+ \circ r_p\, \mathfrak{m} \in S(\mathbf{X}_{X\setminus\{p\}})$. Here $\cot_\kappa^+(t) := \max\{\cot_\kappa(t), 0\}$ and $\cot_\kappa^-(t) := \max\{-\cot_\kappa(t), 0\}$. Then we see

$$\mathcal{E}(r_p, v) = \int_X v \, \mathrm{d}\mu \quad \text{for} \quad v \in \mathcal{C}_c^{\rm Lip}(X \setminus \{p\}).$$

Applying [16, Theorem 6.3] or [17, Corollary 5.5.1] to the signed Radon smooth measure $\mu := \mu_1 - \mu_2$,

$$N_t^{r_p} = (N-1) \int_0^t \cot_\kappa \circ r_p(X_s)\, \mathrm{d}s - A_t^{\nu} \tag{3.10}$$

holds for $t \in [0, \sigma_p[$ under $\mathbb{P}_x$-a.s. for q.e. $x \in X \setminus \{p\}$. Therefore, (2.2) holds for $t \in [0, \sigma_p[$ $\mathbb{P}_x$-a.s. for q.e. $x \in X \setminus \{p\}$. Since $\mathbf{X}$ is a diffusion process, σ_p is predictable $(\mathscr{F}_t)_{t\geq 0}$-stopping time. Thus there is an increasing sequence $\{\sigma_p^n\}$ of $(\mathscr{F}_t)_{t\geq 0}$-stopping times such that $\sigma_p^n < \sigma_p$ and $\lim_{n\to\infty} \sigma_p^n = \sigma_p$ hold under $\mathbb{P}_x$ for all $x \in X$. Since (2.2) holds for all $t \in [0, \sigma_p[$ under $\mathbb{P}_x$ for q.e. $x \in X \setminus \{p\}$, $t \mapsto r_p(X_{t \wedge \sigma_p^n})$ is an $(\mathscr{F}_t)_{t\geq 0}$-semimartingale under $\mathbb{P}_x$ for q.e. $x \in X \setminus \{p\}$. Applying Itô's formula to $r_p(X_{t\wedge\sigma_p^n})$, we can deduce that (3.8) holds for $t \in [0, \sigma_p[$ $\mathbb{P}_x$-a.s. for q.e. $x \in X \setminus \{p\}$.

Under $(\mathsf{R1})$, we already know $\nu \in S(\mathbf{X})$ and $|\cot_\kappa \circ r_p|\, \mathfrak{m} \in S(\mathbf{X})$ by Proposition 3.9 (ii). Then one can deduce that (2.2) holds for all $t \in [0, +\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in X$. Here we use the conservativeness of $\mathbf{X}$ under $\mathsf{RCD}^*(K, N)$-condition for $(X, d, \mathfrak{m})$. $(\mathsf{R1})$ also implies that we can decompose $\cot_\kappa \circ r_p$ into the sum of an integrable function and a continuous function. Thus, by virtue of the Fubini theorem together with L^1-contraction of the heat semigroup P_t, $t \mapsto \int_0^t \cot_\kappa \circ r_p(X_s)\, \mathrm{d}s$ is of bounded variation $\mathbb{P}_x$-a.s. Hence the conclusion follows by the applying the Itô formula as we did in the proof of (i). □

Remark 3.12 *On Riemannian manifolds, we can decompose A^ν as follows:*

$$A_t^\nu = \int_0^t ((N-1)\cot_\kappa \circ r_p(X_s) - \Delta r_p(X_s))\,\mathrm{d}s + L_t,$$

where L_t can be regarded as the local time at the cut locus of p (see [12, 26]). It is not clear whether we can have the same sort of expression. One difficulty may arise from the fact that the cut locus (we may be able to define it somehow) can be dense in X in RCD *spaces. It is also not clear whether we can separate L_t and the additive functional corresponding to Δr_p or not.*

As a first application of Theorem 3.11, we can extend the Laplacian comparison theorem as follows:

Corollary 3.13 (Laplacian comparison II) *We have the following:*

(i) *For $f \in C^2(\mathbb{R})$,*

$$\mathcal{E}(f(r_p), v) = \langle f' \circ r_p\,\nu, v\rangle - (\mathfrak{s}_\kappa^{1-N}(\mathfrak{s}_\kappa^{N-1} f')' \circ r_p, v)_{\mathfrak{m}} \tag{3.11}$$

holds for any $v \in \mathcal{C}_c^{\mathrm{Lip}}(X \setminus \{p\})$. If (R1) *holds, then* (3.11) *holds for $v \in \mathcal{C}_c^{\mathrm{Lip}}(X)$.*

(ii) *For $f \in C^2(\mathbb{R})$ with $f'(t) \geq 0$ for $t \geq 0$,*

$$\mathcal{E}(f(r_p), v) \geq -(\mathfrak{s}_\kappa^{1-N}(\mathfrak{s}_\kappa^{N-1} f')' \circ r_p, v)_{\mathfrak{m}} \tag{3.12}$$

holds for any $v \in \mathcal{C}_c^{\mathrm{Lip}}(X \setminus \{p\})_+$. Moreover, (3.12) *holds for $v \in \mathcal{C}_c^{\mathrm{Lip}}(X)_+$ under* (R1)*.*

To prove Corollary 3.13, we need the following lemma holding for general regular Dirichlet forms:

Lemma 3.14 *Let A_t^μ be a PCAF admitting exceptional set associated to a Radon measure $\mu \in S(\mathbf{X})$. Let $G \subset X$ be a relatively compact open nearly Borel. Suppose that $\mathbf{1}_G\mu \in S_0(\mathbf{X}_G)$. Then for any $f \in \mathcal{C}(X)$*

$$\langle f\mu, v\rangle = \lim_{t\to 0}\frac{1}{t}\mathbb{E}_{v\mathfrak{m}}\left[\int_0^t f(X_s)\,\mathrm{d}A_s^\mu : t < \tau_G\right] \quad \text{for} \quad v \in \mathcal{F}_G \cap \mathcal{C}_c(G),$$

where τ_G is the first exit time from G of $\mathbf{X}$.

Proof. In view of [17, Lemma 5.1.10 (ii)] with [17, Theorem 5.1.3 (i)⇔(iii)], $t \mapsto A_{t\wedge\tau_G}^\mu$ is a PCAF with respect to $\mathbf{X}_G$ associated to $\mathbf{1}_G\mu \in S_0(\mathbf{X}_G)$ (see also [17, Lemma 5.5.2 (iii)] for the additivity of $t \mapsto A_{t\wedge\tau_G}^\mu$). Applying [17, Theorem 5.1.3 (iii)⇔(vi)] to $\mathbf{X}_G$, under $\mathbf{1}_G\mu \in S_0(\mathbf{X}_G)$, we have that

$$\langle f\mu, v\rangle = \lim_{t\to 0}\frac{1}{t}\mathbb{E}_{v\,\mathfrak{m}}\left[\int_0^t f(X_s)\,\mathrm{d}A_{s\wedge\tau_G}^\mu\right]$$

holds for any $v \in \mathcal{F}_G \cap \mathcal{C}_c(G)$. Note here that $\langle \mu, |v| \rangle < \infty$ for $v \in \mathcal{F}_G \cap \mathcal{C}_c(G)$. Applying [17, Lemma 4.5.2(i)] to $\mathbf{X}_G$, we have

$$\lim_{t \to 0} \frac{1}{t} \mathbb{P}_{v^2 \mathfrak{m}}(t \geq \tau_G) = \frac{1}{t} \int_G v(x)^2 (1 - P_t^G 1(x)) \, \mathfrak{m}(\mathrm{d}x) = 0 \tag{3.13}$$

because our process $\mathbf{X}$ has no killing inside, where $(P_t^G)_{t \geq 0}$ is the transition semigroup of $\mathbf{X}_G$. With keeping this fact in mind, we have

$$\begin{aligned}
\frac{1}{t} \mathbb{E}_{|v| \mathfrak{m}} \left[\int_0^t |f(X_s)| \, \mathrm{d}A^\mu_{s \wedge \tau_G} : t \geq \tau_G \right] &\leq \sup_{x \in G} |f(x)| \cdot \frac{1}{t} \mathbb{E}_{\mathfrak{m}} \left[|v(X_0)| A^\mu_{t \wedge \tau_G} : t \geq \tau_G \right] \\
&\leq \sup_{x \in G} |f(x)| \sqrt{\frac{1}{t} \int_G \mathbb{E}_x[(A^\mu_{t \wedge \tau_G})^2] \, \mathfrak{m}(\mathrm{d}x)} \sqrt{\frac{1}{t} \mathbb{P}_{v^2 \mathfrak{m}}(t \geq \tau_G)}.
\end{aligned}$$

Because $\mathbf{1}_G \mu \in S_0(\mathbf{X}_G)$ implies that $t \mapsto A^\mu_{t \wedge \tau_G}$ is a CAF of zero energy with respect to $\mathbf{X}_G$ (see [17, pp. 245]), the right hand side of the last inequality converges to 0 as $t \to 0$. Therefore, we obtain the desired conclusion. $\square$

Proof of Corollary 3.13. (ii) is a simple consequence of (i). We only prove (i). First we suppose $\kappa \leq 0$ or $\mathrm{diam}(X) < \pi/\sqrt{\kappa}$ under $\kappa > 0$. Take $v \in \mathcal{C}_c^{\mathrm{Lip}}(X \setminus \{p\})$. Let G be relatively compact open sets satisfying $\mathrm{supp}[v] \subset G \subset \overline{G} \subset X \setminus \{p\}$. We can construct $r_p^G \in \mathcal{C}_c^{\mathrm{Lip}}(X \setminus \{p\})$ such that $r_p = r_p^G$ on G. Note that we have

$$\begin{aligned}
\lim_{t \to 0} \frac{1}{t} \mathbb{E}_{v\mathfrak{m}} \left[\int_0^t f'(r_p^G(X_s)) \, \mathrm{d}B_s : t < \tau_G \right] &= \lim_{t \to 0} \frac{1}{t} \mathbb{E}_{v\mathfrak{m}} \left[\int_0^t f'(r_p^G(X_s)) \, \mathrm{d}B_s : t \geq \tau_G \right] \\
&= 0,
\end{aligned}$$

because $t \mapsto \int_0^t f'(r_p^G(X_s)) \, \mathrm{d}B_s$ is a martingale additive functional of finite energy and

$$\begin{aligned}
\lim_{t \to 0} \frac{1}{t} &\left| \mathbb{E}_{v\mathfrak{m}} \left[\int_0^t f'(r_p^G(X_s)) \, \mathrm{d}B_s : t \geq \tau_G \right] \right| \\
&\leq \lim_{t \to 0} \sqrt{\frac{1}{t} \mathbb{E}_{\mathfrak{m}} \left[\left(\int_0^t f'(r_p^G(X_s)) \, \mathrm{d}B_s \right)^2 \right]} \sqrt{\frac{1}{t} \mathbb{P}_{v^2 \mathfrak{m}}(t \geq \tau_G)} \\
&= \lim_{t \to 0} \sqrt{\frac{1}{t} \mathbb{E}_{\mathfrak{m}} \left[\int_0^t |f'(r_p^G(X_s))|^2 \, \mathrm{d}s \right]} \sqrt{\frac{1}{t} \mathbb{P}_{v^2 \mathfrak{m}}(t \geq \tau_G)} \\
&= \lim_{t \to 0} \sqrt{\int_G |f'(r_p^G)|^2 \, \mathrm{d}\mathfrak{m}} \cdot \sqrt{0} = 0
\end{aligned} \tag{3.14}$$

holds by virtue of (3.13). From (3.8) and (3.14), we see that

$$\begin{aligned}
\mathcal{E}(f(r_p),v) &= \mathcal{E}(f(r_p^G),v)\\
&= \lim_{t\to0}\frac{1}{t}\mathbb{E}_{v\,\mathfrak{m}}\left[(f(r_p^G(X_0))-f(r_p^G(X_t))) : t<\tau_G\right]\\
&= \lim_{t\to0}\frac{1}{t}\mathbb{E}_{v\,\mathfrak{m}}\left[\int_0^t f'(r_p^G(X_u))\,\mathrm{d}A_u^{\nu}-\sqrt{2}\int_0^t f'(r_p^G(X_u))\,\mathrm{d}B_u\right.\\
&\qquad -(N-1)\int_0^t f'(r_p^G(X_u))\cot_\kappa(r_p^G(X_u))\,\mathrm{d}u\\
&\qquad \left. -\int_0^t f''(r_p^G(X_u))\,\mathrm{d}u : t<\tau_G\right]\\
&= \lim_{t\to0}\frac{1}{t}\mathbb{E}_{v\,\mathfrak{m}}\left[\int_0^t f'(r_p^G(X_u))\,\mathrm{d}A_u^{\nu}\right.\\
&\qquad -(N-1)\int_0^t f'(r_p^G(X_u))\cot_\kappa(r_p^G(X_u))\,\mathrm{d}u\\
&\qquad \left. -\int_0^t f''(r_p^G(X_u))\,\mathrm{d}u : t<\tau_G\right]\\
&= \langle f'(r_p^G)\nu,v\rangle-((N-1)f'(r_p^G)\cot_\kappa(r_p^G)+f''(r_p^G),v)_{\mathfrak{m}}\\
&= \langle f'\circ r_p\,\nu,v\rangle-(\mathfrak{s}_\kappa^{1-N}(\mathfrak{s}_\kappa^{N-1}f')'\circ r_p,v)_{\mathfrak{m}},
\end{aligned}$$

where we use Lemma 3.14 under $\mathbf{1}_G\,\nu\in S_0(\mathbf{X}_G)$, $\mathbf{1}_G|\cot_\kappa\circ r_p|\,\mathfrak{m}\in S_0(\mathbf{X}_G)$ and $\mathbf{1}_G\,\mathfrak{m}\in S_0(\mathbf{X}_G)$ by Remark 3.10 in the fifth equality. Then we obtain (3.11) for $v\in\mathcal{C}_c^{\rm Lip}(X\setminus\{p\})$.

