Zero noise limit of a stochastic differential equation involving a local time

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Abstract

This paper studies the zero noise limit for the solution of a class of one-dimensional stochastic differential equations involving local time with irregular drift. These solutions are expected to approach one of the solutions to the ordinary differential equation formally obtained by cutting off the noise term. By determining the limit, we reveal that the presence of the local time really affects the asymptotic behavior, while it is observed only when intensity of the drift term is close to symmetric around the irregular point. Related with this problem, we also establish the Wentzel-Freidlin type large deviation principle.

1 Introduction

This paper studies the following one-dimensional stochastic differential equation:

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon}dW_t + \beta dL_t^0(X^\varepsilon), \quad t \in (0,T], \quad X_0^\varepsilon = 0, \tag{1.1}$$

where $\varepsilon > 0$, $\beta \in (-1,1)$, the driving noise $(W_t)_{t\geq 0}$ is a standard Brownian motion and $(L_t^0(X^\varepsilon))_{t\geq 0}$ is the (symmetric) local time of X^ε at 0. Roughly speaking, $L_t^0(X^\varepsilon)$ provides a singular drift to X^ε at 0. We are interested in the behavior of X^ε in the limit as $\varepsilon \downarrow 0$ when the drift b is not Lipschitz continuous at 0. We expect that X^ε approaches to a solution of the following ordinary differential equation:

$$\varphi_t' = b(\varphi_t), \quad t \in (0,T], \quad \varphi_0 = 0 \tag{1.2}$$

as $\varepsilon \downarrow 0$. However, (1.2) has infinitely many solutions in general because of the non-Lipschitz irregularity of b at 0 even if (1.1) has a unique solution. We investigate which solutions of (1.2) appears in the asymptotic behavior of X^ε for a class of drifts b carrying such a situation.

Our main focus is in the case $\beta \neq 0$. Indeed, no local time is involved in (1.1) when $\beta = 0$ and this problem has been studied well in such a case. The first result in this direction is given by Bafico and Baldi [1]. Funaki and Mitome [8] and Delarue and Flandoli [3] also studied this problem by other means respectively (see also references therein). To explain why we are interested in the case $\beta \neq 0$, we review the main result of [8]. The class of drifts b they consider satisfies $b(0) = 0$, $xb(x) > 0$ for $x \neq 0$ and $b(x)$ behaves as a fractional power when $x \downarrow 0$ and $x \uparrow 0$ respectively (see (2.1) for more precise assumption). They proved that $\lim_{\varepsilon\downarrow 0} X^\varepsilon$ exists

2010 Mathematics Subject Classification. Primary 60H10, 60F10, 34A12; Secondary 60J55 and only two solutions of (1.2) are able to appear in the limit. These solutions are characterized by the fact that they becomes positive or negative immediately. The choice of solutions are completely determined by asymptotic intensity of the drift $b(x)$ in $x \downarrow 0$ and $x \uparrow 0$. It is particularly interesting that one of these two solutions are chosen randomly if intensities are comparable in $x > 0$ and $x < 0$. Moreover, they consider the case of nonconstant (uniformly elliptic) diffusion coefficient, but the choice of diffusion coefficient does not affect the limit. We can guess that the reason why the asymptotic behavior of X^ε studied in [8] depends only on the drift term comes from the fact that the noise W is spatially symmetric at 0. Thus we consider the case $\beta \neq 0$ in (1.1) to study the influence of a spatial asymmetry of the noise to the limit of X^ε. In our case, the noise when $X^\varepsilon = 0$ is biased according to the sign of β. If $b = 0$ and $\varepsilon = 1$, this process is called the skew Brownian motion (see, e.g., Harrison and Shepp [9]). In this case, intuitively, the process enters $(0, \infty)$ or $(-\infty, 0)$ from 0 with probability $(1 + \beta)/2$ or $(1 - \beta)/2$ respectively.

Our result exhibits that β affects the limit of X^ε as we expected. More precisely, the effect of $\beta \neq 0$ is observed only in the case that intensities of the drift $b(x)$ are comparable in $x > 0$ and $x < 0$ (see Theorems 2.1 and 2.2 below). As a consequence, we conclude that asymmetry of the drift term b affects the limit of X^ε more strongly than asymmetry of the noise at 0 given by the local time. In addition, related with this result, we establish the full large deviation principle (Theorem 2.3) in the zero-noise limit which include the explicit description of the rate function. The case $\beta = 0$ can be regarded as a Wentzel-Freidlin theorem with a less regular drift, and we need additional argument to handle the local time. Our large deviation estimate does not seem to be studied before in the literature, and hence it is also interesting in its own right.

In general, adding a noise to deterministic differential equations is a source of many interesting problems. In some cases, we can obtain the uniqueness of the solution by adding a (small) noise to a differential equation which enjoys no uniqueness. Once it happens, the next natural and interesting question is how solutions are selected in the zero-noise limit. As explained above, we discuss such problems in this paper by restricting ourselves to the simplest one-dimensional (ordinary) differential equation but with a less standard noise. From the viewpoint of partial differential equation, our problem can be regarded as a sort of selection problem in the vanishing viscosity limit. See [6] for recent studies to other (ordinary/partial) differential equations.

The method of the proof of our main results is also an extension of the argument given in [8]. As for the determination of the zero noise limit, it is based on the large deviation principle (as mentioned above) and asymptotic behavior of exit time and exit distribution. As it is well-known, the large deviation principle provides us a powerful tool to study the asymptotic behavior of X^ϵ when b is sufficiently regular (see [4, 7] for instance). However, it is not sufficient for our purpose because of the non-uniqueness of solutions of (1.2). Indeed, it only ensures that the probability that X^ϵ is away from the set of all solutions of (1.2) goes to 0, while it provides detailed information on the exponential rate of convergence. In order to determine the limit, we investigate the asymptotic behavior of exit problems and the large deviation estimate is used only in an auxiliary way (see Remark 2.4). Even in our case, techniques in analysis of one-dimensional diffusion processes such as the use of scale functions are available. We discuss exit problems in such a way by means of the Itô-Tanaka formula instead of Itô formula since the scale function associated with (1.1) does not belong to $C^2(\mathbb{R})$. As for the large deviation principle, we remove the local time by using a transform of the process associated with the scale function for the skew Brownian motion (*not* the one for X^ε itself) as did in [11]. Then we can reduce the problem to that in the framework of [2], where they established the large deviation principle for stochastic differential equations with an irregular coefficient.

The paper is organized as follows. We introduce the precise statement of our main results in Section 2. Section 3 is devoted to explain the argument of the proof in [8]. There are two reasons why we review it in this paper. The first reason is to explain how we give the proof of our result. Indeed, as explained, our proof is based on the same strategy. As the second reason, we partially use their result even in our case. Main results are proved in Section 4. First we reduce the proof of our main theorem (Theorems 2.1 and 2.2) into some assertions (Theorem 2.3 and Proposition 4.4) on the basis of the preparations in Section 3. We review known facts on local times in Subsection 4.1, and discuss exit problems (Proposition 4.4) and the large deviation principle (Theorem 2.3) in Subsections 4.2 and 4.3 respectively. For completeness, we provide proofs of some results in Section 3 in Appendices.

2 Main results

As stated in Section 1, we consider the stochastic differential equation (1.1) with $T \in (0, \infty)$. See Subsection 4.1 for the definition of the local time $(L_t^0(X^\varepsilon))_{t\geq 0}$. We suppose that $b : \mathbb{R} \to \mathbb{R}$ is continuous, locally Lipschitz continuous on $\mathbb{R} \setminus \{0\}$ and

$$\lim_{x\downarrow 0} \frac{b(x)}{x^{\gamma_1}} = C_1, \quad \lim_{x\uparrow 0} \frac{b(x)}{|x|^{\gamma_2}} = -C_2, \tag{2.1}$$

with some constants satisfying $0 < \gamma_1 \leq \gamma_2 < 1$, $C_1, C_2 > 0$. In addition, we assume that b is bounded and $b(x) > 0$ for $x > 0$, and $b(x) < 0$ for $x < 0$ for simplicity. It looks restrictive to assume b to be bounded, but we are only interested in asymptotic behavior of the solution of (1.1) when X_t^ε is close to 0. Note that, unlike (1.2), our stochastic differential equation (1.1) has a pathwise unique strong solution (see [5, Theorem 5]). We can easily see $\lim_{\varepsilon\downarrow 0} L_t^0(X^\varepsilon) = 0$ by a property of $L_t^0(X^\varepsilon)$ (see Proposition 4.3). Thus we can expect that X^ε becomes close to the set of solutions of (1.2) as $\varepsilon \downarrow 0$ by a formal observation. To state our main results, we prepare some notations for solutions of (1.2). Under our condition on b, the differential equation (1.2) has a unique positive solution φ_t^+ and a unique negative solution φ_t^-, where positive or negative means $\varphi^+ > 0$ or $\varphi^- < 0$ on $(0, \infty)$ respectively. More precisely, $(\varphi^+)^{-1}(y) = \int_0^y (1/b(u))du$ and $(\varphi^-)^{-1}(y) = \int_y^0 (1/b(u))du$. In this case, it is not difficult to verify that the set of all solutions of (1.2) consists of 0, $\{\varphi^+((t-t_0)_+)\}_{t_0\geq 0}$ and $\{\varphi^-((t-t_0)_+)\}_{t_0\geq 0}$.

