Quantum Groups and Quantizations of Isomonodromic Systems

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5 March 2007

Exploration of New Structures and Natural Constructions in Mathematical Physics

Graduate School of Mathematics (Room 509), Nagoya University, March 5–8, 2007. On the occasion of Professor Akihiro Tsuchiya's retirement

§1. Introduction

Isomonodromic Systems
= Isomonodromic Deformations + Discrete Symmetries

Jimbo-Miwa-Ueno, Physica 2D, 1981.
Jimbo-Miwa, Physica 2D, 4D, 1981.

- Isomonodromic deformations
  = monodromy preserving deformations (differential equations) of rational connections on $\mathbb{P}^1_{\mathbb{C}}$ (or on compact Riemann surfaces).

◦ Deformation parameters = time variables
  = positions of singularities and irregular types of irregular singularities

- Discrete symmetries
  = discrete group actions compatible with isomonodromic deformations.
  = Bäcklund transformations of deformation differential equations

Quantizations of isomonodromic deformations

- the Schlesinger equations $\longrightarrow$ the Knizhnik-Zamolodchikov equations (Reshetikhin (LMP26, 1992), Harnad (hep-th/9406078))

The KZ equations have hypergeometric integral solutions.

- the generalized Schlesinger equations (rank-1 irreg. sing. at ∞)

$\longrightarrow$ the generalized Knizhnik-Zamolodchikov equations
(Babujian-Kitaev (for sl_2, JMP39, 1998),
Felder-Markov-Tarasov-Varchenko (for any $\mathfrak{g}$, math.QA/0001184))

The gen. KZ equations have confluent hypergeometric integral solutions.

Conjecture. Any quantum isomonodromic system has (confluent or non-confluent) hypergeometric integral solutions.

Problem. Quantize the discrete symmetries (the Schlesinger transformations, the birational Weyl group actions, ...).

Quantizations of discrete symmetries

- the q-difference version of the birational Weyl group action (Kajiwara-Noumi-Yamada (nlin.SI/0012063))

$\longrightarrow$ the quantum q-difference version of the birat. Weyl group action (Koji Hasegawa (math.QA/0703036))

- the higher Painlevé equation of type $A_l^{(1)}$ with $\widetilde{W}(A_l^{(1)})$ symmetry (rank-2 irr. sing. at ∞) (Noumi-Yamada (math.QA/9808003))

$\longrightarrow$ the quantum higher Painlevé equation type $A_l^{(1)}$ with $\widetilde{W}(A_l^{(1)})$ sym. (Hajime Nagoya (math.QA/0402281))

- the birational Weyl group action arising from a nilpotent Poisson algebra (Noumi-Yamada (math.QA/0012028))

$\downarrow$ complex powers of Chevalley generators in the Kac-Moody algebra

the Weyl group action on the quotient skew field of $U(\mathfrak{n}) \otimes U(\mathfrak{h})$

- the dressing chains (Shabat-Yamilov (LMJ2, 1991), (Veselov-Shabat (FAA27, 1993), V. E. Adler (Phys.D73, 1994))

$\longrightarrow$ the quantum dressing chains (Lipan-Rasinariu (hep-th/0006074))

∘ $R(z) := z + P^{12}$, $L_k(z) := \begin{bmatrix} x_k & 1 \\ x_k\partial_k - \varepsilon_k + z & \partial_k \end{bmatrix}$, $\partial_k = \partial/\partial x_k$.

∘ $R(z-w)L_k(z)^1 L_k(w)^2 = L_k(w)^2 L_k(z)^1 R(z-w)$.

∘ Assume $n = 2g+1$, $x_{k+n} = x_k$, $\varepsilon_{k+n} = \varepsilon_k + \kappa$ (quasi-periodicity).

∘ The fundamental algebra of the quantum dressing chain is not the algebra generated by x_k, ∂_k but the algebra generated by $f_k := \partial_k + x_{k+1}$. The Hamiltonian of the dressing chain can be expressed with f_k.

Duality. the quantum quasi-periodic dressing chain with period n
$\cong$ the quantum higher Painlevé equation of type $A^{(1)}_{n-1}$.

∘ Thus the $\widetilde{W}(A^{(1)}_{2g})$ symmetry of the dressing chain is also quantized.

Quantizations of Isomonodromic Systems

Classical	**Quantum**
Poisson algebra $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$	Non-commutative algebra $U(\mathfrak{g})$
(generalized) Schlesinger eq.	(generalized) KZ eq.
$A_l^{(1)}$ higher Painlevé eq. with $\widetilde{W}(A_l^{(1)})$ symmetry	quantum $A_l^{(1)}$ higher Painlevé eq. with $\widetilde{W}(A_l^{(1)})$ symmetry
dressing chain with quasi-period $2g+1$ ($\cong A_{2g}^{(1)}$ higher Painlevé eq.) and its $\widetilde{W}(A_{2g}^{(1)})$-symmetry	quantum dressing chain with quasi-period $2g+1$ ($\cong$ quantum $A_{2g}^{(1)}$ higher Painlevé eq.) and its $\widetilde{W}(A_{2g}^{(1)})$-symmetry
birational Weyl group action arising from nilpotent Poisson algebra of NY	the "$U_q(\mathfrak{g}) \to U(\mathfrak{g})$" limit of the Weyl group action on $Q(U_q(\mathfrak{n}) \otimes U_q(\mathfrak{h}))$ constructed in §2

(As far as the speaker knows, the red-colored results are new.)