Next we assume that $\kappa>0$ and there exists a point $p'\in X$ such that $d(p,p')=\pi/\sqrt{\kappa}$. In this case, we can show that $\mathbf{1}_G\,\nu\in S_0(\mathbf{X}_G)$, $\mathbf{1}_G|\cot_\kappa\circ r_p|\,\mathfrak{m}\in S_0(\mathbf{X}_G)$ and $\mathbf{1}_G\,\mathfrak{m}\in S_0(\mathbf{X}_G)$ for any relatively compact open set G satisfying $\overline{G}\subset X\setminus\{p,p'\}$ by Remark 3.10. Then we obtain (3.11) for $v\in\mathcal{C}_c^{\rm Lip}(X\setminus\{p,p'\})$. The proof of (3.11) for $v\in\mathcal{C}_c^{\rm Lip}(X\setminus\{p\})$ under $\kappa>0$ can be done by approximating v by $v_\ell:=v\psi(1-1\wedge(2-2\ell r_{p'})_+)$, where v_ℓ is a function similarly defined for the point p' instead p as in the proof of Proposition 3.7. Note that p' verifies the condition $(\mathsf{R1})$ in view of (3.1). Indeed, the convergences

$$\lim_{\ell\to\infty}\mathcal{E}(f(r_p),v-v_\ell)=0\quad\text{and}\quad\lim_{\ell\to\infty}(\mathfrak{s}_\kappa^{1-N}(\mathfrak{s}_\kappa^{N-1}f')'\circ r_p,v-v_\ell)_{\mathfrak{m}}=0$$

hold similarly as in the proof of Proposition 3.7. Moreover,

$$\begin{aligned}
|\langle f'(r_p)\nu,v-v_\ell\rangle| &\le \langle |f'(r_p)|\nu,(1\wedge(2-2\ell r_{p'})_+)|v|\rangle\\
&\le \|v\|_\infty\sup_{x\in\operatorname{supp}[v]}|f'(r_p(x))|\nu(B_{\frac{1}{\ell}}(p'))\to0\quad\text{as}\quad\ell\to\infty,
\end{aligned}$$

because $\lim_{\ell\to\infty}\nu(B_{\frac{1}{\ell}}(p'))=\nu(\{p'\})=0$. Thus we obtain (3.11) for $v\in\mathcal{C}_c^{\rm Lip}(X\setminus\{p\})$. The proof of (3.11) for $v\in\mathcal{C}_c^{\rm Lip}(X)$ under $(\mathsf{R1})$ can be done by approximating v by $v_n:=v\psi(1-1\wedge(2-2nr_p)_+)$, where v_n is the function defined in the proof of Proposition 3.7. The proof is similar as above. We omit it. □

Let $(R_\alpha)_{\alpha>0}$ be the resolvent operator of Δ with the integral kernel $\mathfrak{r}_\alpha(x,y)$. In this section, we consider some regularity properties of ν (Lemma 4.4 and Proposition 4.7 below) which are used in Section 5 for a refinement of Theorem 3.11. We begin with the following proposition.

Proposition 4.1 *Let ν be the smooth measure specified in Proposition 3.9* (i). *Then we have the following:*

(i) *There exists $\alpha>0$ such that, for $\varphi\in\mathcal{C}_c^{\mathrm{Lip}}(X\setminus\{p\})_+$,*

$$\sup_{x\in\mathrm{supp}[\varphi]}\int_X \mathfrak{r}_\alpha(x,y)\varphi(y)\,\nu(\mathrm{d}y)<\infty$$

and

$$\sup_{x\in\mathrm{supp}[\varphi]}\int_X \mathfrak{r}_\alpha(x,y)\varphi(y)|\cot_\kappa\circ r_p(y)|\,\mathfrak{m}(\mathrm{d}y)<\infty.$$

(ii) *There exists $\alpha>0$ such that, for $\varphi\in\mathcal{C}_c^{\mathrm{Lip}}(X)_+$,*

$$\sup_{x\in\mathrm{supp}[\varphi]}\int_X \mathfrak{r}_\alpha(x,y)\varphi(y)r_p(y)\,\nu(\mathrm{d}y)<\infty$$

and

$$\sup_{x\in\mathrm{supp}[\varphi]}\int_X \mathfrak{r}_\alpha(x,y)\varphi(y)r_p(y)|\cot_\kappa\circ r_p(y)|\,\mathfrak{m}(\mathrm{d}y)<\infty.$$

We begin with a basic estimate. By (2.9) and the definition of $\mathfrak{r}_\alpha$, for any measurable $g:[0,+\infty[\to[0,+\infty]$,

$$\begin{aligned}\int_X \mathfrak{r}_\alpha(x,y)g(r_p(y))\varphi(y)\,\nu(\mathrm{d}y)&=\int_0^\infty \mathrm{e}^{-\alpha t}\left(\int_X \mathfrak{p}_t(x,y)g(r_p(y))\varphi(y)\,\nu(\mathrm{d}y)\right)\mathrm{d}t\\&\le\int_0^\infty\frac{C_1\mathrm{e}^{-\alpha t+C_3t}}{V_{\sqrt{t}}(x)}\left(\int_X\exp\left(-\frac{d(x,y)^2}{C_2t}\right)g(r_p(y))\varphi(y)\,\nu(\mathrm{d}y)\right)\mathrm{d}t.\end{aligned}\tag{4.1}$$

Since $\exp(-r_x^2/(C_2t))\varphi\in\mathcal{C}_c^{\mathrm{Lip}}(X\setminus\{p\})\subset\mathcal{F}\cap L^\infty(X;\mathfrak{m})$ and $|Dr_z|\le1$ for any $z\in X$, the definition of ν together with (2.6) yields

$$\begin{aligned}0&\le\int_X\exp\left(-\frac{d(x,y)^2}{C_2t}\right)\varphi(y)\,\nu(\mathrm{d}y)\\&=\int_X\left\langle D\left(\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\right),Dr_p\right\rangle\mathrm{d}\mathfrak{m}+(N-1)\int_X\cot_\kappa(r_p)\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m}\\&\le\frac{2}{C_2t}\int_X r_x\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m}+\int_X\exp\left(-\frac{r_x^2}{C_2t}\right)|D\varphi|\,\mathrm{d}\mathfrak{m}\\&\qquad+(N-1)\int_X\cot_\kappa(r_p)\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m}.\end{aligned}\tag{4.2}$$

From (4.2), we have

$$
\begin{aligned}
(N-1)&\int_X |\cot_\kappa(r_p)| \exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m}\\
&\le \frac{2}{C_2t}\int_X r_x\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m} + \int_X \exp\left(-\frac{r_x^2}{C_2t}\right)|D\varphi|\,\mathrm{d}\mathfrak{m}\\
&\qquad + 2(N-1)\int_X \cot_\kappa^+(r_p)\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m}.
\end{aligned}
\tag{4.3}
$$

Similarly, (3.11) with $f(t)=t^2$ and $v=\varphi\exp(-r_x^2/(C_2t))$ yields

$$
\begin{aligned}
0 &\le \int_X \exp\left(-\frac{r_x^2}{C_2t}\right) 2r_p\varphi\,\mathrm{d}\nu\\
&= \int_X \left\langle D\left(\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\right), Dr_p^2\right\rangle \mathrm{d}\mathfrak{m}\\
&\qquad + 2(N-1)\int_X r_p\cot_\kappa(r_p)\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m} + 2\int_X \exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m}\\
&\le \frac{4}{C_2t}\int_X r_x\exp\left(-\frac{r_x^2}{C_2t}\right) r_p\varphi\,\mathrm{d}\mathfrak{m} + 2\int_X \exp\left(-\frac{r_x^2}{C_2t}\right)|D\varphi| r_p\,\mathrm{d}\mathfrak{m}\\
&\quad + 2(N-1)\int_X r_p\cot_\kappa(r_p)\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m} + 2\int_X \exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m}.
\end{aligned}
\tag{4.4}
$$

From (4.4), we have

$$
\begin{aligned}
2(N-1)&\int_X r_p|\cot_\kappa(r_p)|\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m}\\
&\le \frac{4}{C_2t}\int_X r_x\exp\left(-\frac{r_x^2}{C_2t}\right) r_p\varphi\,\mathrm{d}\mathfrak{m} + 2\int_X \exp\left(-\frac{r_x^2}{C_2t}\right)|D\varphi| r_p\,\mathrm{d}\mathfrak{m}\\
&\quad + 4(N-1)\int_X r_p\cot_\kappa^+(r_p)\exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m} + 2\int_X \exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\mathfrak{m}.
\end{aligned}
\tag{4.5}
$$

Note here that $\cot_\kappa^+\circ r_p$ is bounded from above by a positive constant on $\mathrm{supp}[\varphi]$ for $\varphi\in\mathcal{C}_c^{\mathrm{Lip}}(X\setminus\{p\})_+$. Thus (4.2) and (4.3) imply that there is a constant $C_*=C_*(p,\varphi)>0$ satisfying

$$
\int_X \exp\left(-\frac{r_x^2}{C_2t}\right)\varphi\,\mathrm{d}\nu \le \frac{C_*}{\sqrt{t}\wedge 1}\int_{\mathrm{supp}[\varphi]}\exp\left(-\frac{r_x^2}{(C_2+1)t}\right)\mathrm{d}\mathfrak{m}, \tag{4.6}
$$

$$
\int_X \exp\left(-\frac{r_x^2}{C_2t}\right)\varphi|\cot_\kappa\circ r_p|\,\mathrm{d}\mathfrak{m} \le \frac{C_*}{\sqrt{t}\wedge 1}\int_{\mathrm{supp}[\varphi]}\exp\left(-\frac{r_x^2}{(C_2+1)t}\right)\mathrm{d}\mathfrak{m} \tag{4.7}
$$

for all $x\in\mathrm{supp}[\varphi]$ and $t>0$. Similarly, since $r_p\cot_\kappa^+\circ r_p$ is bounded from above above on $\mathrm{supp}[\varphi]$ for $\varphi\in\mathcal{C}_c^{\mathrm{Lip}}(X)$, (4.4) and (4.5) imply that there is a constant $C_*=C_*(p,\varphi)>0$ satisfying

$$
\int_X \exp\left(-\frac{r_x^2}{C_2t}\right)\varphi r_p\,\mathrm{d}\nu \le \frac{C_*}{\sqrt{t}\wedge 1}\int_{\mathrm{supp}[\varphi]}\exp\left(-\frac{r_x^2}{(C_2+1)t}\right)\mathrm{d}\mathfrak{m}, \tag{4.8}
$$

$$
\int_X \exp\left(-\frac{r_x^2}{C_2t}\right)\varphi r_p|\cot_\kappa\circ r_p|\,\mathrm{d}\mathfrak{m} \le \frac{C_*}{\sqrt{t}\wedge 1}\int_{\mathrm{supp}[\varphi]}\exp\left(-\frac{r_x^2}{(C_2+1)t}\right)\mathrm{d}\mathfrak{m} \tag{4.9}
$$

for all $\varphi \in \mathcal{C}_c^{\mathrm{Lip}}(X)_+$, $x \in \mathrm{supp}[\varphi]$ and $t > 0$.

By combining (4.6)–(4.9) with (4.1), we can reduce the proof of Proposition 4.1 into the following two lemmas, by taking $\alpha > C_3$.

Lemma 4.2 *For each $\alpha_0, C > 0$ and any compact set $K \subset X$,*

$$\sup_{x \in K} \int_1^\infty \mathrm{e}^{-\alpha_0 t} \frac{1}{V_{\sqrt{t}}(x)} \int_K \exp\left(-\frac{r_x^2}{Ct}\right) \mathrm{d}\mathfrak{m}\, \mathrm{d}t < \infty. \tag{4.10}$$

Lemma 4.3 *For each $C > 0$ and any compact set K,*

$$\sup_{x \in K} \int_0^1 \frac{1}{\sqrt{t} V_{\sqrt{t}}(x)} \int_K \exp\left(-\frac{r_x^2}{Ct}\right) \mathrm{d}\mathfrak{m}\, \mathrm{d}t < \infty. \tag{4.11}$$

Take $R > 1$ so that $d(z, w) < R$ for any $z, w \in K$. Then $K \subset B_R(x)$ for any $x \in K$.