Let $\alpha, p_\beta \in (0,1)$ be given as follows:

$$\alpha := \frac{\beta+1}{2}, \quad p_\beta := \frac{\alpha C_1^{1/(1+\gamma_1)}}{\alpha C_1^{1/(1+\gamma_1)} + (1-\alpha) C_2^{1/(1+\gamma_1)}}. \tag{2.2}$$

The following are the first main results of this paper.

Theorem 2.1. If $\gamma_1 < \gamma_2$, for any $\delta > 0$ and $T > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}(\sup_{0\leq t\leq T} |X_t^\varepsilon - \varphi_t^+| \leq \delta) = 1.$$

Theorem 2.2. If $\gamma_1 = \gamma_2$, for any $T > 0$ and sufficiently small $\delta > 0$, we have

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\sup_{0 \le t \le T} |X_t^\varepsilon - \varphi_t^+| \le \delta) = p_\beta, \qquad \lim_{\varepsilon \downarrow 0} \mathbb{P}(\sup_{0 \le t \le T} |X_t^\varepsilon - \varphi_t^-| \le \delta) = 1 - p_\beta.$$

Theorem 2.1 means that, if $|b(x)| \gg |b(-x)|$ for $x > 0$, then X^ε converges to φ^+ in probability. Theorem 2.2 means that, if $|b(x)/b(-x)| \in (0, \infty)$ asymptotically, then X^ε goes to either φ^+ or φ^- and both of them can be chosen with some probability given explicitly in terms of $C_1, C_2, \gamma_1 (= \gamma_2)$ and α. In particular, if $\lim_{x \to 0} |b(x)/b(-x)| = 1$, φ^+ or φ^- is chosen with probability α or $1 - \alpha$ respectively.

To state the second main result, that is, the large deviation principle, we prepare some more notations. Let $\mathcal{W}_0 := \{\phi \in C([0,T]) \mid \phi(0) = 0\}$ be the Wiener space equipped with the supremum norm $\|\cdot\|_\infty$ and the Borel σ-field, and

$$H^1 := \{\phi \in \mathcal{W}_0 \mid \phi \text{ is absolutely continuous on } [0,T] \text{ and } \phi' \in L^2[0,T]\}$$

is the Cameron-Martin subspace of $\mathcal{W}_0$.

Theorem 2.3. The family of distributions of $(X^\varepsilon)_{\varepsilon > 0}$ on $\mathcal{W}_0$ satisfies the large deviation principle as $\varepsilon \downarrow 0$ with the rate function

$$I(\phi) := \begin{cases} \dfrac{1}{2} \displaystyle\int_0^T |b(\phi_s) - \phi_s'|^2 ds, & \phi \in H^1, \\ \infty, & \text{otherwise.} \end{cases}$$

That is, for every open set G in $\mathcal{W}_0$ and closed set F in $\mathcal{W}_0$, we have

$$-\inf_{\phi \in G} I(\phi) \le \underline{\lim}_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}[X^\varepsilon \in G] \le \overline{\lim}_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}[X^\varepsilon \in F] \le -\inf_{\phi \in F} I(\phi).$$

Note that Theorem 2.3 is shown in [8, Theorem 3.1] when $\beta = 0$. It is worth mentioning that our rate function is of the same form as the usual Wentzel-Freidlin theorem and in particular it is independent of β.

Remark 2.4. As we see in the proof of Theorem 2.1 and Theorem 2.2 in Appendix, the full statement of Theorem 2.3 is not required in their proof, even in the case $\beta = 0$ in [8]. What we really use is the fact that the probability that X^ε is away from the set of all solutions of (1.2) goes to 0 as $\varepsilon \to 0$. It is much weaker than the precise upper bound in terms of the rate function. In particular, the lower bound plays no role. Nevertheless, we believe that this assertion would be of independent interest and thus we give a full estimate.

3 Asymptotic behavior of the SDE without local time

As mentioned in Section 1, we briefly review the proof of the main result in [8] which is the case $\beta = 0$ as a starting point of our proof in the case $\beta \neq 0$. For $\ell < 0 < r$, define the exit time $T_{\ell,r}^\varepsilon = T_{\ell,r}^\varepsilon(X^\varepsilon)$ as

$$T_{\ell,r}^\varepsilon := \inf\{t \ge 0 \mid X_t^\varepsilon \in (\ell, r)^c\}.$$

Put T_x^+, T_x^- as follows:

$$T_x^+ := \int_0^x \frac{du}{b(u)}, \quad x \geq 0, \qquad T_x^- := \int_0^x \frac{du}{b(u)}, \quad x \leq 0.$$

Note that $T_x^\pm = \inf\{t \geq 0 \mid \varphi_t^\pm = x\}$ holds respectively. In particular, $T_r^+ = T_\ell^-$ when $\ell = \varphi^-_{T_r^+}$. Together with Theorem 2.3, the following two assertions for asymptotic behavior of the exit time $T^\varepsilon_{\ell,r}$ as $\varepsilon \downarrow 0$ are key ingredients of the proof.

Proposition 3.1 ([8, Proposition 2.3])**.** Suppose $\beta = 0$. Then we have the following:

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}[T^\varepsilon_{\ell,r}] = \begin{cases} p_\beta T_r^+ + (1-p_\beta)T_\ell^-, & \gamma_1 = \gamma_2, \\ T_r^+, & \gamma_1 < \gamma_2, \end{cases}$$

$$\lim_{\varepsilon\downarrow 0} \mathbb{P}[X^\varepsilon_{T^\varepsilon_{\ell,r}} = r] = \begin{cases} p_\beta, & \gamma_1 = \gamma_2, \\ 1, & \gamma_1 < \gamma_2, \end{cases} \qquad \lim_{\varepsilon\downarrow 0} \mathbb{P}[X^\varepsilon_{T^\varepsilon_{\ell,r}} = \ell] = \begin{cases} 1-p_\beta, & \gamma_1 = \gamma_2, \\ 0, & \gamma_1 < \gamma_2, \end{cases}$$

where p_β is the constant given in (2.2).

Lemma 3.2 ([8, Lemma 2.4])**.** Suppose $\beta = 0$. Let $r > 0$ and $\ell = \varphi^-_{T_r^+}$. Then, for any $\delta > 0$,

$$\lim_{\varepsilon\downarrow 0} \mathbb{P}[|T^\varepsilon_{\ell,r} - T_r^+| > \delta] = 0.$$

Lemma 3.2 follows from Theorem 2.3 and Proposition 3.1, and Theorem 2.3, Proposition 3.1 and Lemma 3.2 yield Theorems 2.1 and 2.2 (see Appendix B for details). Note that these proofs are independent of a particular choice of the noise as long as we have Theorem 2.3 and Proposition 3.1. Hence we need only the claim corresponding to Proposition 3.1 once we have shown Theorem 2.3, to extend the result to the case $\beta \neq 0$.