Quantum q-difference Versions of Discrete Symmetries

q-difference Classical	q-difference Quantum
Poisson algebra $\mathbb{C}[G^*]$ ($G =$ Poisson Lie group)	Non-commutative algebra $U_q(\mathfrak{g})$ (quantum universal enveloping alg.)
q-difference version of the NY birat. Weyl group action arising from nilp. Poisson alg.	Weyl group action on the quotient skew field $Q(U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{h}))$ constructed in **§2**
q-difference version of birational Weyl Group action of KNY (nlin.SI/0012063)	quantum q-difference version of birational Weyl Group action of Hasegawa (reconstructed in **§2**)
$\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ action of KNY	quantum $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ action of **§3**

(As far as the speaker knows, the red-colored results are new.)

§2. Complex powers of Chevalley generators in quantum groups

Problem 1. Find a quantum q-difference version of the Noumi-Yamada birational Weyl group action arising from a nilpotent Poisson algebra (math.QA/0012028).

Answer. Using complex powers of Chevalley generators in quantum groups, we can naturally construct the quantum q-difference version of the NY birational action arising from a nilpotent Poisson algebra.

Problem 2. Find a quantum group interpretation of the quantum q-difference version of the birational Weyl group action constructed by Koji Hasegawa (math.QA/0703036).

Answer. Using complex powers of Chevalley generators in quantum groups, we can reconstruct the Hasegawa quantum birat. action.

Complex powers of Chevalley generators

- $A = [a_{ij}]_{i,j\in I}$, symmetrizable GCM. $d_i a_{ij} = d_j a_{ji}$. $q_i := q^{d_i}$.
- $U_q(\mathfrak{n}_-) = \langle\, f_i \mid i \in I \,\rangle :=$ maximal nilpotent subalgebra of $U_q(\mathfrak{g}(A))$.
- $U_q(\mathfrak{h}) = \langle\, a_\lambda = q^\lambda \mid \lambda \in \mathfrak{h} \,\rangle :=$ Cartan subalgebra of $U_q(\mathfrak{g}(A))$.
- $\alpha_i^\vee :=$ simple coroot, $\alpha_i :=$ simple root, $a_i := a_{\alpha_i} = q^{\alpha_i} = q_i^{\alpha_i^\vee}$.
- $\mathcal{K}_A := Q(U_q(\mathfrak{n}_-)\otimes U_q(\mathfrak{h})) =$ the quotient skew field of $U_q(\mathfrak{n}_-)\otimes U_q(\mathfrak{h})$.
- $a_\lambda = q^\lambda$ regarded as a central element of $\mathcal{K}_A$ is called a **parameter**.

Complex powers of f_i: (Iohara-Malikov (`hep-th/9305138`))

- The action of $\mathrm{Ad}(f_i^\lambda)x = f_i^\lambda x f_i^{-\lambda}$ on $\mathcal{K}_A$ is well-defined.

◦ $$f_i^\lambda f_j f_i^{-\lambda} = q_i^{-\lambda} f_j + [\lambda]_{q_i}(f_i f_j - q_i^{-1} f_j f_i) f_i^{-1}$$
$$= [1-\lambda]_{q_i} f_j + [\lambda]_{q_i} f_i f_j f_i^{-1} \qquad \text{if } a_{ij} = -1,$$

where $[x]_q := (q^x - q^{-x})/(q - q^{-1})$.

Verma relations $\Longleftrightarrow$ Coxeter relations

Verma relations of Chevalley generators f_i in $U_q(\mathfrak{n}_-)$:

$$f_i^a f_j^{a+b} f_i^b = f_j^b f_i^{a+b} f_j^a \qquad (a, b \in \mathbb{Z}_{\geqq 0}) \qquad \text{if } a_{ij}a_{ji} = 1.$$

(formulae for non-simply-laced cases are omitted)

(Lusztig, Introduction to Quantum Groups, Prop.39.3.7 or Lemma 42.1.2.)

- Verma relations can be extended to the complex powers f_i^λ.
- $\tilde{r}_i \lambda \tilde{r}_i^{-1} = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$ for $\lambda \in \mathfrak{h}$ (Weyl group action on parameters).
- Verma relations of f_i's $\Longleftrightarrow$ Coxeter relations of $R_i := f_i^{\alpha_i^\vee} \tilde{r}_i$'s.