Proof of Lemma 4.2. By the Bishop-Gromov inequality (2.4),

$$\begin{aligned}
&\int_1^\infty \mathrm{e}^{-\alpha_0 t} \frac{1}{V_{\sqrt{t}}(x)} \int_K \exp\left(-\frac{r_x^2}{Ct}\right) \mathrm{d}\mathfrak{m}\, \mathrm{d}t \\
&\quad \le \int_1^\infty \mathrm{e}^{-\alpha_0 t} \frac{1}{V_{\sqrt{t}}(x)} \int_{B_R(x)} \exp\left(-\frac{r_x^2}{Ct}\right) \mathrm{d}\mathfrak{m}\, \mathrm{d}t \\
&\quad \le \frac{\bar{V}_R}{\bar{V}_1} \int_1^\infty \mathrm{e}^{-\alpha_0 t}\, \mathrm{d}t \le \frac{\mathrm{e}^{-\alpha_0}}{\alpha_0} \cdot \frac{\bar{V}_R}{\bar{V}_1}.
\end{aligned}$$

□

Proof of Lemma 4.3. By the Fubini theorem,

$$\begin{aligned}
\int_K \exp\left(-\frac{r_x^2}{Ct}\right) \mathrm{d}\mathfrak{m} &\le \int_{B_R(x)} \exp\left(-\frac{r_x^2}{Ct}\right) \mathrm{d}\mathfrak{m} \\
&\le \int_{B_R(x)} \left(\int_{d(x,y)}^\infty \frac{2u}{Ct} \exp\left(-\frac{u^2}{Ct}\right) \mathrm{d}u \right) \mathfrak{m}(\mathrm{d}y) \\
&= \frac{2}{Ct} \int_0^R \mathfrak{m}(B_u(x)) u \exp\left(-\frac{u^2}{Ct}\right) \mathrm{d}u + V_R(x) \exp\left(-\frac{R^2}{Ct}\right) \\
&= \frac{2}{C} \int_0^{R/\sqrt{t}} V_{\sqrt{t}s}(x) s \mathrm{e}^{-s^2/C}\, \mathrm{d}s + V_R(x) \exp\left(-\frac{R^2}{Ct}\right),
\end{aligned}$$

where the first identity follows from dividing the domain $[d(x, y), \infty[$ of the integral in u-variable into $[d(x, y), R]$ and $[R, \infty[$. By the Bishop-Gromov inequality (2.4),

$$\int_0^1 \frac{V_R(x)}{\sqrt{t} V_{\sqrt{t}}(x)} \exp\left(-\frac{R^2}{Ct}\right) \mathrm{d}t \le \int_0^1 \frac{\bar{V}_R}{\sqrt{t} \bar{V}_{\sqrt{t}}} \exp\left(-\frac{R^2}{Ct}\right) \mathrm{d}t < \infty.$$

Thus it suffices to prove the following claim:

$$\sup_{x\in K}\int_0^1 \frac{1}{\sqrt{t}V_{\sqrt{t}}(x)}\int_0^{R/\sqrt{t}} V_{\sqrt{t}s}(x)s\mathrm{e}^{-s^2/C}\,\mathrm{d}s\,\mathrm{d}t<\infty. \tag{4.12}$$

We now divide the domain of the integral in s variable into two parts. Take $\beta = 1/(2N+2)$ and first consider the integral on $[0, t^{-\beta}]$. By virtue of the Bishop-Gromov inequality (2.4), we have

$$\begin{aligned}
&\int_0^1 \frac{1}{\sqrt{t}V_{\sqrt{t}}(x)}\int_0^{t^{-\beta}} V_{\sqrt{t}s}(x)s\mathrm{e}^{-s^2/C}\,\mathrm{d}s\,\mathrm{d}t\\
&\qquad= \int_0^1 \frac{V_{t^{-\beta+1/2}}(x)}{\sqrt{t}V_{\sqrt{t}}(x)}\int_0^{t^{-\beta}} \frac{V_{\sqrt{t}s}(x)}{V_{t^{-\beta+1/2}}(x)}s\mathrm{e}^{-s^2/C}\,\mathrm{d}s\,\mathrm{d}t\\
&\qquad\le \int_0^1 \frac{V_{t^{-\beta+1/2}}(x)}{\sqrt{t}V_{\sqrt{t}}(x)}\int_0^{t^{-\beta}} s\mathrm{e}^{-s^2/C}\,\mathrm{d}s\,\mathrm{d}t\\
&\qquad\le \frac{C}{2}\int_0^1 \frac{\bar{V}_{t^{-\beta+1/2}}}{\sqrt{t}\bar{V}_{\sqrt{t}}}\,\mathrm{d}t.
\end{aligned} \tag{4.13}$$

Recall $\lim_{s\downarrow 0}\bar{V}_s/s^N = 1$. Thus, by taking $\beta < 1/2$ into account, we have

$$\frac{\bar{V}_{t^{-\beta+1/2}}}{\sqrt{t}\bar{V}_{\sqrt{t}}} = O(t^{-(N\beta+1/2)}) \text{ as } t\to 0.$$

Hence the integral in the right hand side of (4.13) is finite by our choice of β.

Next we deal with the integral on $[t^{-\beta}, R/\sqrt{t}]$. Again by the Bishop-Gromov inequality (2.4),

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{t}V_{\sqrt{t}}(x)}\int_{t^{-\beta}}^{R/\sqrt{t}} V_{\sqrt{t}s}(x)s\mathrm{e}^{-s^2/C}\,\mathrm{d}s\,\mathrm{d}t &\le \int_0^1 \frac{1}{\sqrt{t}}\int_{t^{-\beta}}^{R/\sqrt{t}} \frac{\bar{V}_{\sqrt{t}s}}{\bar{V}_{\sqrt{t}}}s\mathrm{e}^{-s^2/C}\,\mathrm{d}s\,\mathrm{d}t\\
&\le \frac{C\bar{V}_R}{2}\int_0^1 \frac{1}{\sqrt{t}\bar{V}_{\sqrt{t}}}\exp\left(-\frac{1}{Ct^{2\beta}}\right)\,\mathrm{d}t.
\end{aligned}$$

Then the last integral is finite by using the asymptotic behavior of $\bar{V}_s$ as $s\to 0$ again. Hence the claim (4.12) holds by combining these two estimates. □

Let $S_{00}(\mathbf{X})$ or $S_{00}(\mathbf{X}_{X\setminus\{p\}})$ be the family of positive smooth finite measures of finite energy integrals with bounded potential associated to $(\mathcal{E},\mathcal{F})$ or the part $(\mathcal{E}_{X\setminus\{p\}},\mathcal{F}_{X\setminus\{p\}})$ of $(\mathcal{E},\mathcal{F})$ on $X\setminus\{p\}$ (see [17, Section 2.2])) respectively, i.e., $\nu\in S_{00}(\mathbf{X})$ if and only if $\nu\in S_0(\mathbf{X})$, $\nu(X)<\infty$ and $U_\alpha\nu\in L^\infty(X;\mathfrak{m})$ ($\alpha>0$). By definition, $S_{00}(\mathbf{X}_{X\setminus\{p\}})\subset S(\mathbf{X}_{X\setminus\{p\}})$ and $S_{00}(\mathbf{X})\subset S(\mathbf{X})$.

We now turn to the proof of Proposition 4.7 below. We are interested in a refined property of ν in Proposition 3.9. We show the following lemma as an application of Proposition 4.1.

Lemma 4.4 *Let ν be the smooth measure specified in Proposition 3.9* (i). *Then, for any relatively compact open set G with $\overline{G}\subset X\setminus\{p\}$, $\mathbf{1}_G\,\nu\in S_{00}(\mathbf{X}_{X\setminus\{p\}})$ and $\mathbf{1}_G|\cot_\kappa\circ r_p|\,\mathfrak{m}\in S_{00}(\mathbf{X}_{X\setminus\{p\}})$. Moreover, for any relatively compact open set $G\subset X$, $\mathbf{1}_G r_p\,\nu\in S_{00}(\mathbf{X})$ and $\mathbf{1}_G r_p|\cot_\kappa\circ r_p|\,\mathfrak{m}\in S_{00}(\mathbf{X})$.*

Proof. Note that for each relatively compact open G with $\overline{G}\subset X\setminus\{p\}$, there exists $\varphi\in\mathcal{C}_c^{\mathrm{Lip}}(X\setminus\{p\})_+$ such that $\mathbf{1}_G\le\varphi\le 1$. Indeed, $\varphi(x):=(1-nd(x,\overline{G}))_+$ does the job for $n>1/d(p,\overline{G})$.

First we prove $R_\alpha(\varphi\nu)$ is bounded on $X\setminus\{p\}$. By Proposition 4.1, we already know that $R_\alpha(\varphi\nu)$ is bounded on the support of φ. Let A^ν be the PCAF admitting (properly) exceptional set N_ν associated to ν in Revuz correspondence with respect to $\mathbf{X}_{X\setminus\{p\}}$. Let K be the support of φ. Since $\mathbb{P}_x(X_{\sigma_K}\in K,\sigma_K<\infty)=\mathbb{P}_x(\sigma_K<\infty)$ for $x\in X$ (see [17, Lemma A.2.7]), we then see that for $x\in X\setminus(N_\nu\cup\{p\})$

$$\begin{aligned}R_\alpha(\varphi\nu)(x)&=\mathbb{E}_x\left[\int_0^\infty e^{-\alpha t}\varphi(X_t)\,\mathrm{d}A_t^\nu\right]=\mathbb{E}_x\left[\int_{\sigma_K}^\infty e^{-\alpha t}\varphi(X_t)\,\mathrm{d}A_t^\nu\right]\\&=\mathbb{E}_x\left[e^{-\alpha\sigma_K}\mathbb{E}_{X_{\sigma_K}}\left[\int_0^\infty e^{-\alpha t}\varphi(X_t)\,\mathrm{d}A_t^\nu\right]\right]\\&\le\sup_{y\in K}R_\alpha(\varphi\nu)(y)<\infty.\end{aligned}$$

Noting that $R_\alpha(\varphi\nu)$ is a finely continuous function and $\mathfrak{m}$ has full topological support with respect to the fine topology under absolute continuity condition, we can conclude that $\sup_{x\in X\setminus\{p\}}R_\alpha(\varphi\nu)(x)<\infty$. Similarly, we can obtain

$$\sup_{x\in X\setminus\{p\}}R_\alpha(\varphi|\cot_\kappa\circ r_p|\,\mathfrak{m})(x)<\infty.$$

By Proposition 3.9 (i), we know $\varphi\,\nu\in S(\mathbf{X}_{X\setminus\{p\}})$ and $\varphi|\cot_\kappa\circ r_p|\,\mathfrak{m}\in S(\mathbf{X}_{X\setminus\{p\}})$. Thus $\varphi\,\nu$ charges no exceptional set with respect to $\mathbf{X}_{X\setminus\{p\}}$.

Next we claim $\varphi\,\nu\in S_{00}(\mathbf{X}_{X\setminus\{p\}})$. As a first step, we prove $R_\alpha(\varphi\,\nu)\in\mathcal{F}$. Since $R_\alpha(\varphi\,\nu)$ is bounded on $X\setminus\{p\}$, we have $R_\alpha(\varphi\,\nu)\in L^\infty(X;\mathfrak{m})$, because of $\mathfrak{m}(\{p\})=0$ by Lemma 3.1. Then

$$\begin{aligned}\|R_\alpha(\varphi\,\nu)\|_2^2&\le\|R_\alpha(\varphi\,\nu)\|_\infty\|R_\alpha(\varphi\,\nu)\|_1=\|R_\alpha(\varphi\,\nu)\|_\infty\langle\varphi\,\nu,R_\alpha 1\rangle\\&=\frac{1}{\alpha}\|R_\alpha(\varphi\,\nu)\|_\infty\int_X\varphi\,\mathrm{d}\nu<\infty.\end{aligned}$$

Thus, it suffices to show

$$\sup_{\beta>0}\mathcal{E}_\alpha^{(\beta)}(R_\alpha(\varphi\,\nu),R_\alpha(\varphi\,\nu))<\infty,$$

where $\mathcal{E}_\alpha^{(\beta)}(u,v):=\beta(u-\beta R_{\beta+\alpha}u,v)_{\mathfrak{m}}$ for $u,v\in L^2(X;\mathfrak{m})$. It actually holds as follows:

$$\begin{aligned}\sup_{\beta>0}\mathcal{E}_\alpha^{(\beta)}(R_\alpha(\varphi\,\nu),R_\alpha(\varphi\,\nu))&=\sup_{\beta>0}\beta(R_{\beta+\alpha}(\varphi\,\nu),R_\alpha(\varphi\,\nu))_{\mathfrak{m}}=\|R_\alpha(\varphi\,\nu)\|_\infty\sup_{\beta>0}\beta\langle\varphi\,\nu,R_{\beta+\alpha}1\rangle\\&\le\|R_\alpha(\varphi\,\nu)\|_\infty\int_X\varphi\,\mathrm{d}\nu<\infty,\end{aligned}$$

and hence $R_\alpha(\varphi\,\nu) \in \mathcal{F}$. To conclude $\varphi\,\nu \in S_{00}(\mathbf{X}_{X\setminus\{p\}})$, it suffices to show $\mathcal{E}_\alpha(R_\alpha\varphi\nu, v) = \langle \varphi\,\nu, v\rangle$ for $v \in \mathcal{F}\cap\mathcal{C}_c(X)$. It indeed implies $\varphi\,\nu \in S_0(\mathbf{X})$ and hence $R_\alpha(\varphi\,\nu) = U_\alpha(\varphi\,\nu)$. For any $v \in \mathcal{F}\cap\mathcal{C}_c(X)$, we have

$$\mathcal{E}_\alpha(R_\alpha(\varphi\,\nu), v) = \lim_{\beta\to\infty} \mathcal{E}_\alpha^{(\beta)}(R_\alpha(\varphi\,\nu), v) = \lim_{\beta\to\infty} \beta(R_{\beta+\alpha}(\varphi\,\nu), v)_{\mathfrak{m}} = \lim_{\beta\to\infty} \beta\langle \varphi\,\nu, R_{\beta+\alpha}v\rangle.$$

Since $\beta R_\beta v$ converges to v with respect to $\mathcal{E}_1$, there exists a subsequence $\{\beta_n\}_n$ such that $\beta_n R_{\beta_n+\alpha}v \to v$ $\mathfrak{m}$-a.s. By the definition of R_α, we have $\alpha|R_\alpha v(x)| \leq \|v\|_\infty$ for every $x \in X$. Thus the dominated convergence theorem together with the definition of ν yields

$$\lim_{n\to\infty} \beta_n\langle \varphi\,\nu, R_{\beta_n+\alpha}v\rangle = \langle \varphi\,\nu, v\rangle.$$

Hence $U_\alpha(\varphi\,\nu) = R_\alpha(\varphi\,\nu)$ and $\varphi\,\nu \in S_{00}(\mathbf{X}_{X\setminus\{p\}})$ hold. In particular, $\mathbf{1}_G\,\nu \in S_{00}(\mathbf{X}_{X\setminus\{p\}})$ for each relatively compact open set $G \subset X \setminus \{p\}$. All other assertions can be shown in a similar way by using Proposition 4.1. $\square$

Now we introduce the following condition on $p \in X$ to discuss a further precision of our results.

Definition 4.5 *We say that $p \in X$ verifies the condition* (R2) *if there exist $\xi_0 > 0$ and $C_{\xi_0} > 0$ such that*

$$\frac{1}{V_\xi(p)} \int_{B_\xi(p)} \frac{\mathrm{d}\mathfrak{m}}{r_p} \leq \frac{C_{\xi_0}}{\xi}$$

for any $\xi \in]0, \xi_0[$.