In the rest of this section, we review the proof of Proposition 3.1 in [8] in more details, with keeping the intention that we will follow the same strategy in our case in Subsection 4.2. Let us define functions f_β and s^ε_β (scale function) on $\mathbb{R}$, and a measure m^ε_β (speed measure) on $\mathbb{R}$ as follows:

$$f_\beta(x) := (1-\alpha)\mathbf{1}_{(0,\infty)}(x) + \frac{1}{2}\mathbf{1}_{\{0\}}(x) + \alpha\mathbf{1}_{(-\infty,0)}(x),$$
$$s^\varepsilon_\beta(x) := 2\int_0^x f_\beta(y)\exp\left(-\frac{2}{\varepsilon}\int_0^y b(z)dz\right)dy, \qquad m^\varepsilon_\beta(dx) := \frac{1}{\varepsilon f_\beta(x)}\exp\left(\frac{2}{\varepsilon}\int_0^x b(z)dz\right)dx$$

(For later use, we define them for general β; Recall $\alpha = (\beta+1)/2$). Then the following identities are well-known (see [10] for instance):

Proposition 3.3. Suppose $\beta = 0$. Then, for any $\ell < 0 < r$, we have

$$\mathbb{E}[T^\varepsilon_{\ell,r}] = \frac{-s^\varepsilon_\beta(\ell)}{s^\varepsilon_\beta(r) - s^\varepsilon_\beta(\ell)}\int_0^r (s^\varepsilon_\beta(r) - s^\varepsilon_\beta(y))m^\varepsilon_\beta(dy) + \frac{s^\varepsilon_\beta(r)}{s^\varepsilon_\beta(r) - s^\varepsilon_\beta(\ell)}\int_\ell^0 (s^\varepsilon_\beta(y) - s^\varepsilon_\beta(\ell))m^\varepsilon_\beta(dy), \tag{3.1}$$

$$\mathbb{P}[X^\varepsilon_{T^\varepsilon_{\ell,r}} = r] = \frac{-s^\varepsilon_\beta(\ell)}{s^\varepsilon_\beta(r) - s^\varepsilon_\beta(\ell)}, \quad \mathbb{P}[X^\varepsilon_{T^\varepsilon_{\ell,r}} = \ell] = \frac{s^\varepsilon_\beta(r)}{s^\varepsilon_\beta(r) - s^\varepsilon_\beta(\ell)}. \tag{3.2}$$

Based on these identities, we can discuss the asymptotic behavior of $\mathbb{E}[T^\varepsilon_{\ell,r}]$, $\mathbb{P}[X^\varepsilon_{T^\varepsilon_{\ell,r}} = r]$ and $\mathbb{P}[X^\varepsilon_{T^\varepsilon_{\ell,r}} = \ell]$ by studying that of s^ε_0 and m^ε_0. Actually, the following asymptotic estimates (Lemma 3.4 and Lemma 3.5) are established in [8].

Lemma 3.4 ([8, Lemma 2.1])**.** Suppose $\beta = 0$. Then, for any $\ell < 0 < r$,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1/(1+\gamma_1)} s^\varepsilon_\beta(r) = 2f_\beta(r)(2C_1)^{-1/(1+\gamma_1)}(1+\gamma_1)^{-\gamma_1/(1+\gamma_1)}\Gamma\left(\frac{1}{1+\gamma_1}\right),$$
$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1/(1+\gamma_2)} s^\varepsilon_\beta(\ell) = -2f_\beta(\ell)(2C_2)^{-1/(1+\gamma_2)}(1+\gamma_2)^{-\gamma_2/(1+\gamma_2)}\Gamma\left(\frac{1}{1+\gamma_2}\right),$$

where Γ is the gamma function.

Lemma 3.5 ([8, Lemma 2.2])**.** Suppose $\beta = 0$. Then, for any $\ell < 0 < r$,

$$\lim_{\varepsilon\downarrow 0} \int_0^r (s^\varepsilon_\beta(r) - s^\varepsilon_\beta(y)) m^\varepsilon_\beta(dy) = T^+_r, \qquad \lim_{\varepsilon\downarrow 0} \int_\ell^0 (s^\varepsilon_\beta(y) - s^\varepsilon_\beta(\ell)) m^\varepsilon_\beta(dy) = T^-_\ell.$$

Once we know them, Proposition 3.1 follows immediately by applying them to Proposition 3.3. We will prove Lemma 3.4 and Lemma 3.5 in Appendix A since we require them even in our main result.

Remark 3.6. Let $\hat{s}_\beta := (1-\alpha)x_+ - \alpha x_-$. This is the scale function of the skew Brownian motion (the solution of (1.1) with $b = 0$ and $\varepsilon = 1$). The function f_β given above is the mean of the left and right derivatives of $\hat{s}_\beta$.

Remark 3.7. As mentioned in Section 1, in [8], they deals with a more general diffusion coefficient σ (with uniform ellipticity) than our case $\sigma \equiv 1$. Thus all these results in this section are stated in a more general form in [8].

4 Proof of the main theorems

By virtue of the argument in the last section, the proof of our main theorem is reduced to show Theorem 2.3 and Proposition 3.1 for $\beta \neq 0$. By the definition of s^ε_β and m^ε_β, we can easily see that

$$s^\varepsilon_\beta(x) = 2f_\beta(x) s^\varepsilon_0(x), \quad m^\varepsilon_\beta(dx) = \frac{1}{2f_\beta(x)} m^\varepsilon_0(dx). \tag{4.1}$$

Based on this relation, we can extend Lemma 3.4 and Lemma 3.5 to the case $\beta \neq 0$ immediately. Thus, in the same way as mentioned in the last section, Proposition 3.1 will be extended to the case $\beta \neq 0$ once we show that the assertions in Proposition 3.3 is valid even when $\beta \neq 0$. It will be done in Subsection 4.2 after reviewing some properties of the local time in Subsection 4.1. The proof of Theorem 2.3 will be given in Subsection 4.3. Then the proof of Theorems 2.1 and 2.2 will be completed by gathering them with the observation in the last section.

4.1 Basic properties of local time

Definition 4.1 (Local Time)**.** For a continuous semimartingale X and $a \in \mathbb{R}$, define right or left local time $L^{a\pm} = (L_t^{a\pm}(X))_{t\geq 0}$ and symmetric local time $L^a = (L_t^a(X))_{t\geq 0}$ as follows:

$$
\begin{aligned}
L_t^{a+}(X) &:= (X_t - a)_+ - (X_0 - a)_+ - \int_0^t \mathbf{1}_{(a,\infty)}(X_s)dX_s, \\
L_t^{a-}(X) &:= (X_t - a)_- - (X_0 - a)_- + \int_0^t \mathbf{1}_{(-\infty,a)}(X_s)dX_s, \\
L_t^{a}(X) &:= \frac{1}{2}(L_t^{a+}(X) + L_t^{a-}(X)).
\end{aligned}
$$

We refer to [10, Section 3.7] and [12, Chapter 6] for the proof of properties of them reviewed below. Note that they study only the right local time. The corresponding properties for L_t^{a-} follow by considering $-X_t$ instead of X_t. Indeed, $L_t^{a-}(X) = L_t^{(-a)+}(-X)$ holds. Our argument in Subsections 4.2 and 4.3 are based on the following.

Theorem 4.2 (Itô-Tanaka formula)**.** If X is a semimartingale and $f : \mathbb{R} \to \mathbb{R}$ has left and right derivatives which are of bounded variation, then we have the following:

(i) For $t \geq 0$,

$$
f(X_t) = f(X_0) + \int_0^t D_\pm f(X_s)dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^{x\mp}(X)\mu_f(dx),
$$

where $D_+ f$ and $D_- f$ are left and right derivatives of f respectively, and μ_f is a signed measure corresponding to second derivative of f in the following way

$$
\mu_f([\ell, r)) = D_- f(r) - D_- f(\ell) \quad \text{for any } \ell < r.
$$

(ii) For $t \geq 0$,

$$
f(X_t) = f(X_0) + \int_0^t \frac{1}{2}(D_+ f(X_s) + D_- f(X_s))dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^{x}(X)\mu_f(dx).
$$

We obtain (ii) by combining two formulas in (i).

It is well-known that $t \mapsto L_t^{a+}$ is non-decreasing and continuous for each $a \in \mathbb{R}$, and $a \mapsto L_t^{a+}$ is right continuous for each $a \in \mathbb{R}$. L_t^{a-} satisfies the corresponding properties, and $a \mapsto L_t^{a-}$ is indeed left continuous instead of the right continuity. Moreover we have the following:

Proposition 4.3. (i) $L_t^{a\pm}(X)$ or $L_t^a(X)$ can increase only on the set $\{t \in [0, T] \,;\, X_t = a\}$ as a function of t. In particular, for any measurable $g : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$ and $t \geq 0$, we have

$$
\int_0^t g(X_s)dL_s^a(X) = g(a)L_t^a(X).
$$

Here the integral in the left hand side is the Stieltjes integral by $s \mapsto L_s^a(X)$.