∘ $R_i^2 = f_i^{\alpha_i^\vee} \tilde{r}_i f_i^{\alpha_i^\vee} \tilde{r}_i = f_i^{\alpha_i^\vee} f_i^{-\alpha_i^\vee} \tilde{r}_i^2 = 1.$

∘ $R_i R_j R_i = f_i^{\alpha_i^\vee} \tilde{r}_i f_j^{\alpha_j^\vee} \tilde{r}_j f_i^{\alpha_i^\vee} \tilde{r}_i = f_i^{\alpha_i^\vee} f_j^{\alpha_i^\vee + \alpha_j^\vee} f_i^{\alpha_j^\vee} \tilde{r}_i \tilde{r}_j \tilde{r}_i$

$= f_j^{\alpha_j^\vee} f_i^{\alpha_i^\vee + \alpha_j^\vee} f_j^{\alpha_i^\vee} \tilde{r}_j \tilde{r}_i \tilde{r}_j = f_j^{\alpha_j^\vee} \tilde{r}_j f_i^{\alpha_i^\vee} \tilde{r}_i f_j^{\alpha_j^\vee} \tilde{r}_j = R_j R_i R_j$ if $a_{ij}a_{ji} = 1.$

(formulae for non-simply-laced cases are omitted)

Theorem. $\mathrm{Ad}(R_i) = \mathrm{Ad}(f_i^{\alpha_i^\vee}\tilde{r}_i)$ $(i \in I)$ generate the action of the Weyl group on $\mathcal{K}_A$ as algebra automorphisms. This is the quantum q-difference version of the Noumi-Yamada birational Weyl group action arising from a nilpotent Poisson algebra (`math.QA/0012028`).

Example. If $a_{ij} = -1$, then

$$f_i^2 f_j - (q_i + q_i^{-1}) f_i f_j f_i + f_j f_i f_i = 0,$$

$$\begin{aligned}\mathrm{Ad}(R_i) f_j = f_i^{\alpha_i^\vee} f_j f_i^{-\alpha_i^\vee} &= q_i^{-\alpha_i^\vee} f_j + [\alpha_i^\vee]_{q_i} (f_i f_j - q_i^{-1} f_j f_i) f_i^{-1} \\ &= [1-\alpha_i^\vee]_{q_i} f_j + [\alpha_i^\vee]_{q_i} f_i f_j f_i^{-1},\end{aligned}$$

$$\mathrm{Ad}(R_i) a_i = \tilde{r}_i a_i \tilde{r}_i^{-1} = a_i^{-1}, \qquad \mathrm{Ad}(R_i) a_j = \tilde{r}_i a_j \tilde{r}_i^{-1} = a_i a_j.$$

In particular, as the $q \to 1$ limit, we have

$$\mathrm{Ad}(R_i) f_j = f_j + \alpha_i^\vee [f_i, f_j] f_i^{-1} = (1 - \alpha_i^\vee) f_j + \alpha_i^\vee f_i f_j f_i^{-1}.$$

Truncated q-Serre relations and Weyl group actions

Assumptions:

- $k_i k_j = k_j k_i,\ \ k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j.$ (the action of the Cartan subalgebra)
- $f_i f_j = q_i^{\pm(-a_{ij})} f_j f_i \ \ (i \neq j).$ (truncated q-Serre relations)
- $f_{i1} := f_i \otimes 1, \quad f_{i2} := k_i^{-1} \otimes f_i.$ ($f_{i1} + f_{i2} =$ "coproduct of f_i")

Skew field $\mathcal{K}_H$ generated by F_i, a_i:

- $\mathcal{K}_H :=$ the skew field generated by $F_i := a_i^{-1} f_{i1}^{-1} f_{i2},\ \ a_i = q^{\alpha_i}.$
- Then $F_i F_j = q_i^{\pm 2(-a_{ij})} F_j F_i \quad (i \neq j),$ $a_i \in$ center of $\mathcal{K}_H$.
- $\tilde{r}_i a_j \tilde{r}_i^{-1} = a_i^{-a_{ij}} a_j.$ (the action of the Weyl group on parameters).

Theorem. Put $R_i := (f_{i1} + f_{i2})^{\alpha_i^\vee} \tilde{r}_i.$
Then $\mathrm{Ad}(R_i)$'s generate the action of the Weyl group on $\mathcal{K}_H$.

q-binomial theorem and explicit formulae of actions

- Applying the q-binomial theorem to $f_{i1}f_{i2} = q_i^{-2} f_{i2}f_{i1}$, we obtain

$$(f_{i1} + f_{i2})^{\alpha_i^\vee} = \frac{(a_i^{-1}F_i)_{i,\infty}}{(a_iF_i)_{i,\infty}} f_{i1}^{\alpha_i^\vee}, \quad \text{where } (x)_{i,\infty} := \prod_{\mu=0}^{\infty}(1 + q_i^{2\mu}x).$$

Explicit Formulae. If $i \neq j$, then

$$\mathrm{Ad}(R_i)F_i = F_i,$$

$$\mathrm{Ad}(R_i)F_j = \begin{cases} F_j \displaystyle\prod_{\mu=0}^{-a_{ij}-1} \frac{1 + q_i^{2\mu}a_iF_i}{a_i + q_i^{2\mu}F_i} & \text{if } F_iF_j = q_i^{+2(-a_{ij})}F_jF_i, \\ \displaystyle\prod_{\mu=0}^{-a_{ij}-1} \frac{a_i + q_i^{2\mu}F_i}{1 + q_i^{2\mu}a_iF_i} F_j & \text{if } F_iF_j = q_i^{-2(-a_{ij})}F_jF_i. \end{cases}$$

- These formulae coincide with those of the quantum q-difference Weyl group action constructed by Koji Hasegawa (`math.QA/0703036`).