Note that (R2) immediately implies (R1) by Lemma 3.3. Before entering further arguments, we provide a sufficient condition to (R2) in terms of volume growth exponent.

Lemma 4.6 *Suppose $N > 1$ and that there exists $C_V > 0$ and $\delta > 0$ such that $V_r(p) \leq C_V r^N$ holds for $r \in [0, \delta[$. Then $p \in X$ verifies the condition* (R2).

Note that this sufficient condition holds for any $p \in X$ on N-dimensional Alexandrov spaces equipped with N-dimensional Hausdorff measure with $N \geq 2$ (Recall (3.2)).

Proof. By the Fubini theorem, we have

$$\int_{B_\xi(p)} \frac{\mathrm{d}\mathfrak{m}}{r_p} = \int_{B_\xi(p)} \left(\frac{1}{\xi} + \int_{r_p}^{\xi} \frac{\mathrm{d}u}{u^2} \right) \mathrm{d}\mathfrak{m} = \left(\int_0^\xi \frac{V_u(p)}{u^2}\,\mathrm{d}u + \frac{V_\xi(p)}{\xi} \right). \tag{4.14}$$

By the Bishop-Gromov inequality (2.4), there exists $C' > 0$ such that

$$\frac{1}{V_\xi(p)} \leq \frac{\bar{V}_\delta}{V_\delta(p)\bar{V}_\xi} \leq C'\xi^{-N}$$

for $\xi \in]0, \delta[$. Then the assertion holds by this estimate, our assumption and (4.14). $\square$

Proposition 4.7 *Suppose that $p \in X$ verifies* (R2)*. Then, for any relatively compact open set $G \subset X$, $\mathbf{1}_G \nu \in S_{00}(\mathbf{X})$ and $\mathbf{1}_G |\cot_\kappa \circ r_p| \mathfrak{m} \in S_{00}(\mathbf{X})$.*

To prove Proposition 4.7, we prepare the following auxiliary lemma.

Lemma 4.8 *Suppose that $p \in X$ verifies* (R2)*. Then there exists $C'_{\xi_0} > 0$ such that*

$$\int_{B_u(x) \cap B_\xi(p)} \frac{\mathrm{d}\mathfrak{m}}{r_p} \le \frac{C'_{\xi_0} V_{5u}(x)}{u}$$

for any $x \in X$ and $u \in]0, \xi_0/6[$.

Proof. Let $\delta = d(x,p)$. We divide the proof into two cases. We first consider the case $u \in]0, \delta/2]$. Since $r_p(y) \ge \delta/2$ for any $y \in B_u(x)$, we have

$$\int_{B_u(x) \cap B_\xi(p)} \frac{\mathrm{d}\mathfrak{m}}{r_p} \le \frac{2}{\delta} \mathfrak{m}(B_u(x)) \le \frac{V_{5u}(x)}{u}.$$

Next, let $u > \delta/2$. In this case, we have $B_u(x) \subset B_{3u}(p) \subset B_{5u}(x)$. Thus

$$\int_{B_u(x) \cap B_\xi(p)} \frac{\mathrm{d}\mathfrak{m}}{r_p} \le \int_{B_{3u}(p)} \frac{\mathrm{d}\mathfrak{m}}{r_p} \le \frac{C_{\xi_0} V_{3u}(p)}{3u} \le \frac{C_{\xi_0} V_{5u}(x)}{3u},$$

where the second inequality follows from the condition (R2). Hence we complete the proof by combining these two cases. □

Proof of Proposition 4.7. As mentioned, (R2) implies (R1). Thus Proposition 4.1 (ii) ensures $\nu, |\cot_\kappa \circ r_p| \mathfrak{m} \in S(\mathbf{X})$.

We first show $\mathbf{1}_G |\cot_\kappa \circ r_p| \mathfrak{m} \in S_{00}(\mathbf{X})$. Since $\int_G |\cot_\kappa \circ r_p| \mathrm{d}\mathfrak{m} < \infty$, it suffices to show

$$\|R_\alpha(\mathbf{1}_G |\cot_\kappa \circ r_p|)\|_\infty < \infty. \tag{4.15}$$

Indeed, the rest of arguments, namely, showing $\mathbf{1}_G |\cot_\kappa \circ r_p| \mathfrak{m} \in S_0(\mathbf{X})$, goes along the same line as in the proof of Lemma 4.4. For this claim, we may assume $G = B_\xi(p)$ for sufficiently small $\xi > 0$ by virtue of Lemma 4.4. In addition, we can reduce the proof of (4.15) to the following estimate:

$$\sup_{x \in \overline{B_\xi(p)}} R_\alpha \left(\mathbf{1}_{B_\xi(p)} \frac{1}{r_p} \right)(x) < \infty$$

for some $\xi \in]0, \xi_0/12[$. Indeed, the change of the range of the supremum can be done as in the proof of Lemma 4.4. By the Gaussian heat kernel upper bound, this estimate holds if we have the following:

$$\sup_{x \in \overline{B_\xi(p)}} \int_0^\infty \frac{\mathrm{e}^{-(\alpha - C_3)t}}{V_{\sqrt{t}}(x)} \left(\int_{B_\xi(p)} \exp\left(-\frac{r_x^2}{C_2 t} \right) \frac{\mathrm{d}\mathfrak{m}}{r_p} \right) \mathrm{d}t < \infty \tag{4.16}$$

for $\alpha > C_3$. Set $\alpha_0 = \alpha - C_3$ and let $x \in \overline{B_\xi(p)}$. By the Fubini theorem and Lemma 4.8,

$$\begin{aligned}\int_{B_\xi(p)} \exp\left(-\frac{d(x,y)^2}{C_2 t}\right)\frac{1}{r_p(y)}\mathfrak{m}(\mathrm{d}y) &= \int_{B_\xi(p)}\left(\int_{r_x}^{\infty}\frac{2u}{C_2 t}\exp\left(-\frac{u^2}{C_2 t}\right)\mathrm{d}u\right)\frac{\mathrm{d}\mathfrak{m}}{r_p}\\ &= \frac{2}{C_2 t}\int_0^{2\xi} u\exp\left(-\frac{u^2}{C_2 t}\right)\left(\int_{B_u(x)\cap B_\xi(p)}\frac{\mathrm{d}\mathfrak{m}}{r_p}\right)\mathrm{d}u\\ &\le \frac{2C'_{\xi_0}}{C_2 t}\int_0^{2\xi} V_{5u}(x)\exp\left(-\frac{u^2}{C_2 t}\right)\mathrm{d}u. \end{aligned} \tag{4.17}$$

Thus the proof of (4.16) is reduced to the following two estimates:

$$\sup_{y\in\overline{B_\xi(p)}}\int_1^\infty \frac{\mathrm{e}^{-(\alpha-C_3)t}}{tV_{\sqrt{t}}(y)}\left(\int_0^{2\xi} V_{5u}(y)\exp\left(-\frac{u^2}{C_2 t}\right)\mathrm{d}u\right)\mathrm{d}t < \infty, \tag{4.18}$$

$$\sup_{y\in\overline{B_\xi(p)}}\int_0^1 \frac{1}{\sqrt{t}V_{\sqrt{t}}(y)}\left(\int_0^{10\xi/\sqrt{t}} V_{\sqrt{t}s}(y)\exp\left(-\frac{s^2}{25C_2}\right)\mathrm{d}s\right)\mathrm{d}t < \infty. \tag{4.19}$$

We may suppose $\xi < 1/10$ without loss of generality, and thus

$$\int_1^\infty \frac{\mathrm{e}^{-(\alpha-C_3)t}}{tV_{\sqrt{t}}(x)}\left(\int_0^{2\xi} V_{5u}(x)\exp\left(-\frac{u^2}{C_2 t}\right)\mathrm{d}u\right)\mathrm{d}t \le 2\xi\int_1^\infty \mathrm{e}^{-(\alpha-C_3)t}\,\mathrm{d}t.$$

This means (4.18). We can show (4.19) in the same way as we did for (4.12).

We next prove $\mathbf{1}_G\,\nu \in S_{00}(\mathbf{X})$. It suffices to show $\varphi\,\nu \in S_{00}(\mathbf{X})$ for a suitably chosen $\varphi \in \mathcal{C}_c^{\mathrm{Lip}}(X)_+$. As above, we can reduce the proof to showing $\|R_\alpha(\varphi\,\nu)\|_\infty < \infty$. By (4.1) and (4.2) together with Lemmas 4.2 and 4.3, (4.16) also implies it. Thus we complete the proof. □

5 Refinement of stochastic expression of radial processes

In this section, we establish a refined stochastic expression of radial process. Let $S_1(\mathbf{X})$ or $S_1(\mathbf{X}_{X\setminus\{p\}})$ be the family of positive smooth measures in the strict sense (see [17, p. 238]) associated to $(\mathcal{E},\mathcal{F})$ or the part $(\mathcal{E}_{X\setminus\{p\}},\mathcal{F}_{X\setminus\{p\}})$ of $(\mathcal{E},\mathcal{F})$ on $X\setminus\{p\}$, respectively. Note that $S_1(\mathbf{X}) \subset S(\mathbf{X})$ and $S_1(\mathbf{X}_{X\setminus\{p\}}) \subset S(\mathbf{X}_{X\setminus\{p\}})$. It is known that for any $\mu \in S_1(\mathbf{X})$, its associated PCAF A^μ under Revuz correspondence (3.3) can be taken to be in the strict sense (i.e. $(A_t^\mu)_{t\ge 0}$ can be defined under $\mathbb{P}_x$-a.s. for any $x \in X$) in our present framework (see [17, Theorems 5.1.6 and 5.1.7]). By definition, if for any relatively compact open subset G satisfying $\overline{G} \subset X$ (resp. $\overline{G} \subset X\setminus\{p\}$), $\mathbf{1}_G\nu \in S_{00}(\mathbf{X})$ (resp. $\nu \in S_{00}(\mathbf{X}_{X\setminus\{p\}})$) holds, then $\nu \in S_1(\mathbf{X})$ (resp. $\nu \in S_1(\mathbf{X}_{X\setminus\{p\}})$).

Theorem 5.1 *The measure ν in Proposition 3.9* (i) *and $|\cot_\kappa\circ r_p|\,\mathfrak{m}$ belong to $S_1(\mathbf{X}_{X\setminus\{p\}})$. Moreover, $r_p\,\nu \in S_1(\mathbf{X})$ and $r_p|\cot_\kappa\circ r_p|\,\mathfrak{m} \in S_1(\mathbf{X})$. If* (R2) *is verified at p, then ν and $|\cot_\kappa\circ r_p|\,\mathfrak{m}$ belong to $S_1(\mathbf{X})$.*

Proof. Let $\{G_n\}$ be an increasing sequence of relatively compact open subsets satisfying $\overline{G}_n \subset X\setminus\{p\}$, $n\in\mathbb{N}$. By Lemma 4.4, $\mathbf{1}_{G_n}\nu \in S_{00}(\mathbf{X}_{X\setminus\{p\}})$ and $\mathbf{1}_{G_n}\cot_\kappa\circ r_p \in$

$S_{00}(\mathbf{X}_{X\setminus\{p\}})$. Therefore, we obtain $\nu \in S_1(\mathbf{X}_{X\setminus\{p\}})$ and $|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S_1(\mathbf{X}_{X\setminus\{p\}})$ by [17, Theorem 5.1.7 (iii)]. Similarly, we can obtain $r_p\nu \in S_1(\mathbf{X})$ and $r_p|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S_1(\mathbf{X})$. □

Before stating our main theorem, we provide a condition alternative to (R2) in Theorem 5.1.

Proposition 5.2 *Suppose that p verifies* (R1) *and that $\{p\}$ is non-polar. Then ν and $|\cot_\kappa \circ r_p|\,\mathfrak{m}$ belong to $S_1(\mathbf{X})$.*

As we see in Theorem 7.1, $\{p\}$ is typically polar. Thus there seems to be less opportunity to apply Proposition 5.2, while it states a supplementary result to Theorem 5.1.

Proof. By (R1), $|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S(\mathbf{X})$ and $\nu \in S(\mathbf{X})$ holds by Proposition 3.9 (ii). Then the associated PCAF $\int_0^t \cot_\kappa \circ r_p(X_s)\,\mathrm{d}s$ and A_t^ν admitting common exceptional set $N_{p,\nu}$ are defined. We already know $|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S_1(\mathbf{X}_{X\setminus\{p\}})$ and $\nu \in S_1(\mathbf{X}_{X\setminus\{p\}})$. This means that $\int_0^t |\cot_\kappa \circ r_p|(X_s)\,\mathrm{d}s$ and A_t^ν can be regarded to be PCAFs of $\mathbf{X}_{X\setminus\{p\}}$ in the strict sense. Since $\{p\}$ is non-polar, $p \in X \setminus N_{p,\nu}$. Therefore, $\int_0^t |\cot_\kappa \circ r_p|(X_s)\,\mathrm{d}s$ and A_t^ν are defined to be PCAFs of $\mathbf{X}$ in the strict sense, consequently, $|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S_1(\mathbf{X})$ and $\nu \in S_1(\mathbf{X})$ by [17, Theorem 5.1.7(i)]. □

Now the following refinement follows immediately from Theorem 5.1.