(ii) Let $g:\mathbb{R}\to\mathbb{R}$ be bounded from below and measurable. Then, for $t\geq 0$,

$$\int_{\mathbb{R}} g(x)L_t^{x\pm}(X)dx = \int_{\mathbb{R}} g(x)L_t^x(X)dx = \int_0^t g(X_s)d\langle X\rangle_s.$$

In particular, for $a\in\mathbb{R}$ and $t>0$,

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^t \mathbf{1}_{[a,a+\delta)}(X_s)d\langle X\rangle_s = L_t^{a+}(X), \qquad \lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^t \mathbf{1}_{(a-\delta,a]}(X_s)d\langle X\rangle_s = L_t^{a-}(X),$$
$$\lim_{\delta\downarrow 0}\frac{1}{2\delta}\int_0^t \mathbf{1}_{(a-\delta,a+\delta)}(X_s)d\langle X\rangle_s = L_t^{a}(X).$$

4.2 Extension of Proposition 3.3 to the case $\beta\neq 0$

The goal of this section is to show the following:

Proposition 4.4. (3.1) and (3.2) in Proposition 3.3 hold even when $\beta\neq 0$. In particular, the conclusion of Proposition 3.1 is valid even when $\beta\neq 0$.

The latter assertion is already discussed at the beginning of this section and thus we prove only the former one. Though it is more or less standard, we will give a proof for completeness.

In this section, we abbreviate superscript ε for s_β^ε, m_β^ε, $T_{\ell,r}^\varepsilon$ and X^ε for simplicity of notations. That is, we denote s_β^ε, m_β^ε, $T_{\ell,r}^\varepsilon$ and X^ε by s_β, m_β, $T_{\ell,r}$ and X. We also denote $L_t^x(X)$ by L_t^x for simplicity. We prepare two lemmas (Lemma 4.5 and Lemma 4.6) for the proof of Proposition 4.4

Lemma 4.5. $s_\beta(X_t)$ is a local martingale localized by $(T_{-n,n})_{n\in\mathbb{N}}$.

Proof. Let us define S by

$$S(x) := 2f_\alpha(x)\exp\left(-\frac{2}{\varepsilon}\int_0^x b(z)dz\right) = \frac{D_+s_\beta(x)+D_-s_\beta(x)}{2}.$$

Let us denote $T_{-n,n}$ by τ. By applying the Itô-Tanaka formula (Theorem 4.2) to $s_\beta(X_t)$, we obtain

$$\begin{aligned} s_\beta(X_{t\wedge\tau}) = s_\beta(X_0) + \sqrt{\varepsilon}\int_0^{t\wedge\tau} S(X_s)dW_s + \int_0^{t\wedge\tau} S(X_s)b(X_s)ds \\ + \beta\int_0^{t\wedge\tau} S(X_s)dL_s^0 + \frac{1}{2}\int_{\mathbb{R}} L_{t\wedge\tau}^x\,\mu_{s_\beta}(dx). \end{aligned} \tag{4.2}$$

By Proposition 4.3, we have

$$\int_0^{t\wedge\tau} S(X_s)dL_s^0 = S(0)L_{t\wedge\tau}^0 = L_{t\wedge\tau}^0. \tag{4.3}$$

To proceed, we calculate μ_{s_β}. For $x\in\mathbb{R}$, we have

$$\begin{aligned} D_-s_\beta(x) &= 2\exp\left(-\frac{2}{\varepsilon}\int_0^x b(z)dz\right)\left(\alpha\mathbf{1}_{(-\infty,0]}(x)+(1-\alpha)\mathbf{1}_{(0,\infty)}(x)\right) \\ &= 2\left(\exp\left(-\frac{2}{\varepsilon}\int_0^x b(z)dz\right)-1\right)f_\beta(x) + 2\left(\alpha\mathbf{1}_{(-\infty,0]}(x)+(1-\alpha)\mathbf{1}_{(0,\infty)}(x)\right). \end{aligned}$$

Thus, for an interval $[\ell, r)$, the definition of μ_{s_β} yields

$$\mu_{s_\beta}([\ell, r)) = -\frac{4}{\varepsilon}\int_{[\ell,r)} b(x)\exp\left(-\frac{2}{\varepsilon}\int_0^x b(z)dz\right) f_\beta(x)dx + 2(1-2\alpha)\delta_0([\ell, r)).$$

Here we have used the fact that f_β is constant on $(0, \infty)$ or $(-\infty, 0)$ respectively. By rephrasing the last identity,

$$\mu_{s_\beta}(dx) = -\frac{2}{\varepsilon}b(x)S(x)dx + 2(1-2\alpha)\delta_0(dx).$$

Note that $L^x_{t\wedge\tau} = 0$ if $|x| > n$ by Proposition 4.3 (i). Since $\langle X^\varepsilon\rangle_s = \varepsilon s$ and $\alpha = (\beta+1)/2$, by Proposition 4.3 (ii), we obtain

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} L^x_{t\wedge\tau}\mu_{s_\beta}(dx) &= -\frac{1}{\varepsilon}\int_{\mathbb{R}} L^x_{t\wedge\tau}b(x)S(x)dx + (1-2\alpha)\int_{\mathbb{R}} L^x_{t\wedge\tau}\delta_0(dx)\\ &= -\frac{1}{\varepsilon}\int_0^{t\wedge\tau} b(X_s)S(X_s)d\langle X\rangle_s + (1-2\alpha)L^0_{t\wedge\tau}\\ &= -\int_0^{t\wedge\tau} b(X_s)S(X_s)ds - \beta L^0_{t\wedge\tau}. \qquad (4.4)\end{aligned}$$

Thus the conclusion follows by substituting (4.3) and (4.4) into (4.2). □

Next we consider a function $\psi : \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x) := \int_\ell^x (s_\beta(x) - s_\beta(y))m_\beta(dy).$$

Lemma 4.6. $\psi(X_t) - t$ is a local martingale localized by $(T_{-n,n})_{n\in\mathbb{N}}$.

Proof. Set $\tilde{\psi}(x) := \int_0^x (s_\beta(x) - s_\beta(y))m_\beta(dy)$. Then we have

$$\psi(x) = \tilde{\psi}(x) + s_\beta(x)\int_\ell^0 m_\beta(dy) + \int_\ell^0 s_\beta(y)m_\beta(dy).$$

Thus, by virtue of Lemma 4.5, it suffices to show $\tilde{\psi}(X_t) - t$ is a local martingale. By (4.1), we know $\tilde{\psi}$ is independent of β. Thus, by a standard argument in the case $\beta = 0$, we can verify $\tilde{\psi} \in C^2(\mathbb{R})$, $\tilde{\psi}'(0) = 0$ and

$$\frac{\varepsilon}{2}\tilde{\psi}''(x) + b(x)\tilde{\psi}'(x) = 1.$$

Hence the Itô-Tanaka formula together with Proposition 4.3 (i)(ii) implies the conclusion. □

Proof of Propositon 4.4. We first prove (3.1). Let us define $M : \mathbb{R} \to \mathbb{R}$ by

$$M(x) := -\psi(x) + \frac{s_\beta(x) - s_\beta(\ell)}{s_\beta(r) - s_\beta(\ell)}\int_\ell^r (s_\beta(r) - s_\beta(y))m_\beta(dy).$$

Note that $M(\ell) = M(r) = 0$. Then Lemma 4.5 and Lemma 4.6 yields

$$\mathbb{E}[M(X_{t\wedge T_{\ell,r}})] = M(0) - \mathbb{E}[t \wedge T_{\ell,r}].$$

By the monotone convergence theorem, we obtain

$$\mathbb{E}[T_{\ell,r}] \leq M(0) - \min_{x\in[\ell,r]} M(x) < \infty.$$

Thus $T_{\ell,r} < \infty$ a.s. Hence we obtain $\mathbb{E}[T_{\ell,r}] = M(0) - \mathbb{E}[M(X_{T_{\ell,r}})] = M(0)$ by the dominated convergence theorem because $M(\ell) = M(r) = 0$. By an easy rearrangement, we conclude (3.1).