§3. Quantization of the $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ action of KNY

Problem 3. For any integers $m, n \geqq 2$, construct
(a) a non-commutative skew field $\mathcal{K}_{m,n}$ and
(b) an action of $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ on $\mathcal{K}_{m,n}$ as alg. automorphisms which is a quantization of the Kajiwara-Noumi-Yamada action of $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ on $\mathbb{C}(x_{ik} | 1 \leqq i \leqq m, 1 \leqq k \leqq n)$.

Answer. If m, n are **mutually prime**,
then we can construct a quantization of the KNY action.

Tools.
(a) Gauge invariant subalgebras of quotients of affine quantum groups,
(b) Complex powers of corrected Chevalley generators.

The KNY discrete dynamical systems

Kajiwara-Noumi-Yamada, nlin.SI/0106029,
Discrete dynamical systems with $W\bigl(A^{(1)}_{m-1} \times A^{(1)}_{n-1}\bigr)$ symmetry.
Kajiwara-Noumi-Yamada, nlin.SI/0112045.
Noumi-Yamada, math-ph/0203030.

(1) Action of $\widetilde{W}\bigl(A^{(1)}_{m-1}\bigr) \times \widetilde{W}\bigl(A^{(1)}_{n-1}\bigr)$ as algebra automorphisms on the rational function field $\mathbb{C}(x_{ik}|1 \leqq i \leqq m, 1 \leqq k \leqq n)$.

(2) Lax representations $\Longrightarrow$ q-difference isomonodromic systems.

(3) Poisson brackets are, however, **not** given.

First Problem. Usually quantization replaces Poisson brackets by commutators. How to find an appropriate quantization of $\mathbb{C}(x_{ik}|1 \leqq i \leqq m, 1 \leqq k \leqq n)$ without Poisson brackets?

Minimal representations of Borel subalgebra of $U_q(\widehat{\mathrm{gl}}_m)$

- $\mathcal{B}_{m,n}$:= the associative algebra over $\mathbb{F}' := \mathbb{C}(q, r', s')$ generated by $a_{ik}^{\pm 1}, b_{ik}^{\pm 1}$ $(i, k \in \mathbb{Z})$ with following fundamental relations:

$$a_{i+m,k} = r' a_{ik}, \quad a_{i,k+n} = s' a_{ik}, \quad b_{i+m,k} = r' b_{ik}, \quad b_{i,k+n} = s' b_{ik},$$
$$a_{ik} b_{ik} = q^{-1} b_{ik} a_{ik}, \qquad a_{ik} b_{i-1,k} = q b_{i-1,k} a_{ik}.$$

All other combinations from $\{a_{ik}, b_{ik}\}_{1 \leqq i \leqq m,\, 1 \leqq k \leqq n}$ commute.

- $U_q(\mathfrak{b}_-) = \langle\, t_i, f_i \mid i \in \mathbb{Z} \,\rangle$:= the lower Borel subalgebra of $U_q(\widehat{\mathrm{gl}}_m)$ with fundamental relations: $t_{i+m} = r' t_i$, $f_{i+m} = f_i$,

$$t_i t_j = t_j t_i, \qquad t_i f_i = q^{-1} f_i t_i, \qquad t_i f_{i-1} = q f_{i-1} t_i,$$
$$f_i f_j = f_j f_i \qquad (j \not\equiv i \pm 1 \pmod{m}),$$
$$f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0 \qquad (q\text{-Serre relations}).$$

- For each k, the algebra homomorphism $U_q(\mathfrak{b}_-) \to \mathcal{B}_{m,n}$ is given by

$$t_i \mapsto a_{ik}, \qquad f_i \mapsto a_{ik}^{-1} b_{ik}. \qquad \text{(minimal representations of } U_q(\mathfrak{b}_-))$$

$RLL = LLR$ relations (Quantum group)

R-matrix:

$$R(z) := \sum_{i=1}^{m} (q - z/q) E_{ii} \otimes E_{ii} + \sum_{i \neq j} (1 - z) E_{ii} \otimes E_{jj} + \sum_{i<j} \left((q - q^{-1}) E_{ij} \otimes E_{ji} + (q - q^{-1}) z E_{ji} \otimes E_{ij} \right).$$

L-operators:

$$L_k(z) := \begin{bmatrix} a_{1k} & b_{1k} & & \\ & a_{2k} & \ddots & \\ & & \ddots & b_{m-1,k} \\ b_{mk}\, z & & & a_{mk} \end{bmatrix}.$$

$RLL = LLR$ relations:

$$R(z/w) L_k(z)^1 L_k(w)^2 = L_k(w)^2 L_k(z)^1 R(z/w),$$

$$L_k(z)^1 L_l(w)^2 = L_l(w)^2 L_k(z)^1 \qquad (k \not\equiv l \pmod{n}),$$

where $L_k(z)^1 := L_k(z) \otimes 1$, $L_k(w)^2 := 1 \otimes L_k(w)$.