Theorem 5.3 (Stochastic expression of radial process II) *Let A_t^ν be the PCAF in the strict sense associated to $\nu \in S_1(\mathbf{X}_{X\setminus\{p\}})$. Then, for $f \in C^2(\mathbb{R})$, we have that* (3.8) *holds for all $t \in [0, \sigma_p[$ $\mathbb{P}_x$-a.s. for all $x \in X \setminus \{p\}$, where B is given in Theorem 3.11 (i). In particular, $f(r_p)$ is a semimartingale up to σ_p. Moreover, if* (R2) *is verified at p, then* (3.8) *holds for $t \in [0, +\infty[$ $\mathbb{P}_x$-a.s. for $x \in X$. In particular, $f(r_p)$ is a semimartingale.*

Proof. In the proof of Theorem 3.11, we already know that (3.7) holds for all $x \in X$ and (3.10) holds for $t \in [0, \sigma_p[$ for q.e. $x \in X \setminus \{p\}$. By Lemma 4.4, we have $\mathbf{1}_G\,\nu \in S_{00}(\mathbf{X}_{X\setminus\{p\}})$ and $\mathbf{1}_G|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S_{00}(\mathbf{X}_{X\setminus\{p\}})$ for any relatively compact open set G with $\overline{G} \subset X \setminus \{p\}$. We have $\mathbf{1}_G\,\mu_{\langle r_p\rangle} = 2\mathbf{1}_G\,\mathfrak{m} \in S_{00}(\mathbf{X}_{X\setminus\{p\}})$ for such G. Then, we can apply [17, Theorem 5.5.5] to the part process $\mathbf{X}_{X\setminus\{p\}}$. At this stage, N^{r_p} can be redefined as a local additive functional in the strict sense locally of zero energy with respect to $\mathbf{X}_{X\setminus\{p\}}$ such that (3.10) holds for $t \in [0, \sigma_p[$ under $\mathbb{P}_x$-a.s. for all $x \in X \setminus \{p\}$. Therefore, (2.2) with $A = A^\nu$ holds for $t \in [0, \sigma_p[$ $\mathbb{P}_x$-a.s. for $x \in X \setminus \{p\}$. Then we can derive (3.8) for $t \in [0, \sigma_p[$ $\mathbb{P}_x$-a.s. for $x \in X \setminus \{p\}$ in the same manner as in the proof of Theorem 3.11. The latter assertion under (R2) follows similarly. □

Although we require (R2) in Theorem 5.3 not to exclude $p \in X$, we do not require such an additional condition for a class of f.

Definition 5.4 *We say that $f \in C^2(\mathbb{R})$ enjoys the condition* (F) *if $\mathbf{1}_G|f'(r_p)\cot_\kappa(r_p)|\,\mathfrak{m}$ and $\mathbf{1}_G|f'(r_p)|\,\nu$ belong to $S_{00}(\mathbf{X})$ for each relatively compact open set G. Under* (F), *$\int_0^t f'(r_p(X_s))\,\mathrm{d}A_s^\nu$ can be regarded as a CAF in the strict sense with respect to $\mathbf{X}$.*

Corollary 5.5 (Stochastic expression III) *Let A_t^ν be the PCAF in the strict sense associated to $\nu \in S_1(\mathbf{X}_{X\setminus\{p\}})$. For $f \in C^2(\mathbb{R})$, we have that* (3.8) *holds for all $t \in [0,+\infty[$ $\mathbb{P}_x$-a.s. for all $x \in X$ and $f(r_p)$ is a semimartingale, provided* (F) *is verified. In particular, we always have that*

$$\begin{aligned} r_p^2(X_t) - r_p^2(X_0) = 2\sqrt{2}\int_0^t r_p(X_s)\,\mathrm{d}B_s + 2(N-1)\int_0^t r_p(X_s)\cot_\kappa \circ r_p(X_s)\,\mathrm{d}s \\ - 2\int_0^t r_p(X_s)\,\mathrm{d}A_s^\nu + 2t \end{aligned} \tag{5.1}$$

for $t \in [0,+\infty[$ $\mathbb{P}_x$-a.s. for $x \in X$.

Remark 5.6 *The second term of the right hand side of* (3.8) *and* (5.1) *does not appear respectively, provided $N = 1$.*

Proof of Corollary 5.5. First we will show that our assumption enables us to give a Fukushima decomposition of $f(r_p)$ in the strict sense. Suppose $\mathbf{1}_G|f'(r_p)\cot_\kappa \circ r_p|\,\mathfrak{m} \in S_{00}(\mathbf{X})$ and $\mathbf{1}_G|f'(r_p)|\,\nu \in S_{00}(\mathbf{X})$ for each relatively compact open set G. In particular, $|f'(r_p)\cot_\kappa \circ r_p|\,\mathfrak{m}$ and $|f'(r_p)|\,\nu$ are positive Radon smooth measures on X in the strict sense. Since (2.7) holds, we see $\mu_{\langle f(r_p)\rangle} = |f'(r_p)|^2\,\mathfrak{m}$. Set $\mu^f := \mu_1^f - \mu_2^f$ with $\mu_1^f := f'(r_p)_+\,\nu + (\mathfrak{s}_\kappa^{1-N}(\mathfrak{s}_\kappa^{N-1}f')')_- \circ r_p\,\mathfrak{m}$ and $\mu_2^f := f'(r_p)_-\,\nu + (\mathfrak{s}_\kappa^{1-N}(\mathfrak{s}_\kappa^{N-1}f')')_+ \circ r_p\,\mathfrak{m}$. The estimate $|\mu_i^f| \le |f'(r_p)|\nu + (N-1)|f'(r_p)\cot_\kappa(r_p)|\,\mathfrak{m} + |f''(r_p)|\,\mathfrak{m}$, $(i=1,2)$ shows that $\mu_1^f, \mu_2^f \in S_1(\mathbf{X})$, hence μ^f is a signed Radon smooth measure on X in the strict sense. By Corollary 3.13 (i), we see

$$\mathcal{E}(f(r_p), v) = \int_X v\,\mathrm{d}\mu^f \quad \text{for} \quad v \in \mathcal{C}_c^{\mathrm{Lip}}(X).$$

Note that $\mathbf{1}_G\,\mu_{f(r_p)} = \mathbf{1}_G|f'(r_p)|^2\,\mathfrak{m} \in S_{00}(\mathbf{X})$ for each relatively compact open set G. Applying [17, Theorem 5.5.5] to $\mathbf{X}$ for $\mu^f = \mu_1^f - \mu_2^f$, we have

$$f(r_p(X_t)) - f(r_p(X_0)) = M_t^{f(r_p)} + N_t^{f(r_p)} \quad t \in [0,+\infty[, \quad \mathbb{P}_x\text{-a.s. for all } x \in X.$$

Here

$$\begin{aligned} N_t^{f(r_p)} &= \int_0^t (\mathfrak{s}_\kappa^{1-N}(\mathfrak{s}_\kappa^{N-1}f')')(X_s)\,\mathrm{d}s - \int_0^t f'(X_s)\,\mathrm{d}A_s^\nu \\ &= (N-1)\int_0^t f'(r_p(X_s))\cot_\kappa(r_p(X_s))\,\mathrm{d}s + \int_0^t f''(r_p(X_s))\,\mathrm{d}s - \int_0^t f'(X_s)\,\mathrm{d}A_s^\nu \\ &\text{for } t \in [0,+\infty[\quad \mathbb{P}_x\text{-a.s. for all } x \in X \end{aligned}$$

and $M_t^{f(r_p)}$ is a local additive functional in the strict sense such that, for any relatively compact open set G,

$$\begin{aligned} \mathbb{E}_x\left[M_{t\wedge\tau_G}^{f(r_p)}\right] &= 0 \quad x \in G, \\ \mathbb{E}_x\left[(M_{t\wedge\tau_G}^{f(r_p)})^2\right] &= 2\mathbb{E}_x\left[\int_0^{t\wedge\tau_G} |f'(r_p(X_s))|^2\,\mathrm{d}s\right] \quad x \in G. \end{aligned}$$

From this, $M_t^{f(r_p)}$ is a locally square integrable MAF and its quadratic variational process $\langle M^{f(r_p)}\rangle$ has the expression

$$\langle M^{f(r_p)}\rangle_t = 2\int_0^t |f'(r_p(X_s))|^2\,\mathrm{d}s.$$

Next we will observe that we have an alternative expression of $M^{f(r_p)}$. By applying the generalized Itô's formula proved in [28, Theorem 4.3] to (3.9), we have

$$\begin{aligned} f(r_p(X_t)) - f(r_p(X_0)) = \int_0^t f'(r_p(X_s))\,\mathrm{d}M_s^{r_p} + \int_0^t f'(r_p(X_s))\,\mathrm{d}N_s^{r_p} \\ + \frac{1}{2}\int_0^t f''(r_p(X_s))\,\mathrm{d}\langle M^{r_p}\rangle_s, \quad t\in[0,+\infty[\quad \mathbb{P}_x\text{-a.s.} \end{aligned}$$

for q.e. $x\in X$. Here the second term in the right hand side is the stochastic integral by CAF locally of zero energy (see [28, Definition 3.1], where the stochastic integral by $\Gamma(M)_t$ for local MAF M is defined and note $\Gamma(M^{r_p})_t = N_t^{r_p}$). By the uniqueness of the Fukushima decomposition, $M_t^{f(r_p)} = \int_0^t f'(r_p(X_s))\,\mathrm{d}M_s^{r_p}$, $t\in[0,+\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x\in X$. Since $M_t^{r_p} = \sqrt{2}B_t$, $t\in[0,+\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x\in X$, we can conclude

$$M_t^{f(r_p)} = \sqrt{2}\int_0^t f'(r_p(X_s))\,\mathrm{d}B_s, \quad t\in[0,+\infty[\tag{5.2}$$

$\mathbb{P}_x$-a.s. for q.e. $\in X$. The both hands of (5.2) are continuous local additive functionals in the strict sense. This implies that (5.2) holds under $\mathbb{P}_x$ for all $x\in X$. Therefore we have that (3.8) holds for all $t\in[0,+\infty[$ $\mathbb{P}_x$-a.s. for all $x\in X$.

The final assertion follows from $\mathbf{1}_G r_p|\cot_\kappa\circ r_p|\,\mathfrak{m}\in S_{00}(\mathbf{X})$ and $\mathbf{1}_G r_p\,\nu\in S_{00}(\mathbf{X})$ for each relatively compact open set G by Lemma 4.4. □

6 Applications

In this section we will turn to discuss applications of our main theorem. In Subsection 6.1, we show comparison theorems. Starting from the comparison theorem for the radial process (Corollary 6.1), we show the heat kernel comparison theorem (Corollary 6.2) under the Bishop inequality and Cheng's eigenvalue comparison theorem (Corollary 6.3). In Subsection 6.2, we prove Cheng's Liouville theorem for harmonic functions of sublinear growth on non-negatively curved spaces.

6.1 Comparison theorems

Corollary 6.1 (Comparison of radial process) *We have the following:*

(i) *Let $f\in C^2(\mathbb{R})$ satisfying $f'(t)\geq 0$ for $t\geq 0$. Then*

$$\begin{aligned} f(r_p(X_t)) - f(r_p(X_0)) \leq \sqrt{2}\int_0^t f'(r_p(X_s))\,\mathrm{d}B_s \\ + (N-1)\int_0^t f'(r_p(X_s))\cot_\kappa(r_p(X_s))\,\mathrm{d}s + \int_0^t f''(r_p(X_s))\,\mathrm{d}s \end{aligned} \tag{6.1}$$

holds for all $t \in [0, \sigma_p[$ $\mathbb{P}_x$-a.s. for all $x \in X \setminus \{p\}$. If p verifies (R1)*, then* (6.1) *holds for all $t \in [0, +\infty[$ $\mathbb{P}_x$-a.s. for all q.e. $x \in X \setminus \{p\}$. If p verifies* (R2) *or f verifies* (F)*, then* (6.1) *holds for all $t \in [0, +\infty[$ $\mathbb{P}_x$-a.s. for all $x \in X$. In particular, we always have*

$$r_p^2(X_t) - r_p^2(X_0) \leq 2\sqrt{2} \int_0^t r_p(X_s)\, \mathrm{d}B_s \\ + 2(N-1) \int_0^t r_p(X_s) \cot_\kappa \circ r_p(X_s)\, \mathrm{d}s + 2t \tag{6.2}$$

for $t \in [0, +\infty[$ $\mathbb{P}_x$-a.s. for $x \in X$.

(ii) *Let ρ_t be the unique non-negative strong solution of the stochastic differential equation*

$$\rho_t = r_p(x) + \sqrt{2} B_t + (N-1) \int_0^t \cot_\kappa(\rho_s)\, \mathrm{d}s \tag{6.3}$$

under $\mathbb{P}_x$. Here B_t is as given in Theorem 3.11 (i). Then we have

$$r_p(X_t) \leq \rho_t \tag{6.4}$$

holds for all $t \in [0, \sigma_p[$ $\mathbb{P}_x$-a.s. for all $x \in X \setminus \{p\}$. If (R2) *is verified, then* (6.4) *holds for all $t \in [0, +\infty[$ $\mathbb{P}_x$-a.s. for all $x \in X$.*

Note that the corresponding result on the basis of the latter assertion of Theorem 3.11 or Proposition 5.2 also holds. We just preferred to state our result in a simplified form.

Proof. (i) is an easy consequence of Theorem 5.3.
(ii) We set R_κ by

$$R_\kappa := \begin{cases} \pi/\sqrt{\kappa} & \text{if } \kappa > 0, \\ +\infty & \text{if } \kappa \leq 0. \end{cases}$$

So $r_p(x) \leq R_\kappa$ always holds by (2.3). The SDE (6.3) can make sense for $N \geq 2$. Similar to the corresponding property of Bessel processes, ρ_t does not hit neither 0 or R_κ. The explosion time of ρ_t is infinite. According to the same way of the proof of [22, Theorem 3.5.3 (ii)], we can conclude that $r_p(X_t) \leq \rho_t$ for all $t \in [0, \sigma_p[$ $\mathbb{P}_x$-a.s. for all $x \in X \setminus \{p\}$. This implies the conclusion. The case under (R2) is similar. □

In the framework of N-dimensional Alexandrov space X with $\mathrm{curv}(X) \geq \kappa$, Corollary 6.1 extends [37, Theorem III], which show the corresponding result under some additional conditions.