Next, by Lemma 4.5,

$$\mathbb{E}\left[\frac{s_\beta(X_{t\wedge T_{\ell,r}}) - s_\beta(\ell)}{s_\beta(r) - s_\beta(\ell)}\right] = \frac{-s_\beta(\ell)}{s_\beta(r) - s_\beta(\ell)}.$$

Since $T_{\ell,r} < \infty$ a.s., the dominated convergence theorem yields that the left hand side converges to $\mathbb{P}[X_{T_{\ell,r}} = r]$. Thus we obtain the former half of (3.2). By using the fact $T_{\ell,r} < \infty$ a.s. again, we obtain $\mathbb{P}[X_{T_{\ell,r}} = r] + \mathbb{P}[X_{T_{\ell,r}} = l] = 1$. Hence the latter half of (3.2) holds. $\square$

4.3 Large deviation principle

In the case $\beta = 0$, our large deviation principle corresponds to the Wentzel-Freidlin theorem. It is based on the Schilder theorem which deals with the case $b = 0$. Even in the case $\beta \neq 0$, there is a result corresponding to the Schilder theorem by Krykun [11]. The idea in [11] is to use the scale function $\hat{s}_\beta$ of the skew Brownian motion (see Remark 3.6) to transfer the problem to the one with irregular diffusion coefficient and no local time. We show Theorem 2.3 by following this idea. Even in the case $\beta = 0$ studied in [8, Theorem 1.3], the assertion is less trivial in the sense that b is not smooth but just continuous. See Remark 4.10 below for more details of the proof in [8].

We apply the Itô-Tanaka formula to $\hat{s}_\beta(X_t^\varepsilon)$. It is easy to verify $\mu_{\hat{s}_\beta} = -\beta\delta_0$. Thus, with the aid of Proposition 4.3 (i), we obtain

$$\begin{aligned}\hat{s}_\beta(X_t^\varepsilon) &= \hat{s}_\beta(X_0^\varepsilon) + \sqrt{\varepsilon}\int_0^t f_\alpha(X_s^\varepsilon)dW_s + \int_0^t b(X_s^\varepsilon)f_\alpha(X_s^\varepsilon)ds \\ &\qquad + \beta\int_0^t f_\alpha(X_s^\varepsilon)dL_s^0(X^\varepsilon) + \frac{1}{2}\int_{\mathbb{R}} L_t^x(X^\varepsilon)\mu_{\hat{s}_\beta}(dx) \\ &= \sqrt{\varepsilon}\int_0^t f_\alpha(\hat{s}_\beta(X_s^\varepsilon))dW_s + \int_0^t (b\cdot f_\alpha)\circ\hat{s}_\beta^{-1}(\hat{s}_\beta(X_s^\varepsilon))ds.\end{aligned}$$

Let $Y^\varepsilon := \hat{s}_\beta(X^\varepsilon)$ and $b^* := (b\cdot f_\beta)\circ\hat{s}_\beta^{-1}$. Then the above computation implies that Y^ε is a solution of the stochastic differential equation

$$dY_t^\varepsilon = \sqrt{\varepsilon}f_\beta(Y_t^\varepsilon)dW_t + b^*(Y_t^\varepsilon)dt, \quad t\in(0,T], \quad Y_0^\varepsilon = 0. \tag{4.5}$$

with a discontinuous diffusion coefficient $\sqrt{\varepsilon}f_\beta$ but without local time. Therefore if the law of Y^ε satisfies the large deviation principle on $\mathcal{W}_0$, then the contraction principle yields that the law of $X^\varepsilon = \hat{s}_\beta^{-1}(Y^\varepsilon)$ also satisfies the large deviation principle.

Remark 4.7. We can use s^ε_β instead of $\hat{s}_\beta$ if we merely want to delete the local time. However, it is not suitable for our purpose since s^ε_β depends on ε.

Since b^* is not smooth, we approximate it by smooth functions. For $\tau > 0$, let us define functions g_τ and $\tilde{b}^*$ on $\mathbb{R}$ by $g_\tau(x) := (2\pi\tau)^{-1/2}\exp(-|x|^2/(2\tau))$ and $\tilde{b}^*(x) = \tilde{b}^*_\tau(x) := b^* * g_\tau(x) - b^* * g_\tau(0)$. Note that $\tilde{b}^*$ converges to b^* uniformly as $\tau \to 0$ on any compact set of $\mathbb{R}$ since b^* is continuous. Then we consider a solution $\tilde{Y}^\varepsilon$ of the stochastic differential equation obtained by replacing b^* in (4.5) with $\tilde{b}^*$. That is,

$$d\tilde{Y}^\varepsilon_t = \sqrt{\varepsilon} f_\beta(\tilde{Y}^\varepsilon_t)dW_t + \tilde{b}^*(\tilde{Y}^\varepsilon_t)dt, \quad t \in (0,T], \quad \tilde{Y}^\varepsilon_0 = 0. \tag{4.6}$$

Proposition 4.8. The law of $\tilde{Y}^\varepsilon$ satisfies the large deviation principle with the rate function $\tilde{J}$ given by

$$\tilde{J}(\phi) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \frac{1}{f^2_\beta(\phi_s)}|\tilde{b}^*(\phi_s) - \phi'_s|^2 ds, & \phi \in H^1, \\ \infty, & \text{otherwise.} \end{cases} \tag{4.7}$$

Before entering the proof, we remark that, for $\phi \in H^1$,

$$\mathrm{Leb}(\{t \in [0,T] \mid \phi(t) = 0,\ \phi'(t) \neq 0\}) = 0, \tag{4.8}$$

where $\mathrm{Leb}(A)$ stands for the Lebesgue measure of $A \subset \mathbb{R}$. Indeed, the set of $t \in [0,T]$ with $\phi(t) = 0$ and $\phi'(t) \neq 0$ has no accumulation point.

Proof. In order to apply [2, Theorem B], we claim that $\tilde{Y}^\varepsilon$ solves the following stochastic differential equation:

$$d\tilde{Y}^\varepsilon_t = \sqrt{\varepsilon}\tilde{f}_\alpha(\tilde{Y}^\varepsilon_t)dW_t + \tilde{b}^*(\tilde{Y}^\varepsilon_t)dt, \quad t \in (0,T], \quad \tilde{Y}^\varepsilon_0 = 0, \tag{4.9}$$

where $\tilde{f}_\alpha := (1-\alpha)\mathbf{1}_{(0,\infty)} + \alpha\mathbf{1}_{(-\infty,0]}$. Though it is almost obvious, we give a proof for completeness. Let us define N_t by

$$N_t := \int_0^t \mathbf{1}_{\{0\}}(\tilde{Y}^*_s)dW_s.$$

Once we show $\langle N\rangle_T = 0$, then $N \equiv 0$ by a basic property of martingales and hence (4.9) holds. We can easily see that the following holds:

$$\langle N\rangle_T = \mathrm{Leb}(\{s \in [0,T] \mid \tilde{Y}^\varepsilon_s = 0\}) = \lim_{\delta\downarrow 0}\int_0^T \mathbf{1}_{[0,\delta)}(\tilde{Y}^\varepsilon_s)ds. \tag{4.10}$$

Since $\tilde{Y}^*_t$ is a semimartingale, by Proposition 4.3 (ii), we have

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^T \mathbf{1}_{[0,\delta)}(\tilde{Y}^\varepsilon_s)d\langle\tilde{Y}^\varepsilon\rangle_s = \lim_{\delta\downarrow 0}\frac{\varepsilon}{\delta}\int_0^T \mathbf{1}_{[0,\delta)}(\tilde{Y}^\varepsilon_s)f^2_\beta(\tilde{Y}^\varepsilon_s)ds = L^{0+}_T(\tilde{Y}^\varepsilon) < \infty \quad \text{a.s.}$$

It implies that the right hand side of (4.10) is 0 since $f_\beta \geq (1-\alpha)\wedge\alpha$. Hence our claim holds.

By virtue of the above claim, $\tilde{Y}^\varepsilon$ fulfills the assumption of [2, Theorem B]. Thus it yields that the law of $\tilde{Y}^\varepsilon$ satisfies the large deviation principle with the rate function given by replacing f_β in (4.7) with $\tilde{f}_\alpha$. By (4.8) and the fact $\tilde{b}^*(0) = 0$, for $\phi \in H^1$, we have

$$\frac{1}{\tilde{f}_\alpha^2(\phi_s)}|\tilde{b}^*(\phi_s) - \phi'_s|^2 = \frac{1}{f_\beta^2(\phi_s)}|\tilde{b}^*(\phi_s) - \phi'_s|^2$$

for a.e. $s \in [0, T]$. Thus the desired assertion holds. □

We now turn our attention to Y^ε from $\tilde{Y}^\varepsilon$.