Gauge invariant subalgebra $\mathcal{A}_{m,n} = \mathcal{B}_{m,n}^{\mathcal{G}}$ of $\mathcal{B}_{m,n}$

Gauge group: $\mathcal{G} := (\mathbb{F}'^{\times})^{mn} \ni g = (g_{ik})$. $g_{i+m,k} = g_{ik}$, $g_{i,k+n} = g_{ik}$.

Gauge transformation: The algebra automorphism of $\mathcal{B}_{m,n}$ is given by

$$a_{ik} \mapsto g_{ik} a_{ik} g_{i,k+1}^{-1}, \qquad b_{ik} \mapsto g_{ik} b_{ik} g_{i+1,k+1}^{-1},$$

i.e. $$L_k(z) \mapsto g_k L_k(z) g_{k+1}^{-1} \qquad (g_k := \operatorname{diag}(g_{1k}, g_{2k}, \dots, g_{mk})).$$

- Assume that m, n are **mutually prime** integers $\geqq 2$.
- $\widetilde{m}$:= mod-n inverse of m $(\widetilde{m}m \equiv 1 \pmod{n},\ \widetilde{m} = 1, 2, \dots, n-1)$.
- The gauge invariant subalgebra $\mathcal{B}_{m,n}^{\mathcal{G}}$ of $\mathcal{B}_{m,n}$ is generated by

$$x_{ik}^{\pm 1} := \left(a_{ik}(b_{ik} b_{i+1,k+1} \cdots b_{i,k+\widetilde{m}m-1})^{-1}\right)^{\pm 1},$$
$$b_{\mathrm{all}}^{\pm 1} := \left(\prod_{i=1}^{m} \prod_{k=1}^{n} b_{ik}\right)^{\pm 1} \in \text{center of } \mathcal{B}_{m,n}.$$

- $\mathcal{A}_{m,n}$:= the algebra gen. by x_{ik}'s over $\mathbb{F} = \mathbb{C}(q^2, r, s)$.
- $\mathcal{K}_{m,n} := Q(\mathcal{A}_{m,n})$ is an appropriate quantization of $\mathbb{C}(\{x_{ik}\})$.

q-commutation relations of x_{ik}'s

- $B := \{\, (\mu \bmod m,\ \mu \bmod n) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \mid \mu = 0, 1, \dots, \widetilde{m}m - 1 \,\}$.
- $p_{\mu\nu} := \begin{cases} q & \text{if } (\mu \bmod m,\ \nu \bmod n) \in B, \\ 1 & \text{otherwise.} \end{cases}$
- $q_{\mu\nu} := (p_{\mu\nu}/p_{\mu-1,\nu})^2 \in \{1, q^{\pm 2}\}$. (definition of $q_{\mu\nu}$)

Fundamental relations of x_{ik}'s:

$$x_{i+m,k} = r x_{ik}, \quad x_{i,k+n} = s x_{ik} \qquad (r := r'^{1-\widetilde{m}m},\ s := s'^{1-\widetilde{m}m}),$$

$$x_{i+\mu,k+\nu} x_{ik} = q_{\mu\nu} x_{ik} x_{i+\mu,k+\nu} \qquad (0 \leqq \mu < m,\ 0 \leqq \nu < n).$$

Example. If $(m, n) = (2, 3)$, then $\widetilde{m} = 2$ and

$$[p_{\mu\nu}] = \begin{bmatrix} q & 1 & q \\ q & q & 1 \end{bmatrix}, \quad [q_{\mu\nu}] = \begin{bmatrix} 1 & q^{-2} & q^2 \\ 1 & q^2 & q^{-2} \end{bmatrix} \quad \begin{pmatrix} \mu = 0, 1 \\ \nu = 0, 1, 2 \end{pmatrix}.$$

$$x_{11}x_{11} = x_{11}x_{11}, \quad x_{12}x_{11} = q^{-2}x_{11}x_{12}, \quad x_{13}x_{11} = q^2 x_{11}x_{13},$$

$$x_{21}x_{11} = x_{11}x_{21}, \quad x_{22}x_{11} = q^2 x_{11}x_{22}, \quad x_{23}x_{11} = q^{-2}x_{11}x_{23}.$$