Hereafter, we assume $N \in \mathbb{N}$ until the end of this subsection. Let $\mathbb{M}_\kappa^N$ be the be the simply connected N-dimensional Riemannian manifold of constant sectional curvature κ. We denote the heat kernel on $\mathbb{M}_\kappa^N$ (resp. the Dirichlet heat kernel on $B_r(\bar{p}) (\subset \mathbb{M}_\kappa^N)$ by $\mathfrak{p}_t^\kappa(\bar{p}, \bar{q})$ (resp. $\mathfrak{p}_t^{\kappa,r}(\bar{p}, \bar{q})$). By symmetry, $\mathfrak{p}_t^\kappa$ and $\mathfrak{p}_t^{\kappa,r}$ are functions of

t and the distance between $\bar{p}$ and $\bar{q}$; hence there exist functions $\mathfrak{p}^{\kappa}(t,s)$ and $\mathfrak{p}^{\kappa,r}(t,s)$, $t,s \in [0,+\infty[$ such that $\mathfrak{p}^{\kappa,r}(t,s)=0$ for $s>r$ and

$$\mathfrak{p}_t^{\kappa}(\bar{p},\bar{q}) = \mathfrak{p}^{\kappa}(t, d_{\mathbb{M}_\kappa^N}(\bar{p},\bar{q})) \quad \text{and} \quad \mathfrak{p}_t^{\kappa,r}(\bar{p},\bar{q}) = \mathfrak{p}^{\kappa,r}(t, d_{\mathbb{M}_\kappa^N}(\bar{p},\bar{q})).$$

When $\kappa > 0$, $\mathbb{M}_\kappa^N$ is a sphere of radius $1/\sqrt{\kappa}$, and we adopt the convention that $\mathfrak{p}^{\kappa}(t,s) := \mathfrak{p}^{\kappa}(t,\pi/\sqrt{\kappa})$ if $s \geq \pi/\sqrt{\kappa}$. Let $\varpi_N \bar{V}_r$ be the volume of r-ball in $\mathbb{M}_\kappa^N$.

Corollary 6.2 (Comparison of heat kernels) *Suppose $N \in \mathbb{N}$ with $N \geq 2$ and the Bishop inequality* (3.2) *holds. Let $\mathfrak{p}_t^G$ be the Dirichlet heat kernel on some domain $G \subset X$ and let $q \in B_r(p) \subset G$. Then, we have*

$$\mathfrak{p}_t^G(p,q) \geq \mathfrak{p}^{\kappa,r}(t, d(p,q)). \tag{6.5}$$

In particular,

$$\mathfrak{p}_t(p,q) \geq \mathfrak{p}^{\kappa}(t, d(p,q)). \tag{6.6}$$

Proof. Since the measure $\mathfrak{m}$ is (local) doubling and a local weak Poincaré inequality applies, one can show the local Hölder continuity for the Dirichlet heat kernel $\mathfrak{p}^G$ by applying [29, Corollary 8.1] and Moser's iteration method. Here we remark that [29, Section 8] treats the general framework of strongly local Dirichlet forms and the intrinsic distance derived from $(\mathcal{E},\mathcal{F}) = (2\mathsf{Ch}, \mathcal{D}(\mathsf{Ch}))$ on $L^2(X;\mathfrak{m})$ coincides with the given distance, which was proved in [6, Theorem 3.9].

Recall that the Bishop inequality implies $(\mathsf{R2})$ by Lemma 4.6. By Corollary 6.1 (ii),

$$r_p(X_t) \leq \rho_t \quad \text{for } t \in [0,+\infty[$$

under $\mathbb{P}_q$-a.s. for all $q \in X$. Note here that ρ_t under $\mathbb{P}_q$ has the same law with the radial process $\bar{r}_{\bar{p}}(\bar{X}_t)$ for the Brownian motion $\bar{X}_t$ on $\mathbb{M}_\kappa^N$ starting at $\bar{q}$ satisfying $d(\bar{p},\bar{q}) = d(p,q)$. Then

$$\begin{aligned}\int_{B_\varepsilon(p)} \mathfrak{p}_t^G(q,z)\,\mathfrak{m}(\mathrm{d}z) &= \mathbb{P}_q(r_p(X_t) \leq \varepsilon, t < \tau_G) \\ &\geq \mathbb{P}_q(r_p(X_t) \leq \varepsilon, t < \tau_{B_r(p)}) \\ &\geq \mathbb{P}_q(\rho_t \leq \varepsilon, t < \tau_{B_r(p)}) \\ &= \bar{\mathbb{P}}_{\bar{q}}(\bar{r}_{\bar{p}}(\bar{X}_t) \leq \varepsilon, t < \tau_{B_r(\bar{p})}) = \int_{B_\varepsilon(\bar{p})} \mathfrak{p}_t^{\kappa, B_r(\bar{p})}(\bar{q},\bar{z})\,\mathrm{vol}_{\mathbb{M}_\kappa^N}(\mathrm{d}\bar{z}).\end{aligned}$$

Dividing the both side by $\varpi_N \bar{V}_\varepsilon$ with the Bishop inequality, we have

$$\frac{1}{\mathfrak{m}(B_\varepsilon(p))}\int_{B_\varepsilon(p)} \mathfrak{p}_t^G(q,z)\,\mathfrak{m}(\mathrm{d}z) \geq \frac{1}{\varpi_N \bar{V}_\varepsilon}\int_{B_\varepsilon(\bar{p})} \mathfrak{p}_t^{\kappa, B_r(\bar{p})}(\bar{q},\bar{z})\,\mathrm{vol}_{\mathbb{M}_\kappa^N}(\mathrm{d}\bar{z}).$$

Letting $\varepsilon \to 0$ with the continuity of the heat kernels, we obtain the conclusion. □

Corollary 6.3 (Cheng's eigenvalue comparison) *Suppose that $N \in \mathbb{N}$ and the Bishop inequality* (3.2) *holds. Let $G_0 := B_r(p)$ and let $\lambda_j(G_0)$ denote the j-th (counted with multiplicity) Dirichlet eigenvalue of $B_r(p)$ with $0 = \lambda_0(G_0) < \lambda_1(G_0) \le \lambda_2(G_0) \le \cdots$. Then*

$$\lambda_j(G_0) \le \lambda_1^{\kappa}(\mathrm{diam}(G_0)/2j).$$

In particular,

$$\lambda_1(B_r(p)) \le \lambda_1^{\kappa}(r),$$

where $\lambda_1^{\kappa}(s)$ denotes the first Dirichlet eigenvalue for any s-ball in $\mathbb{M}_{\kappa}^{N}$.

Proof. The proof of estimate for the first eigenvalue can be done along the standard argument by using eigenvalue expansion of heat kernels and Corollary 6.2 (see [40, p. 104]). The proof of estimate for the j-th eigenvalue is based on the estimate for the first eigenvalue and the min-max-principle for the higher eigenvalues of the Laplace operator (see [40, p. 105]). □

6.2 Liouville property for sublinear harmonic functions

In this section, we assume that the metric measure space $(X, d, \mathfrak{m})$ satisfies $\mathsf{RCD}^*(0, N)$ condition for $N \in [1, +\infty[$. As an application of the expression of radial process, we give a stochastic proof of Cheng's Liouville property of sublinear $\mathcal{E}$-harmonic functions $(X, d, \mathfrak{m})$. A function $f \in \mathcal{F}_{\mathrm{loc}}$ is said to be *$\mathcal{E}$-harmonic* if $\mathcal{E}(f, v) = 0$ for any $\mathcal{C}_c^{\mathrm{Lip}}(X)$. A function f is said to be have *sublinear growth* if

$$\varlimsup_{a\to\infty} m_f(a)/a = 0,$$

where $m_f(a) := \sup_{r_p(x)<a} |f(x)|$. Our main theorem in this section is the following:

Theorem 6.4 (Cheng's Liouville theorem) *Any continuous $\mathcal{E}$-harmonic function having sublinear growth is a constant.*

Note that an analytic proof of Theorem 6.4 is given in [23]. To prove Theorem 6.4, we need the following two lemmas.

Lemma 6.5 *Let $f \in \mathcal{F}_{\mathrm{loc}}$ be a continuous $\mathcal{E}$-harmonic function having sublinear growth. Then $P_t f(x) = f(x)$ holds for q.e. $x \in X$.*

Proof. First note that $r_p|\cot_\kappa \circ r_p|\,\mathfrak{m} \in S_1(\mathbf{X})$ by Theorem 5.1. From Corollary 6.1 (i), we have

$$r_p^2(X_t) - r_p^2(X_0) \le 2\sqrt{2}\int_0^t r_p(X_s)\,\mathrm{d}B_s + 2Nt$$

holds for $t \in [0, +\infty[$ $\mathbb{P}_x$-a.s. for all $x \in X$. In particular,

$$\mathbb{E}_x[r_p^2(X_{t\wedge\tau_{G_n}})] \le r_p^2(x) + 2Nt$$

for all $x \in X$. By [17, Corollary 5.5.1] with the $\mathcal{E}$-harmonicity of f, we have the following Fukushima's decomposition:

$$f(X_t) - f(X_0) = M_t^f \quad t \in [0, +\infty[\tag{6.7}$$

holds $\mathbb{P}_x$-a.s. for q.e. $x \in X$. Here M^f is a local martingale additive functional locally of finite energy. Let $\{G_n\}$ be an increasing sequence of relatively compact open sets satisfying $\overline{G}_n \subset G_{n+1}$ for $n \in \mathbb{N}$ with $X = \bigcup_{n=1}^{\infty} G_n$. Then $\{M_{t\wedge\tau_{G_n}}^f\}_{t\in[0,+\infty[}$ is a $\mathbb{P}_x$-martingale for q.e. $x \in X$. By (6.7),

$$\mathbb{E}_x[f(X_{t\wedge\tau_{G_n}})] = f(x) \quad \text{for q.e. } x \in X. \tag{6.8}$$

Let $m_f(a) := \sup_{r_p(x)<a} |f(x)|$. The sublinear growth of f yields that for any $\varepsilon > 0$ there exists $A > 0$ such that $m_f(a) < a\varepsilon$ for any $a > A$. Then

$$\begin{aligned}
\mathbb{E}_x\left[f^2(X_{t\wedge\tau_{G_n}})\right] &= \mathbb{E}_x\left[f^2(X_{t\wedge\tau_{G_n}}) : r_p(X_{t\wedge\tau_{G_n}}) \le A\right] \\
&\qquad + \mathbb{E}_x\left[f^2(X_{t\wedge\tau_{G_n}}) : r_p(X_{t\wedge\tau_{G_n}}) > A\right] \\
&\le m_{f^2}(A) + \varepsilon^2 \mathbb{E}_x\left[r_p^2(X_{t\wedge\tau_{G_n}})\right] \\
&\le m_{f^2}(A) + \varepsilon^2 (r_p^2(x) + 2Nt)
\end{aligned} \tag{6.9}$$

for $x \in X$. From this, $\{f(X_{t\wedge\tau_{G_n}})\}_{n\in\mathbb{N}}$ is uniformly $\mathbb{P}_x$-integrable for all $x \in X$. Since $\mathbb{P}_x(\lim_{n\to\infty} \tau_{G_n} = \infty) = 1$ for all $x \in X$, (6.8) yields $P_t f(x) = f(x)$ for q.e. $x \in X$. $\square$

Lemma 6.6 *Let $f \in \mathcal{F}_{\mathrm{loc}}$ be a continuous $\mathcal{E}$-harmonic function having sublinear growth. Then $|Df|^2 \le P_t |Df|^2$ $\mathfrak{m}$-a.e. In particular, $|Df|^2 \le \frac{1}{t}\int_0^t P_s |Df|^2 \,\mathrm{d}s$ $\mathfrak{m}$-a.e.*

Proof. If $f \in \mathcal{F}$, this is a direct consequence of $P_t f = f$ and Bakry-Émery's gradient estimate. Our f may not have this regularity and thus we need an additional approximation argument.

We first prove the following inequality

$$|Df|_w^2 \le \frac{1}{2t} P_t(f^2) \tag{6.10}$$

for any $t > 0$, which is a weaker version of the reverse Poincaré inequality [8, (4.7.6)] for f. We will use this inequality to ensure a required integrability for a candidate of weak upper gradient. Let $x_* \in X$ be a reference point and $\varphi_n \in \mathcal{C}_c^{\mathrm{Lip}}(X)$ a cut-off function satisfying $0 \le \varphi_n \le 1$, $\varphi_n|_{B_n(x_*)} = 1$, $\varphi_n|_{B_{n+1}(x_*)} = 0$ and $|D\varphi_n| \le 1$. Since $f\varphi_n \in L^2(X; \mathfrak{m})$, by [2, Theorem 7.3], we have

$$|DP_t(f\varphi_n)|_w^2 \le \frac{1}{2t} P_t(|f\varphi_n|^2) \quad \mathfrak{m}\text{-a.e.} \tag{6.11}$$

Let $\pi \in \mathscr{P}(\mathcal{C}([0,1] \to X))$ be a 2-test plan as defined in Section 2.3. Note that $\pi(e_0^{-1}(B_R(x_*)) \cap e_1^{-1}(B_R(x_*)))^{-1} \pi|_{e_0^{-1}(B_R(x_*))\cap e_1^{-1}(B_R(x_*))}$ is still a 2-test plan for sufficiently large $R > 0$. By the definition of (minimal) weak upper gradient, (6.11) yields

$$\begin{aligned}
&\int_{e_0^{-1}(B_R(x_*))\cap e_1^{-1}(B_R(x_*))} |P_t(f\varphi_n) \circ e_1 - P_t(f\varphi_n) \circ e_0| \,\mathrm{d}\pi \\
&\qquad \le \frac{1}{\sqrt{2t}} \int_{e_0^{-1}(B_R(x_*))\cap e_1^{-1}(B_R(x_*))} \left(\int_0^1 \sqrt{P_t(|f\varphi_n|^2)(\gamma_s)} |\dot{\gamma}_s| \,\mathrm{d}s\right) \pi(\mathrm{d}\gamma) \\
&\qquad \le \frac{1}{\sqrt{2t}} \int \left(\int_0^1 \sqrt{P_t(f^2)(\gamma_s)} |\dot{\gamma}_s| \,\mathrm{d}s\right) \pi(\mathrm{d}\gamma).
\end{aligned} \tag{6.12}$$

Since we have $W_2(P_t\delta_x, P_t\delta_y) \le d(x,y)$ by (2.10), the sublinear growth condition of f implies that $f \in L^1(X; \mathfrak{p}_t(x,\cdot)\mathfrak{m})$ for every $x \in X$ and that $P_t f$ is locally bounded. Thus, by the dominated convergence theorem, letting $n \to \infty$ and $R \to \infty$ in (6.12) implies

$$\int |f \circ e_1 - f \circ e_0| \,\mathrm{d}\pi \le \frac{1}{\sqrt{2t}} \int \left(\int_0^1 \sqrt{P_t(f^2)(\gamma_s)} |\dot{\gamma}_s| \,\mathrm{d}s \right) \pi(\mathrm{d}\gamma).$$

Here we used the fact $P_t f = f$ from Lemma 6.5. We can show $P_t(f^2)$ is locally bounded by a similar argument as above, and hence $\sqrt{P_t(f^2)} \in L^2_{\mathrm{loc}}(X; \mathfrak{m})$. Thus, by the definition of minimal weak upper gradient, we obtain (6.10) from this inequality since π is arbitrary.