Proposition 4.9. The law of Y^ε satisfies the large deviation principle with the rate function J given by

$$J(\phi) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \frac{1}{f_\beta^2(\phi_s)}|b^*(\phi_s) - \phi'_s|^2 ds, & \phi \in H^1, \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. We first prove the lower bound. Take an open set $G \subset \mathcal{W}_0$ and $\phi \in G$. Since G is open, there is $\delta > 0$ such that $B(\phi, \delta) \subset G$, where $B(\phi, \delta)$ is the δ-neighborhood of ϕ with respect to $\|\cdot\|_\infty$. It suffices to show

$$\varliminf_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}[Y^\varepsilon \in B(\phi, \delta)] \geq -J(\phi). \tag{4.11}$$

since $\mathbb{P}[Y^\varepsilon \in G] \geq \mathbb{P}[Y^\varepsilon \in B(\phi, \delta)]$ and $\phi \in G$ is arbitrary. We may assume $J(\phi) < \infty$ since the claim is obvious if $J(\phi) = \infty$.

For $t \in [0, T]$ and $u \in \mathbb{R}$, let ρ_t and $Z_t(u)$ be given by

$$\begin{aligned} \rho_t :=& \frac{1}{f_\beta(\tilde{Y}_t^\varepsilon)}(b^*(\tilde{Y}_t^\varepsilon) - \tilde{b}^*(\tilde{Y}_t^\varepsilon)), \\ Z_t(u) :=& \exp\left(\frac{u}{\sqrt{\varepsilon}}\int_0^t \rho_s dW_s - \frac{u^2}{2\varepsilon}\int_0^t \rho_s^2 ds\right). \end{aligned}$$

Note that $Z_t(u)$ is integrable and $\mathbb{E}[Z_t(u)] = 1$ since $(\rho_s)_{s \in [0,T]}$ is bounded. We denote $Z_t(1)$ by Z_t for simplicity of notations. We also define a measure $\tilde{\mathbb{P}}$ on $\mathcal{W}_0$ by $d\tilde{\mathbb{P}} := Z_T d\mathbb{P}$. Set $\tilde{W}_t := W_t - \varepsilon^{-1/2}\int_0^t \rho_s ds$. By the Girsanov formula, $\tilde{W}_t$ is a Brownian motion under $\tilde{\mathbb{P}}$. By a rearrangement of (4.6),

$$\tilde{Y}_t^\varepsilon = \sqrt{\varepsilon}\int_0^t f_\beta(\tilde{Y}_s^\varepsilon) d\tilde{W}_s + \int_0^t f_\beta(\tilde{Y}_s^\varepsilon)\rho_s ds + \int_0^t \tilde{b}^*(\tilde{Y}_s^\varepsilon) ds = \sqrt{\varepsilon}\int_0^t f_\beta(\tilde{Y}_s^\varepsilon) d\tilde{W}_s + \int_0^t b^*(\tilde{Y}_s^\varepsilon) ds.$$

Therefore $\tilde{Y}_t^\varepsilon$ solves (4.5) under $\tilde{\mathbb{P}}$.

In order to reduce the claim to that for $\tilde{Y}^\varepsilon$, we prepare an estimate for Z_T. Take $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$ and $q > p$. Then, by a rearrangement,

$$Z_T^{-1/p} = Z_T\left(-\frac{q}{p}\right)^{1/q} \exp\left(\frac{1}{2p\varepsilon}\left(\frac{q}{p} - 1\right)\int_0^T \rho_s^2 ds\right).$$

Set $M_0 := ||\phi||_\infty + \delta$ and $\eta_0 := \sup_{x\in[-M_0,M_0]} |\tilde{b}^*(x) - b^*(x)|$. Then, on $\{\tilde{Y}^\varepsilon \in B(\phi,\delta)\}$, we have $\sup_{0\le s\le t} |\rho_s| < \eta_0 \|f_\beta^{-1}\|_\infty$. Therefore, by applying the Hölder inequality, we obtain

$$\begin{aligned}
\mathbb{P}[\tilde{Y}^\varepsilon \in B(\phi,\delta)] &= \mathbb{E}\left[\mathbf{1}_{B(\phi,\delta)}(\tilde{Y}^\varepsilon) Z_T^{1/p} \cdot Z_T^{-1/p}\right] \\
&\le \mathbb{E}\left[\mathbf{1}_{B(\phi,\delta)}(\tilde{Y}^\varepsilon) Z_T^{1/p} Z_T\left(-\frac{q}{p}\right)\right]^{1/q} \exp\left(\frac{T}{2p\varepsilon}\left(\frac{q}{p}-1\right)\|f_\beta^{-1}\|_\infty^2 \eta_0^2\right) \\
&\le \mathbb{E}\left[\mathbf{1}_{B(\phi,\delta)}(\tilde{Y}^\varepsilon) Z_T\right]^{1/p} \exp\left(\frac{T}{2p\varepsilon}\left(\frac{q}{p}-1\right)\|f_\beta^{-1}\|_\infty^2 \eta_0^2\right) \\
&= \mathbb{P}\left[Y^\varepsilon \in B(\phi,\delta)\right]^{1/p} \exp\left(\frac{T}{2p\varepsilon}\left(\frac{q}{p}-1\right)\|f_\beta^{-1}\|_\infty^2 \eta_0^2\right).
\end{aligned}$$

Here we used the fact that the law of $\tilde{Y}^\varepsilon$ under $\tilde{\mathbb{P}}$ is the same as that of Y^ε under $\mathbb{P}$ in the last equality. Consequently, by virtue of Proposition 4.8, we obtain

$$\begin{aligned}
-\tilde{J}(\phi) &\le \varliminf_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{P}[\tilde{Y}^\varepsilon \in B(\phi,\delta)] \\
&\le \frac{1}{p} \varliminf_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{P}[Y^\varepsilon \in B(\phi,\delta)] + \frac{T}{2p}\left(\frac{q}{p}-1\right)\|f_\beta^{-1}\|_\infty^2 \eta_0^2. \quad (4.12)
\end{aligned}$$

To replace $\tilde{J}$ in the left hand side with J, we provide an estimate of $|J(\phi) - \tilde{J}(\phi)|$ as follows:

$$\begin{aligned}
|\tilde{J}(\phi) - J(\phi)| &= \left|\frac{1}{2}\int_0^T \frac{1}{f_\beta^2(\phi_s)}(\tilde{b}^*(\phi_s) - b^*(\phi_s))(\tilde{b}^*(\phi_s) - b^*(\phi_s) + 2(b^*(\phi_s) - \phi_s'))ds\right| \\
&\le \frac{\eta_0}{2}\int_0^T \frac{1}{f_\beta^2(\phi_s)}(1 + |b^*(\phi_s) - \phi_s'|^2)ds + \frac{\eta_0^2}{2}\int_0^T \frac{ds}{f_\beta^2(\phi_s)} \\
&\le \eta_0\left(\frac{\|f_\beta^{-1}\|_\infty^2 T(1+\eta_0)}{2} + J(\phi)\right). \quad (4.13)
\end{aligned}$$

Thus by combining (4.13) with (4.12), we obtain

$$\varliminf_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{P}[Y^\varepsilon \in B(\phi,\delta)] \ge -p(1+\eta_0)J(\phi) - \frac{\|f_\beta^{-1}\|_\infty^2 T p \eta_0(1+\eta_0)}{2} - \frac{T}{2}\left(\frac{q}{p}-1\right)\|f_\beta^{-1}\|_\infty^2 \eta_0^2.$$

Since $\eta_0 \to 0$ as $\tau \to 0$, (4.11) holds by taking $\tau \to 0$ and $p \downarrow 1$ after it.