Example. (1) If $(m,n) = (2, 2g+1)$, then $\widetilde{m} = g+1$ and

$$[q_{\mu\nu}] = \begin{bmatrix} 1 & q^{-2} & q^2 & \cdots & q^{-2} & q^2 \\ 1 & q^2 & q^{-2} & \cdots & q^2 & q^{-2} \end{bmatrix} \quad \begin{pmatrix} \mu = 0,1 \\ \nu = 0,1,2,\dots,2g-1,2g \end{pmatrix}.$$

$$1 < k \leqq n \implies x_{1k}x_{11} = q^{(-1)^{k-1}2} x_{11}x_{1k},\ x_{2k}x_{11} = q^{(-1)^k 2} x_{11}x_{2k}.$$

(2) If $(m,n) = (2g+1, 2)$, then $\widetilde{m} = 1$ and

$$[p_{\mu\nu}] = \begin{bmatrix} q & 1 \\ 1 & q \\ q & 1 \\ \vdots & \vdots \\ 1 & q \\ q & 1 \end{bmatrix}, \quad [q_{\mu\nu}] = \begin{bmatrix} 1 & 1 \\ q^{-2} & q^2 \\ q^2 & q^{-2} \\ \vdots & \vdots \\ q^{-2} & q^2 \\ q^2 & q^{-2} \end{bmatrix} \quad \begin{pmatrix} \mu = 0,1,2,\dots,2g-1,2g \\ \nu = 0,1 \end{pmatrix}.$$

Observation: $\mathcal{A}_{2,n} \cong \mathcal{A}_{n,2}$, $x_{ik} \leftrightarrow x_{ki}$, $q \leftrightarrow q$, $r \leftrightarrow s$, $s \leftrightarrow r$.

Example. (1) If $(m,n)=(3,4)$, then $\widetilde{m}=3$ and

$$[p_{\mu\nu}]=\begin{bmatrix} q & 1 & q & q \\ q & q & 1 & q \\ q & q & q & 1 \end{bmatrix}, \quad [q_{\mu\nu}]=\begin{bmatrix} 1 & q^{-2} & 1 & q^2 \\ 1 & q^2 & q^{-2} & 1 \\ 1 & 1 & q^2 & q^{-2} \end{bmatrix} \quad \left(\begin{array}{l} \mu=0,1,2 \\ \nu=0,1,2,3 \end{array} \right).$$

$$x_{12}x_{11}=q^{-2}x_{11}x_{12},\ x_{13}x_{11}=x_{11}x_{13},\ x_{14}x_{11}=q^2x_{11}x_{14},\ \ldots$$

(2) If $(m,n)=(4,3)$, then $\widetilde{m}=1$ and

$$[p_{\mu\nu}]=\begin{bmatrix} q & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & q \\ q & 1 & 1 \end{bmatrix}, \quad [q_{\mu\nu}]=\begin{bmatrix} 1 & 1 & 1 \\ q^{-2} & q^2 & 1 \\ 1 & q^{-2} & q^2 \\ q^2 & 1 & q^{-2} \end{bmatrix} \quad \left(\begin{array}{l} \mu=0,1,2,3 \\ \nu=0,1,2 \end{array} \right).$$

Observation: $\mathcal{A}_{3,4}\cong\mathcal{A}_{4,3}$, $\quad x_{ik}\leftrightarrow x_{ki}$, $q\leftrightarrow q$, $r\leftrightarrow s$, $s\leftrightarrow r$.

Example. (1) If $(m,n)=(3,5)$, then $\widetilde{m}=2$ and

$$[p_{\mu\nu}]=\begin{bmatrix} q & 1 & 1 & q & 1 \\ 1 & q & 1 & 1 & q \\ q & 1 & q & 1 & 1 \end{bmatrix}, \quad [q_{\mu\nu}]=\begin{bmatrix} 1 & 1 & q^{-2} & q^2 & 1 \\ q^{-2} & q^2 & 1 & q^{-2} & q^2 \\ q^2 & q^{-2} & q^2 & 1 & q^{-2} \end{bmatrix}.$$

$x_{12}x_{11}=x_{11}x_{12}$, $x_{13}x_{11}=q^{-2}x_{11}x_{13}$, $x_{14}x_{11}=q^2x_{11}x_{14}$, ...

(2) If $(m,n)=(5,3)$, then $\widetilde{m}=2$ and

$$[p_{\mu\nu}]=\begin{bmatrix} q & 1 & q \\ q & q & 1 \\ 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{bmatrix}, \quad [q_{\mu\nu}]=\begin{bmatrix} 1 & q^{-2} & q^2 \\ 1 & q^2 & q^{-2} \\ q^{-2} & 1 & q^2 \\ q^2 & q^{-2} & 1 \\ 1 & q^2 & q^{-2} \end{bmatrix}.$$

Observation: $\mathcal{A}_{3,5}\cong\mathcal{A}_{5,3}$, $\quad x_{ik}\leftrightarrow x_{ki}$, $q\leftrightarrow q$, $r\leftrightarrow s$, $s\leftrightarrow r$.