We are now in turn to prove the assertion. Actually the proof is similar to the one for (6.10). For the same cut-off function φ_n, we have $f\varphi_n \in \mathcal{F}$. Thus (2.8) yields

$$|DP_t(f\varphi_n)|_w \le P_t(|D(f\varphi_n)|_w) \quad \mathfrak{m}\text{-a.e.}$$

Set $E_R \subset \mathcal{C}([0,1] \to X)$ by

$$E_R := \left\{ \gamma \in \mathrm{AC}^2([0,1] \to X) \;\middle|\; \gamma_0 \in B_R(x_*), \gamma_1 \in B_R(x_*), \int_0^1 |\dot{\gamma}_s|^2 \,\mathrm{d}s \le R \right\}.$$

Then $\pi(E_R)^{-1}\pi|_{E_R}$ is 2-test plan. Thus, by the definition of (minimal) weak upper gradient,

$$\int_{E_R} |P_t(f\varphi_n) \circ e_1 - P_t(f\varphi_n) \circ e_0| \,\mathrm{d}\pi \le \int_{E_R} \left(\int_0^1 P_t(|D(f\varphi_n)|)(\gamma_s) |\dot{\gamma}_s| \,\mathrm{d}s \right) \pi(\mathrm{d}\gamma). \tag{6.13}$$

Since

$$|D(f\varphi_n)|_w \le \varphi_n |Df|_w + f|D\varphi_n| \le |Df|_w + 1_{B_{n+1}(x_*) \setminus B_n(x_*)} f,$$

we have

$$\begin{aligned} \int_{E_R} \left(\int_0^1 P_t(|D(f\varphi_n)|_w)(\gamma_s) |\dot{\gamma}_s| \,\mathrm{d}s \right) \pi(\mathrm{d}\gamma) &\le \int \left(\int_0^1 P_t(|Df|_w)(\gamma_s) |\dot{\gamma}_s| \,\mathrm{d}s \right) \pi(\mathrm{d}\gamma) \\ &\quad + \int_{E_R} \left(\int_0^1 P_t(1_{B_{n+1}(x_*) \setminus B_n(x_*)} f)(\gamma_s) |\dot{\gamma}_s| \mathrm{d}s \right) \pi(\mathrm{d}\gamma). \end{aligned} \tag{6.14}$$

We can easily see that $\{\gamma_s \mid \gamma \in E_R, s \in [0,1]\}$ is bounded. Thus the dominated convergence theorem implies that, by letting $n \to \infty$ in (6.14),

$$\varlimsup_{n\to\infty} \int_{E_R} \left(\int_0^1 P_t(|D(f\varphi_n)|_w)(\gamma_s) |\dot{\gamma}_s| \,\mathrm{d}s \right) \pi(\mathrm{d}\gamma) \le \int \left(\int_0^1 P_t(|Df|_w)(\gamma_s) |\dot{\gamma}_s| \,\mathrm{d}s \right) \pi(\mathrm{d}\gamma).$$

Thus, by letting $n \to \infty$ and $R \to \infty$ in (6.13),

$$\int |f \circ e_1 - f \circ e_0| \,\mathrm{d}\pi \le \int \left(\int_0^1 P_t(|Df|_w)(\gamma_s) |\dot{\gamma}_s| \,\mathrm{d}s \right) \pi(\mathrm{d}\gamma).$$

Here we used $P_t f = f$. By (6.10), we have

$$P_t(|Df|_w) \leq \frac{1}{\sqrt{2}} P_t \sqrt{P_1(f^2)} \leq \frac{1}{\sqrt{2}} \sqrt{P_{t+1}(f^2)}.$$

Since $P_{t+1}(f^2)$ is locally bounded as we mentioned above, we have $P_t(|Df|_w) \in L^2_{\mathrm{loc}}(X; \mathfrak{m})$. Thus the definition of minimal weak upper gradient implies the assertion. □

Proof of Theorem 6.4. Applying Itô's formula to (6.7), we have

$$\begin{aligned} f^2(X_t) - f^2(X_0) &= 2\int_0^t f(X_s)\, \mathrm{d}M_s^f + \langle M^f \rangle_t \\ &= 2\int_0^t f(X_s)\, \mathrm{d}M_s^f + 2\int_0^t |Df|^2(X_s)\, \mathrm{d}s \end{aligned}$$

holds $\mathbb{P}_x$-a.s. for q.e. $x \in X$. Let $\{G_n\}$ be an increasing sequence of relatively compact open sets as appeared above. Then

$$\mathbb{E}_x\left[f^2(X_{t\wedge\tau_{G_n}})\right] = f^2(x) + 2\mathbb{E}_x\left[\int_0^{t\wedge\tau_{G_n}} |Df|^2(X_s)\, \mathrm{d}s\right].$$

Combining this with (6.9) and Lemma 6.6, we have that for each $t > 0$

$$\begin{aligned} 2t|Df|^2(x) &\leq 2\textstyle\int_0^t P_s|Df|^2(x)\, \mathrm{d}s = 2\lim_{n\to\infty} \mathbb{E}_x\left[\int_0^{t\wedge\tau_{G_n}} |Df|^2(X_s)\, \mathrm{d}s\right] \\ &\leq m_{f^2}(A) + \varepsilon^2(r_p^2(x) + 2Nt) - f^2(x) \end{aligned} \tag{6.15}$$

for $x \in X \setminus N_t$ with some $\mathfrak{m}$-null set N_t. Set $N = \bigcup_{t\in\mathbb{Q}_+} N_t$. Then (6.15) holds for all $t > 0$ and $x \in X \setminus N$. If $|Df|^2(x) > 0$ for some $x \in X \setminus N$ and choose $\varepsilon > 0$ so that $\varepsilon N < |Df|^2(x)$, then the linear function $2t(|Df|^2(x) - \varepsilon N)$ is bounded above. This is a contradiction. Therefore, $|Df|^2(x) = 0$ for all $x \in X \setminus N$, i.e., f is a constant $\mathfrak{m}$-a.e., consequently f is a constant. □

7 Polarity

In view of Theorem 3.11 or Theorem 5.3, without conditions (R1) or (R2), we have stochastic expression of $r_p(X_t)$ only until σ_p. Thus it becomes important to know whether $\sigma_p = \infty$ $\mathbb{P}_x$-a.s. or not, when (R2) is not available. In our framework, this question is equivalent to the polarity of $\{p\}$. The goal of this section is to prove the following two assertions:

Theorem 7.1 *Suppose Assumption 1 below. Then $\{p\}$ is polar for $\mathfrak{m}$-a.e. $p \in X$.*

Proposition 7.2 *Suppose Assumption 1 below. Then $\mathfrak{m}$-a.e. $p \in X$ verifies* (R2).

In the proof of them below, we will know more information of negligible sets in the assertions (See Lemmas 7.8, 7.9 and 7.12). For our purpose, Proposition 7.2 would be sufficient and we may not require Theorem 7.1. Nevertheless, we have decided to state both of them since there might be of interest for a different purpose. The proof of Proposition 7.2 is based on a similar idea as we use in the proof of Theorem 7.1. It requires some known results on local structure of RCD spaces, and we introduce them before entering the proof.

By [17, Theorems 4.1.2 and 4.2.4], it suffices to show the $\mathfrak{m}$-polarity of $\{p\}$ for $\mathfrak{m}$-a.e. $p \in X$. To achieve this, we review the notion of tangent cone defined through convergence of metric measure spaces in the pointed measured Gromov-Hausdorff sense.

Definition 7.3 *Let $(X_n, d_n, \mathfrak{m}_n, p_n)$ ($n \in \mathbb{N} \cup \{\infty\}$) be pointed metric measure spaces. That is, $(X_n, d_n, \mathfrak{m}_n)$ is a metric measure space satisfying conditions state in Subsection 2.1 and $p_n \in X_n$ is a reference point. We say $(X_n, d_n, \mathfrak{m}_n, p_n) \to (X_\infty, d_\infty, \mathfrak{m}_\infty, p_\infty)$ as $n \to \infty$ in the pointed measured Gromov-Hausdorff sense if for any $\varepsilon, R > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ there exists a Borel map $f_n = f_n^{R,\varepsilon} : B_R(p_n) \to X_\infty$ such that the following holds:*

(i) $f_n(p_n) = p_\infty$,

(ii) $\displaystyle\sup_{x,y \in B_R(p_n)} |d_n(x,y) - d_\infty(f_n(x), f_n(y))| < \varepsilon$,

(iii) $B_{R-\varepsilon}(p_\infty) \subset B_\varepsilon(f_n(B_R(p_n)))$,

(iv) $(f_n)_\sharp \mathfrak{m}_n|_{B_R(p_n)}$ *weakly converges to* $\mathfrak{m}_\infty|_{B_R(p_\infty)}$ *as* $n \to \infty$.

This is not the same definition as given in [20,33], but it is equivalent (see [33, Proposition 2.3] and references therein). In order to discuss measured tangent cones, we introduce the following normalization of $\mathfrak{m}$: For $p \in X$ and $R > 0$, we define $\mathfrak{m}_R^p$ by

$$\mathfrak{m}_R^p := \left(\int_{B_R(p)} \left(1 - \frac{r_p}{R}\right) \mathrm{d}\mathfrak{m} \right)^{-1} \mathrm{d}\mathfrak{m}.$$

For $k \in \mathbb{N}$, we denote the Euclidean distance on $\mathbb{R}^k$ and the origin by d_{E} and o^k respectively. Let $\mathfrak{L}^k$ be a k-dimensional Lebesgue measure on $\mathbb{R}^k$ normalized to satisfy $\int_{B_1(o^k)} (1 - d_E(o^k, x))\, \mathfrak{L}^k(dx) = 1$, that is, $\mathfrak{L}^k(B_1(o^k)) = k + 1$. Let us define $E_k \subset X$ for $k \in \mathbb{N}$ by

$$E_k := \left\{ p \in X \;\middle|\; \begin{array}{l} \text{For any sequence } \xi_n \in]0,\infty[\ (n \in \mathbb{N}) \text{ with } \lim_{n\to\infty} \xi_n = 0, \\ (X, \xi_n^{-1} d, \mathfrak{m}_{\xi_n}^p, p) \text{ converges to } (\mathbb{R}^k, d_{\mathrm{E}}, \mathfrak{L}^k, o^k) \text{ as } n \to \infty \\ \text{in the pointed measured Gromov-Hausdorff sense.} \end{array} \right\}$$

and set $E = \bigcup_{k \in \mathbb{N},\, k \leq N} E_k$. By definition, $p \in E_k$ means that the measured tangent cone at p is unique and identical to $(\mathbb{R}^k, d_{\mathrm{E}}, \mathfrak{L}^k, o^k)$. The following is a consequence of [33, Corollary 1.2 and Proposition 2.2]:

Proposition 7.4 $\mathfrak{m}(E^c) = 0$.

On the basis of this property, we are ready to state the following assumption in Theorem 7.1.

Assumption 1 $E_1 = \emptyset$.

Remark 7.5 *By [27, Corollary 1.2], the assumption $E_1 = \emptyset$ excludes the case that $(X, d, \mathfrak{m})$ is a one-dimensional space. Indeed, if Assumption 1 does not hold, (X, d) is isometric to an interval $I \subset \mathbb{R}$ or $\mathbb{S}^1$ with the canonical distance. We can easily verify that the conclusion of Theorem 7.1 is no longer true in such a case. Our assumption is sharp in this sense.*

The proof of Theorem 7.1 will be divided into two cases: $p \in E_k$, $k \geq 3$ and $p \in E_2$. Actually, $k = 2$ is a boarderline as known in a classical result on the canonical Euclidean space $\mathbb{R}^k$, and it is indeed subtle in our framework (see Example 7.10 below) since we have some degree of freedom on the choice of the measure $\mathfrak{m}$. To overcome this difficulty, we will use the rectifiability of $\mathsf{RCD}^*(K, N)$ spaces as a metric *measure* space ([21, Theorem 3.5] or [25, Theorem 1.2]).

Proposition 7.6 *There exists $R_j \subset X$ and $k_j \in \mathbb{N} \cap [1, N]$ $(j \in \mathbb{N})$ such that the following property holds:*

(i) $\mathfrak{m}\left(X \setminus \bigcup_{j\in\mathbb{N}} R_j\right) = 0$.

(ii) *Each R_j is bi-Lipschitz to a measurable subset of $\mathbb{R}^{k_j}$.*

(iii) $\mathfrak{m}\left(R_j \setminus E_{k_j}\right) = 0$.

(iv) *Each $\mathfrak{m}|_{R_j}$ is absolutely continuous with respect to the k_j-dimensional Hausdorff measure $\mathcal{H}^{k_j}$.*

Usually, (iii) is not included in the definition of rectifiability. We can verify this property from the proof of [25, Theorem 1.2] (and [33, Theorem 1.1]).

Now let us turn to the proof of Theorem 7.1. The following lemma, which asserts that the ratio of volumes of metric balls of small radii typically behaves like a Euclidean one, plays a key role in the sequel.