We next show the upper bound. Indeed, it goes in a similar way as the lower bound, while we additionally need a tail estimate of $\tilde{Y}^\varepsilon$ in order to apply a sort of localization. Take a closed set $F \subset \mathcal{W}_0$. We denote the expectation with respect to $\tilde{\mathbb{P}}$ by $\mathbb{E}^{\tilde{\mathbb{P}}}$. Then, for $p, q \in (1,\infty)$ with

$p^{-1} + q^{-1} = 1$ and $q > p$, the Hölder inequality yields that

$$\begin{aligned}
\mathbb{P}[Y^\varepsilon \in F] &= \tilde{\mathbb{P}}[\tilde{Y}^\varepsilon \in F] \\
&\le \mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_F(\tilde{Y}^\varepsilon) Z_T^{-1}\right]^{1/p} \mathbb{E}^{\tilde{\mathbb{P}}}\left[Z_T^{q/p}\right]^{1/q} \\
&= \mathbb{P}[\tilde{Y}^\varepsilon \in F]^{1/p} \mathbb{E}\left[Z_T^q\right]^{1/q} \\
&= \mathbb{P}[\tilde{Y}^\varepsilon \in F]^{1/p} \mathbb{E}\left[Z_T (2q)^{1/2} \exp\left(\frac{q}{\varepsilon}\left(q - \frac{1}{2}\right)\int_0^t \rho_s^2 ds\right)\right]^{1/q} \\
&\le \mathbb{P}[\tilde{Y}^\varepsilon \in F]^{1/p} \mathbb{E}\left[\exp\left(\frac{2q}{\varepsilon}\left(q - \frac{1}{2}\right)\int_0^t \rho_s^2 ds\right)\right]^{1/(2q)}. \qquad (4.14)
\end{aligned}$$

To estimate the remainder term, we prepare an estimate of $\mathbb{P}[\|\tilde{Y}^\varepsilon\|_\infty \ge M_1]$ for a sufficiently large $M_1 > 0$, which will be specified later. By Proposition 4.8, we have

$$\overline{\lim_{\varepsilon \downarrow 0}} \varepsilon \log \mathbb{P}[\|\tilde{Y}^\varepsilon\|_\infty \ge M_1] \le - \inf_{\|\phi\|_\infty \ge M_1} \tilde{J}(\phi). \qquad (4.15)$$

For $\phi \in H^1$ with $\|\phi\|_\infty \ge M_1$,

$$\begin{aligned}
\tilde{J}(\phi) &\ge \frac{1}{2\|f_\beta\|_\infty^2} \int_0^T |\tilde{b}^*(\phi_s) - \phi_s'|^2 ds \ge \frac{1}{2\|f_\beta\|_\infty^2}\left(\frac{1}{2}\sup_{0 \le t \le T}\int_0^t |\phi_s'|^2 ds - T\|b^*\|_\infty^2\right) \\
&\ge \frac{1}{2\|f_\beta\|_\infty^2}\left(\frac{1}{2T}\sup_{0 \le t \le T}\left|\int_0^t \phi_s' ds\right|^2 - T\|b^*\|_\infty^2\right) \ge \frac{1}{2\|f_\beta\|_\infty^2}\left(\frac{M_1^2}{2T} - T\|b^*\|_\infty^2\right) =: M_2. \qquad (4.16)
\end{aligned}$$

Let $\eta_1 := \sup_{x \in [-M_1, M_1]} |\tilde{b}^*(x) - b^*(x)|$. Then, by dividing the range of the expectation in (4.14) by $\{\|\tilde{Y}^\varepsilon\|_\infty \ge M_1\}$ and its complement, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{2q}{\varepsilon}\left(q - \frac{1}{2}\right)\int_0^T \rho_s^2 ds\right)\right] \\
&\quad = \mathbb{E}\left[\exp\left(\frac{2q}{\varepsilon}\left(q - \frac{1}{2}\right)\int_0^T \rho_s^2 ds\right)\left(\mathbf{1}_{\{\|\tilde{Y}^\varepsilon\|_\infty \ge M_1\}} + \mathbf{1}_{\{\|\tilde{Y}^\varepsilon\|_\infty < M_1\}}\right)\right] \\
&\quad \le \mathbb{P}[\|\tilde{Y}^\varepsilon\|_\infty \ge M_1] \exp\left(\frac{8qT}{\varepsilon}\left(q - \frac{1}{2}\right)\|f_\beta^{-1}\|_\infty^2 \|b^*\|_\infty^2\right) + \exp\left(\frac{2qT}{\varepsilon}\left(q - \frac{1}{2}\right)\|f_\beta^{-1}\|_\infty^2 \eta_1^2\right).
\end{aligned}$$

Take $M_1 > 0$ so large that $M_2 > 8qT\left(q - \frac{1}{2}\right)\|f_\beta^{-1}\|_\infty^2 \|b^*\|_\infty^2$. Then, by virtue of (4.15) and (4.16), we have

$$\overline{\lim_{\varepsilon \downarrow 0}} \varepsilon \log \mathbb{E}\left[\exp\left(\frac{2q}{\varepsilon}\left(q - \frac{1}{2}\right)\int_0^T \rho_s^2 ds\right)\right] \le 2qT\left(q - \frac{1}{2}\right)\|f_\beta^{-1}\|_\infty^2 \eta_1^2.$$

With keeping this estimate and Proposition 4.8 in mind, by (4.14), we obtain

$$\overline{\lim_{\varepsilon \downarrow 0}} \varepsilon \log \mathbb{P}[Y^\varepsilon \in F] \le -\frac{1}{p}\inf_{\phi \in F} \tilde{J}(\phi) + T\left(q - \frac{1}{2}\right)\|f_\beta^{-1}\|_\infty^2 \eta_1^2. \qquad (4.17)$$

By (4.16) again, we have

$$\inf_{\phi\in C} \tilde{J}(\phi) = \inf_{\substack{\phi\in C\\ \|\phi\|_\infty<M_1}} \tilde{J}(\phi) \wedge \inf_{\substack{\phi\in C\\ \|\phi\|_\infty\geq M_1}} \tilde{J}(\phi) \geq \inf_{\substack{\phi\in C\\ \|\phi\|_\infty<M_1}} \tilde{J}(\phi) \wedge M_2. \tag{4.18}$$

The same argument as in (4.13) implies

$$\inf_{\substack{\phi\in C\\ \|\phi\|_\infty<M_1}} \tilde{J}(\phi) \geq -\frac{\|f_\beta^{-1}\|_\infty^2 T\eta_1(1+\eta_1)}{2} + (1-\eta_1) \inf_{\substack{\phi\in C\\ \|\phi\|_\infty<M_1}} J(\phi). \tag{4.19}$$

Then, substituting (4.18) and (4.19) into (4.17), letting $\tau \to 0$, $M_1 \to \infty$ and finally $p \downarrow 1$, we obtain the desired estimate. □

Proof of Theorem 2.3. We can extend $\hat{s}_\beta^{-1}$ to a map from $\mathcal{W}_0$ to itself. Then we can apply the contraction principle (see [4, Theorem 4.2.1] for instance) to $X^\varepsilon = \hat{s}_\beta^{-1}(Y^\varepsilon)$ to obtain the large deviation principle for the law of X^ε from Proposition 4.9. The rate function is given by $J \circ \hat{s}_\beta$. By virtue of (4.8), it is straightforward to verify this function coincides with I. □

Remark 4.10. The proof of the upper bound of Proposition 4.9 is similar to that of [8, Theorem 1.3]. Our proof of the lower bound is based on the same spirit, while the proof in [8] goes in a different way.

Appendix A Proof of Lemmas 3.4 and 3.5

Here, for completeness, we give the proof of Lemmas 3.4 and 3.5, which is shown in [8].

Proof of Lemma 3.4. By symmetry, it suffices to show only the first assertion. Note first that, for each $\delta \in (0, r)$, there exists $c_\delta > 0$ such that

$$\int_\delta^r \exp\left(-\frac{2}{\varepsilon}\int_0^y b(u)\,du\right)\,dy \leq (r-\delta)\mathrm{e}^{-c_\delta/\varepsilon}.$$

Since the right hand side decays faster than $\varepsilon^{1/(1+\gamma_1)}$, we may assume r to be sufficiently small without loss of generality. Thus, for any $C_1', C_1'' > 0$ with $C_1' < C_1 < C_1''$, we may assume

$$C_1' u^{\gamma_1} \leq b(u) \leq C_1'' u^{\gamma_1}$$

for $u \in [0, r]$. Therefore, the problem can be reduced to show the following: for $C > 0$,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1/(1+\gamma_1)} \int_0^r \exp\left(-\frac{Cy^{1+\gamma_1}}{\varepsilon}\right)\,dy = \frac{C^{-1/(1+\gamma_1)}}{1+\gamma_1}\Gamma\left(\frac{1}{1+\gamma_1}\right). \tag{A.1}$$

By the change of variable $u = Cy^{1+\gamma_1}/\varepsilon$, we have

$$\varepsilon^{-1/(1+\gamma_1)} \int_0^r \exp\left(-\frac{Cy^{1+\gamma_1}}{\varepsilon}\right)\,dy = \frac{C^{-1/(1+\gamma_1)}}{1+\gamma_1} \int_0^{Cr^{1+\gamma_1}/\varepsilon} u^{-\gamma_1/(1+\gamma_1)}\mathrm{e}^{-u}\,du$$

and hence (A.1) follows immediately from this expression. □

Proof of Lemma 3.5. As in the proof of Lemma 3.4, we show only the first assertion. By the definition of s_0^ε and m_0^ε,

$$\int_0^r (s_0^\varepsilon(r) - s_0^\varepsilon(y))\, m_0^\varepsilon(dy) = \frac{2}{\varepsilon}\int_0^r \int_y^r \exp\left(-\frac{2}{\varepsilon}\int_y^z b(u)\, du\right)\, dz\, dy.$$