Symmetries of $\mathcal{A}_{m,n}$

Duality. The algebra isomorphism $\mathcal{A}_{m,n} \cong \mathcal{A}_{n,m}$ is given by

$$x_{ik} \leftrightarrow x_{ki}, \quad q \leftrightarrow q, \quad r \leftrightarrow s, \quad s \leftrightarrow r.$$

Reversal. The algebra involution of $\mathcal{A}_{m,n}$ is given by

$$x_{ik} \leftrightarrow x_{-i,-k}, \quad q \leftrightarrow q^{-1}, \quad r \leftrightarrow s^{-1}, \quad s \leftrightarrow r^{-1}.$$

Translation. For any integers μ, ν,
the algebra automorphism of $\mathcal{A}_{m,n}$ is given by

$$x_{ik} \mapsto x_{i+\mu,k+\nu}, \quad q \mapsto q, \quad r \mapsto r, \quad s \mapsto s.$$

Extended affine Weyl groups $\widetilde{W}(A^{(1)}_{m-1})$, $\widetilde{W}(A^{(1)}_{n-1})$

- $\widetilde{W}(A^{(1)}_{m-1}) := \langle r_0, r_1, \dots, r_{m-1}, \omega \rangle$ with fundamental relations:
  $r_i r_j = r_j r_i \ \ (j \not\equiv i, i+1 \pmod{m})$, $\ r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$, $\ r_i^2 = 1$,
  $\omega r_i \omega^{-1} = r_{i+1} \ \ (r_{i+m} = r_i)$.

◦ $T_i := r_{i-1} \cdots r_2 r_1 \omega r_{m-1} \cdots r_{i+1} r_i$ (translations).

◦ $\widetilde{W}(A^{(1)}_{m-1}) = \langle r_1, r_2, \dots, r_{m-1} \rangle \ltimes \langle T_1, T_2, \dots, T_m \rangle \cong S_m \ltimes \mathbb{Z}^m$.

- $\widetilde{W}(A^{(1)}_{n-1}) := \langle s_0, s_1, \dots, s_{n-1}, \varpi \rangle$ with fundamental relations:
  $s_k s_l = s_l s_k \ \ (l \not\equiv k, k+1 \pmod{n})$, $\ s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$, $\ s_k^2 = 1$,
  $\varpi s_k \varpi^{-1} = s_{k+1} \ \ (s_{k+n} = s_k)$.

◦ $U_k := s_{k-1} \cdots s_2 s_1 \varpi s_{n-1} \cdots s_{k+1} s_k$ (translations).

◦ $\widetilde{W}(A^{(1)}_{n-1}) = \langle s_1, s_2, \dots, s_{n-1} \rangle \ltimes \langle U_1, U_2, \dots, U_n \rangle \cong S_n \ltimes \mathbb{Z}^n$.

Explicit formulae of the action of $\widetilde{W}(A^{(1)}_{m-1})$ on $\mathcal{K}_{m,n}$

- $\widetilde{W}(A^{(1)}_{m-1}) = \langle r_0, r_1, \dots, r_{m-1}, \omega\rangle$ acts on $\mathcal{K}_{m,n} = Q(\mathcal{A}_{m,n})$ by

$$
\begin{aligned}
r_i(x_{il}) &= x_{il} - s^{-1}\frac{c_{i,l+1} - c_{i+1,l+2}}{P_{i,l+1}} = sP_{il}x_{i+1,l}P_{i,l+1}^{-1}, \\
r_i(x_{i+1,l}) &= x_{i+1,l} + s^{-1}\frac{c_{il} - c_{i+1,l+1}}{P_{il}} = s^{-1}P_{il}^{-1}x_{il}P_{i,l+1}, \\
r_i(x_{jl}) &= x_{jl} \quad (j \not\equiv i, i+1 \pmod{m}), \\
\omega(x_{jl}) &= x_{j+1,l},
\end{aligned}
$$

where $\quad c_{ik} := x_{ik}x_{i,k+1}\cdots x_{i,k+n-1} \quad$ and

$$
P_{ik} := \sum_{l=1}^{n} \overbrace{x_{ik}x_{i,k+1}\cdots x_{i,k+l-2}}^{l-1}\overbrace{x_{i+1,k+l}x_{i+1,k+l+1}\cdots x_{i+1,k+n-1}}^{n-l}.
$$

Explicit formulae of the action of $\widetilde{W}(A^{(1)}_{n-1})$ on $\mathcal{K}_{m,n}$