Lemma 7.7 *Let $p \in E_k$. Then, for each $\alpha > 0$,*

$$\lim_{\xi\downarrow 0} \frac{\mathfrak{m}(B_{\alpha\xi}(p))}{\mathfrak{m}(B_\xi(p))} = \alpha^k.$$

Proof. Let $\xi_n \in \mathbb{R}$ $(n \in \mathbb{N})$ with $\lim_{n\to\infty} \xi_n = 0$. It suffices to show

$$\lim_{n\to\infty} \mathfrak{m}^p_{\xi_n}(B_{\alpha\xi_n}(p)) = (k + 1)\alpha^k$$

for each $\alpha > 0$. Let $\varepsilon > 0$ and take $f_n : X \to \mathbb{R}^k$ associated with the convergence of $(X, \xi_n^{-1} d, \mathfrak{m}^p_{\xi_n}, p)$ to $(\mathbb{R}^k, d_{\mathrm{E}}, \mathfrak{L}^k, o^k)$ in the pointed measured Gromov-Hausdorff sense, according to Definition 7.3. Then, Definition 7.3 (iv) yields

$$\lim_{n\to\infty} \int_{\mathbb{R}^k} \mathrm{d}(f_n)_\sharp(\mathfrak{m}^p_{\xi_n}|_{B_{\alpha\xi_n}(p)}) = \lim_{n\to\infty} \mathfrak{m}^p_{\xi_n}(B_{\alpha\xi_n}(p)) = \mathfrak{L}^k(B_\alpha(o^k)) = (k + 1)\alpha^k.$$

Hence the conclusion holds. $\square$

By virtue of Proposition 7.4 and Assumption 1, the proof of Theorem 7.1 is reduced to the following two lemmas:

Lemma 7.8 *For $k \geq 3$, every $p \in E_k$ is polar.*

Lemma 7.9 $\mathfrak{m}$*-a.e. $p \in E_2$ is polar.*

Proof of Lemma 7.8. Let $\alpha, \alpha_1 \in]0,1[$ with $\alpha_1 > \alpha$ and $\alpha^2 > \alpha_1^k$ by taking α_1 to be sufficiently close to α. By Lemma 7.7, there exists $\xi_0 > 0$ such that, for any $\xi \in]0, \xi_0[$, we have

$$\frac{\mathfrak{m}(B_{\alpha\xi}(p))}{\mathfrak{m}(B_\xi(p))} \leq \alpha_1^k. \tag{7.1}$$

By [41, Theorem 1], the conclusion follows once we show the following:

$$\int_0^{\xi_0} \frac{u \,\mathrm{d}u}{\mathfrak{m}(B_u(p))} = \infty. \tag{7.2}$$

The change of variable together with an iteration of (7.1) yields

$$\int_{\alpha^n\xi_0}^{\alpha^{n-1}\xi_0} \frac{u\,\mathrm{d}u}{\mathfrak{m}(B_u(p))} = \alpha^{2n-2}\int_{\alpha\xi_0}^{\xi_0} \frac{u'\,\mathrm{d}u'}{\mathfrak{m}(B_{\alpha^{n-1}u'}(p))} \geq \frac{\alpha^{2n-2}}{\alpha_1^{k(n-1)}}\int_{\alpha\xi_0}^{\xi_0} \frac{u'\,\mathrm{d}u'}{\mathfrak{m}(B_{u'}(p))} > 0. \tag{7.3}$$

Thus, by the choice of α_1, (7.2) holds by taking a sum in $n \in \mathbb{N}$ in (7.3). $\square$

Proof of Lemma 7.9. It suffices to show that $\mathfrak{m}$-a.e. $p \in \bigcup_{j\in\mathbb{N}, k_j=2} R_j \cap E_2$ is polar. Let $j_0 \in \mathbb{N}$, $k_{j_0} = 2$ and set $E := R_{j_0} \cap E_2$. Let $\psi : E \to \mathbb{R}^2$ be a bi-Lipschitz map to its image $\psi(E)$ and c_1 the maximum of Lipschitz constants of ψ and ψ^{-1}. Note that there exists a universal constant $c > 0$ such that $\psi^{-1}_\sharp \mathfrak{L}^2 \leq cc_1^2 \mathcal{H}^2|_E$ and $\psi_\sharp \mathcal{H}^2|_E \leq cc_1^2 \mathfrak{L}^2$. We denote the density of $\mathfrak{m}|_E$ with respect to $\mathcal{H}^2|_E$ by ρ. By the Bishop-Gromov inequality (2.4), $\mathfrak{m}$ is locally doubling and hence the Lebesgue differentiation theorem is applicable to $\mathfrak{m}$. With keeping this fact in mind, we define the subset E' of E as follows:

$$E' := \left\{ p \in E \;\middle|\; \begin{array}{l} \rho(p) < \infty,\ p \text{ is an } \mathfrak{m}\text{-Lebesgue point of } 1_E, \text{ and} \\ \psi(p) \text{ is a } \mathfrak{L}^2\text{-Lebesgue point of } \rho \circ \psi^{-1} 1_{\psi(E)} \end{array} \right\}.$$

Apparently, $\mathcal{H}^2(E \setminus E') = \mathfrak{m}(E \setminus E') = 0$. Thus the proof is reduced to show that each $p \in E'$ is polar. Let $B^*_r(q) \subset \mathbb{R}^2$, $q \in \mathbb{R}^2$ and $r > 0$ be a Euclidean ball of radius r centered at q. Then, for $p \in E$ and $r > 0$,

$$\begin{aligned}\mathfrak{m}(B_r(p) \cap E) = \int_{B_r(p)\cap E} \rho \,\mathrm{d}\mathcal{H}^2 &\leq cc_1^2 \int_{\psi(B_r(p)\cap E)} \rho \circ \psi^{-1} \,\mathrm{d}\mathfrak{L}^2 \\ &\leq cc_1^2 \int_{B^*_{c_1 r}(\psi(p))} \rho \circ \psi^{-1} 1_{\psi(E)} \,\mathrm{d}\mathfrak{L}^2.\end{aligned}$$

Therefore, for $p \in E'$, we have

$$\varlimsup_{r\downarrow 0} \frac{\mathfrak{m}(B_r(p))}{\mathfrak{L}^2(B^*_{c_1 r}(\psi(p)))} = \varlimsup_{r\downarrow 0} \frac{\mathfrak{m}(B_r(p)\cap E)}{\mathfrak{L}^2(B^*_{c_1 r}(\psi(p)))} \leq cc_1^2 \rho(p) < \infty.$$

This means that there exists $c_2 > 0$ such that $\mathfrak{m}(B_r(p)) \leq c_2 r^2$ holds for sufficiently small $r > 0$. Thus (7.2) holds and hence the proof is completed. $\square$

The next example is not strictly in our framework, but it suggests the possibility that "$\mathfrak{m}$-a.e." in Theorem 7.1 is not able to be omitted in general.

Example 7.10 *Let* $(X, d, \mathfrak{m}) = (\mathbb{R}^2, d_{\mathrm{E}}, \mathrm{e}^{-V}\mathfrak{L}^2)$*, where*

$$V = -\log((\log|x|)^2 - \log|x|) \cdot 1_{\{|x| \le \mathrm{e}^{-1/2}\}} - 2\log 2 \cdot 1_{\{|x| > \mathrm{e}^{-1/2}\}}.$$

In this case, we can easily see

$$\mathfrak{m}(B_\xi(o^2)) = \pi\xi^2 \left(\log\frac{1}{\xi}\right)^2$$

for $\xi \le \mathrm{e}^{-1/2}$*. This implies*

$$\lim_{\xi \downarrow 0} \frac{\mathfrak{m}(B_{R\xi}(o^2))}{\mathfrak{m}(B_\xi(o^2))} = R^2 \text{ for all } R > 0, \qquad \int_0^{\mathrm{e}^{-1/2}} \frac{u\,\mathrm{d}u}{\mathfrak{m}(B_u(o^2))} < \infty.$$

Thus the same argument as in the proof of Theorem 7.1 does not work in the case $k = 2$*. Actually, we can show that* $\{o^2\}$ *is not polar. By considering the radial process* $d_E(X_t, o^2)$*, we can reduce the problem into the one for a one-dimensional diffusion processes. Then, by using a famous integral test, we can see that* $d_E(X_t, o^2)$ *hits 0.Moreover, since* $u \mapsto u\log u$ *is positive and increasing on* $[\mathrm{e}, 1]$*, we have*

$$\frac{\mathfrak{m}(B_R(o^2))}{\mathfrak{m}(B_r(o^2))} \le \frac{R^2}{r^2}$$

for $0 < r < R < \mathrm{e}^{-1/2}$*. That is, the Bishop-Gromov inequality with* $K = 0$ *and* $N = 2$ *holds locally at* o^2*.*

Remark 7.11 *It seems possible to show that* $(X, d, \mathfrak{m})$ *enjoys* $CD(0, \infty)$ *condition, while this space does not satisfy* $CD(0, N)$ *for any finite* N*. It also seems possible to show that the tangent cone at* o^2 *is unique and identical to* $(\mathbb{R}^2, d_E, \mathfrak{L}^2, o^2)$*. We leave them as future problems.*

Finally we prove Proposition 7.2. It is reduced to the next lemma, which refines the statement by using $(E_k)_k$.

Lemma 7.12 *Suppose* $p \in E_k$ *with* $k \ge 2$*. Then* $p \in X$ *verifies the condition* (R2)*.*

Proof of Lemma 7.12. Take $\alpha, \alpha_1 \in]0, 1[$ satisfying $\alpha_1 > \alpha > \alpha_1^k$. As in the proof of Theorem 7.1 (or Lemma 7.8), we take $\xi_0 > 0$ so that (7.1) holds for any $\xi \in]0, \xi_0[$. Then, by combining (4.14) with the same decomposition we have used in the proof of

Theorem 7.1,

$$\begin{aligned}\int_0^{\xi} \frac{V_u(p)}{u^2}\,\mathrm{d}u &= \sum_{n\in\mathbb{N}} \int_{\alpha^n\xi_0}^{\alpha^{n-1}\xi_0} \frac{V_u(p)}{u^2}\,\mathrm{d}u \\ &= \sum_{n\in\mathbb{N}} \frac{1}{\alpha^{(n-1)}} \int_{\alpha\xi}^{\xi} \frac{V_{\alpha^{n-1}v}(p)}{v^2}\,\mathrm{d}v \\ &\le \sum_{n\in\mathbb{N}} \left(\frac{\alpha_1^k}{\alpha}\right)^{n-1} \int_{\alpha\xi}^{\xi} \frac{V_v(p)}{v^2}\,\mathrm{d}v \\ &\le \frac{1}{\xi}\left(\frac{1}{\alpha}-1\right) V_\xi(p) \sum_{n\in\mathbb{N}} \left(\frac{\alpha_1^k}{\alpha}\right)^{n-1}.\end{aligned}$$

Hence the conclusion follows from these estimates. □

A Appendix: $\mathsf{RCD}^*(K,1)$ spaces

We have tried to include the case $N=1$ in our results for completeness, while the statement of usual Laplacian comparison theorem on weighted Riemannian manifolds looks problematic. Such a problem can be avoided since the case $N=1$ heavily restrict the space $(X,d,\mathfrak{m})$ as follows:

Proposition A.1 *Suppose that $(X,d,\mathfrak{m})$ is an $\mathsf{RCD}^*(K,1)$ space. Then $(X,d,\mathfrak{m})$ is isomorphic as a metric measure space to either $\mathbb{R}$, $[0,+\infty[$, $\mathbb{S}^1(r)=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=r^2\}$ for some $r>0$, or $[0,\ell]$ for some $\ell>0$, where we consider the canonical metric and measure (up to multiplicative constants) on these spaces. In particular, $K\le 0$.*

Proof. By virtue of [27, Corollary 1.2], we already know $(X,d,\mathfrak{m})$ is isomorphic to either one of the candidates with the canonical metric and the measure is of the form $\mathrm{e}^{-f}\mathcal{H}^1$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure and f is $(K,1)$-convex. Thus it suffices to show that f is a constant function. We will show it in the case $X=\mathbb{R}$. All other cases can be discussed similarly. Let $x\in X$ be a point where f is differentiable and $g,h\in\mathcal{C}_c^\infty(X)$ supported on a neighbourhood of x and $g\ge 0$. Note that $\Delta u = u''-u'f'$ for any smooth functions u. By the $(K,1)$-Bochner inequality, we have

$$\frac{1}{2}\int_X |Dh|^2 \Delta g \mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 - \int_X \langle Dh, D\Delta h\rangle g\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 \ge K\int_X |Dh|^2 g\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 + \int_X |\Delta h|^2 g\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1. \quad \text{(A.1)}$$

Here we have

$$\begin{aligned}
-\int_X \langle Dh, D\Delta h\rangle g\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 &= \int_X |\Delta h|^2 g\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 + \int_X \langle Dh, Dg\rangle \Delta h\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 \\
&= \int_X |\Delta h|^2 g\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 - \int_X h'h''g'\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 - \int_X |h'|^2 g''\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 \\
&= \int_X |\Delta h|^2 g\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 + \frac{1}{2}\int_X |h'|^2 \Delta g\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 - \int_X |h'|^2 g''\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1
\end{aligned}$$

Since $\Delta g = g'' - g'f'$, (A.1) implies

$$-\int_X |h'|^2 g'f'\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1 \geq K\int_X |h'|^2 g\mathrm{e}^{-f}\,\mathrm{d}\mathcal{H}^1.$$

First, by choosing g, h so that $\operatorname{supp} h \subset \operatorname{supp} g$ and g is constant on $\operatorname{supp} h$, we can easily deduce $K \leq 0$. Next, by considering a sequence h_n so that $|h_n'|^2 \to \delta_x$ in an appropriate sense, we obtain

$$g'(x)f'(x) \geq Kg(x).$$

Since $g \geq 0$ can be arbitrary, $f'(x) = 0$ must hold. Hence the conclusion follows. $\square$

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