By the change of variable $z' = 2(z-y)/\varepsilon$ in z-variable, we have

$$\frac{2}{\varepsilon}\int_y^r \exp\left(-\frac{2}{\varepsilon}\int_y^z b(u)\, du\right)\, dz = \int_0^\infty \mathbf{1}_{[0,2(r-y)/\varepsilon]}(z') \exp\left(-\frac{2}{\varepsilon}\int_y^{y+\varepsilon z'/2} b(u)\, du\right)\, dz'.$$

By the definition of b, there exists $C > 0$ such that $b(u) \geq Cu^{\gamma_1}$ for $u \in [0,r]$. This lower bound of b implies

$$\mathbf{1}_{[0,2(r-y)/\varepsilon]}(z') \exp\left(-\frac{2}{\varepsilon}\int_y^{y+\varepsilon z'/2} b(u)\, du\right) \leq \exp\left(-Cz'y^{\gamma_1}\right).$$

Since the right hand side is integrable on $(y,z') \in [0,r]\times[0,\infty)$, we can apply the dominated convergence theorem to conclude

$$\lim_{\varepsilon\downarrow 0}\int_0^r (s_0^\varepsilon(r) - s_0^\varepsilon(y))\, m_0^\varepsilon(dy) = \int_0^r \int_0^\infty \exp(-z' b(y))\, dz'\, dy = T_r^+.$$

□

Appendix B Proof of Theorems 2.1 and 2.2

Here we show our main theorems in the case $\beta = 0$ on the basis of Theorem 2.3 and Proposition 3.1 by following the argument in [8]. We first prove Lemma 3.2 and the main theorem will be shown after it. We provide the proof for completeness so that we can verify that the proof in the case $\beta \neq 0$ goes in exactly the same way.

To begin with, we prepare some notations. Set $\varphi_t^{\eta,\lambda} := \varphi_{(t-\lambda)_+}^\eta$ for $\eta \in \{+,-\}$ and $\lambda \geq 0$. For $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $\delta > 0$, we define $A^\eta_{[t_1,t_2]}(\delta)$ ($\eta \in \{+,-\}$) and $G(\delta)$ as follows:

$$A^\eta_{[t_1,t_2]}(\delta) := \left\{\phi \in \mathcal{W}_0 \;\middle|\; \inf_{\lambda\in[t_1,t_2]} \|\phi - \varphi^{\eta,\lambda}\|_\infty < \delta\right\},$$
$$G(\delta) := A^+_{[0,T]}(\delta) \cup A^-_{[0,T]}(\delta).$$

Proof of Lemma 3.2. We first prove

$$\lim_{\varepsilon\downarrow 0}\mathbb{P}[T^\varepsilon_{\ell,r} < T_r^+ - \delta] = 0. \tag{B.1}$$

Take $\delta_1 > 0$ so small that $\varphi^-_{t+\delta} \leq \phi_t \leq \varphi^+_{t+\delta}$ holds for any $\phi \in G(\delta_1)$ and $t \in [0,T]$. Then it implies $\inf\{t \geq 0 \mid \phi_t \notin (\ell, r)\} \geq T_r^+ - \delta$ (Recall $T_r^+ = T_\ell^-$ in this case). Thus

$$\mathbb{P}[T^\varepsilon_{\ell,r} < T_r^+ - \delta] \leq \mathbb{P}[X^\varepsilon \notin G(\delta_1)]$$

and hence Theorem 2.3 yields (B.1). Next, we prove

$$\lim_{\varepsilon\downarrow 0}\mathbb{P}[T^\varepsilon_{\ell,r} > T^+_r + \delta] = 0. \tag{B.2}$$

Take $\delta_1 > 0$. By (B.1), we have

$$\varlimsup_{\varepsilon\downarrow 0}\mathbb{P}[T^\varepsilon_{\ell,r} > T^+_r + \delta] = \varlimsup_{\varepsilon\downarrow 0}\mathbb{P}[T^\varepsilon_{\ell,r} > T^+_r + \delta \,|\, T^\varepsilon_{\ell,r} \geq T^+_r - \delta_1].$$

Then, the Chebyshev inequality, (B.1) and Proposition 3.1 yield

$$\begin{aligned}\varlimsup_{\varepsilon\downarrow 0}\mathbb{P}[T^\varepsilon_{\ell,r} > T^+_r + \delta \,|\, T^\varepsilon_{\ell,r} \geq T^+_r - \delta_1] &\leq \varlimsup_{\varepsilon\downarrow 0}\frac{1}{\delta+\delta_1}\mathbb{E}[T^\varepsilon_{\ell,r} - T^+_r + \delta_1 \,|\, T^\varepsilon_{\ell,r} \geq T^+_r - \delta_1] \\ &= \varlimsup_{\varepsilon\downarrow 0}\frac{1}{\delta+\delta_1}\mathbb{E}[T^\varepsilon_{\ell,r} - T^+_r + \delta_1] = \frac{\delta_1}{\delta+\delta_1}\end{aligned}$$

(again, recall $T^+_r = T^-_\ell$). Thus (B.2) holds since $\delta_1 > 0$ is arbitrary, and hence the proof is completed. □

Remark B.1. (i) A similar result as Lemma 3.2 is also discussed in [1]. (ii) When $\gamma_1 < \gamma_2$, the conclusion of Lemma 3.2 holds for arbitrary $\ell < 0$ instead of $\ell = \varphi^-_{T^+_r}$, since $\mathbb{P}[X^\varepsilon_{T^\varepsilon_{\ell,r}} = \ell] \to 0$ by Proposition 3.1.

Proof of Theorems 2.1 and 2.2. For $\delta > 0$, take $t_0 > 0$ and $\delta_0 > 0$ so small that the following holds for any $\delta' \in (0, \delta_0)$:

$$A^\eta_{[0,t_0]}(\delta') \subset \{\phi \in \mathcal{W}_0 \mid \|\phi - \varphi^\eta\|_\infty \leq \delta\}, \quad \eta \in \{+,-\}.$$

Let $r \in (0, \varphi^+_T)$ and $\ell := \varphi^-_{T^+_r} \in (\varphi^-_T, 0)$. For $\delta_1 \in (0, \delta_0)$ and $\delta_2 \in (0, T - T^+_r)$, we consider the event E_c ($c \in \{r, \ell\}$) given by

$$E_c := \{X^\varepsilon \in G(\delta_1)\} \cap \{|T^\varepsilon_{\ell,r} - T^+_r| \leq \delta_2\} \cap \{X^\varepsilon_{T^\varepsilon_{\ell,r}} = c\}.$$

If δ_1 is sufficiently small, then $X^\varepsilon_{T^\varepsilon_{\ell,r}} = r$ never occurs on $A^-_{[0,T]}(\delta_1)$. Furthermore, we can take $\delta_1, \delta_2 > 0$ so small that $|T^\varepsilon_{\ell,r} - T^+_r| \leq \delta_2$ never occurs on $A^+_{[t_0,T]}(\delta_1)$. Thus, for appropriately chosen δ_1, δ_2, we have

$$E_r \subset \{X^\varepsilon \in A^+_{[0,t_0]}(\delta_1)\} \subset \{\|X^\varepsilon - \varphi^+\|_\infty \leq \delta\}. \tag{B.3}$$

By the same argument, we also have

$$E_\ell \subset \{X^\varepsilon \in A^-_{[0,t_0]}(\delta_1)\} \subset \{\|X^\varepsilon - \varphi^-\|_\infty \leq \delta\}. \tag{B.4}$$

by taking smaller δ_1, δ_2 if necessary (recall $T^-_\ell = T^+_r$). When $\gamma_1 < \gamma_2$, Theorem 2.3, Proposition 3.1 and Lemma 3.2 yield $\mathbb{P}[E_r] \to 1$ as $\varepsilon \downarrow 0$ and hence the assertion of Theorem 2.1 follows from (B.3). Similarly, when $\gamma_1 = \gamma_2$, we have $\mathbb{P}[E_r] \to p$ and $\mathbb{P}[E_\ell] \to 1 - p$ as $\varepsilon \downarrow 0$ and hence (B.3) and (B.4) conclude Theorem 2.2. □

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