- $\widetilde{W}(A^{(1)}_{n-1}) = \langle s_0, s_1, \dots, s_{n-1}, \varpi \rangle$ acts on $\mathcal{K}_{m,n} = Q(\mathcal{A}_{m,n})$ by

$$s_k(x_{jk}) = x_{jk} - r^{-1}\frac{d_{j+1,k} - d_{j+2,k+1}}{Q_{j+1,k}} = rQ^{-1}_{j+1,k}x_{j,k+1}Q_{jk},$$

$$s_k(x_{j,k+1}) = x_{j,k+1} + r^{-1}\frac{d_{jk} - d_{j+1,k+1}}{Q_{jk}} = r^{-1}Q_{j+1,k}x_{jk}Q_{jk},$$

$$s_k(x_{jl}) = x_{jl} \quad (l \not\equiv k, k+1 \pmod{n}),$$

$$\varpi(x_{jl}) = x_{j,l+1},$$

where $d_{ik} := x_{i+m-1,k} \cdots x_{i+1,k}x_{ik}$ and

$$Q_{ik} := \sum_{j=1}^{m} \overbrace{x_{i+m-1,k+1} \cdots x_{i+j+1,k+1}x_{i+j,k+1}}^{m-j}\overbrace{x_{i+j-2,k} \cdots x_{i+1,k}x_{ik}}^{j-1}.$$

Duality of the extended affine Weyl group actions

- $x_{ik}^{(m,n)} := x_{ik} \in \mathcal{A}_{m,n}$, $c_{ik}^{(m,n)} := c_{ik} \in \mathcal{A}_{m,n}$, $P_{ik}^{(m,n)} := P_{ik} \in \mathcal{A}_{m,n}$, $s_i^{(m,n)} :=$ (s_i-action on $\mathcal{K}_{m,n}$), $\omega^{(m,n)} :=$ (ω-action on $\mathcal{K}_{m,n}$), etc.

- The algebra isomorphism $\theta : \mathcal{A}_{m,n} \xrightarrow{\sim} \mathcal{A}_{n,m}$ is defined by

$$\theta(x_{ik}^{(m,n)}) = x_{-k,-i}^{(n,m)},\ \theta(q) = q^{-1},\ \theta(r) = s^{-1},\ \theta(s) = r^{-1}.$$

- Then

$$\theta(c_{ik}^{(m,n)}) = d_{-k-n+1,-i}^{(n,m)}, \qquad \theta(P_{ik}^{(m,n)}) = Q_{-k-n+1,-i-1}^{(n,m)},$$
$$\theta(d_{ik}^{(m,n)}) = c_{-k,-i-m+1}^{(n,m)}, \qquad \theta(Q_{ik}^{(m,n)}) = P_{-k-1,-i-m+1}^{(n,m)}.$$

- Therefore

$$\theta \circ r_i^{(m,n)} = s_{-i-1}^{(n,m)} \circ \theta, \qquad \theta \circ \omega^{(m,n)} = (\varpi^{(n,m)})^{-1} \circ \theta,$$
$$\theta \circ s_k^{(m,n)} = r_{-k-1}^{(n,m)} \circ \theta, \qquad \theta \circ \varpi^{(m,n)} = (\omega^{(n,m)})^{-1} \circ \theta.$$

Lax representations of the actions of r_i and s_k

X-operators: $X_{ik} = X_{ik}(z) := \begin{bmatrix} x_{ik} & 1 & & \\ & x_{i+1,k} & \ddots & \\ & & \ddots & 1 \\ r^{-k}z & & & x_{i+m-1,k} \end{bmatrix}.$

(1) The action of r_i on $\{x_{1k}, \dots, x_{mk}\}$ is uniquely characterized by

$$r_i(X_{1k}) = G_k^{(i)} X_{1k} \bigl(G_{k+1}^{(i)}\bigr)^{-1}.$$

$$G_k^{(i)} := 1 + s^{-1}\frac{c_{ik} - c_{i+1,k+1}}{P_{ik}} E_{i+1,i} \qquad (c_{ik} = x_{ik}x_{i+1,k}\cdots x_{i+m-1,k}),$$

$$G_k^{(0)} := 1 + r^{k-1}z^{-1}s^{-1}\frac{c_{mk} - c_{m+1,k+1}}{P_{mk}} E_{1m}. \quad (E_{ij}\text{'s are matrix units.})$$

(2) The action of s_k is uniquely characterized by

$$s_k(X_{ik}X_{i,k+1}) = X_{ik}X_{i,k+1}, \; s_k(X_{il}) = X_{il} \; (l \not\equiv k \pmod n),$$

$$s_k : d_{ik} \leftrightarrow d_{i+1,k+1} \quad (d_{ik} = x_{i+m-1,k}\cdots x_{i+1,k}x_{ik}).$$

Quantum q-difference isomonodromic systems

Monodromy matrix: $\mathbb{X}_{ik}(z) := X_{ik}(z)X_{i,k+1}(z)\cdots X_{i,k+n-1}(z)$.

Matrix q-difference shift operator (shift parameter $= s$):

$T_{z,s}v(s) := \mathrm{diag}(s^{-1}, s^{-2}, \ldots, s^{-m})v(s^m z)$ ($v(z)$ is m-vector valued).

Linear q-difference equation: $T_{z,s}v(z) = \mathbb{X}_{11}(z)v(z)$.

Connection matrix preserving transformations:

(1) $s_k(\mathbb{X}_{11}(z)) = \mathbb{X}_{11}(z)$ for $k = 1, 2, \ldots, n-1$.

(2) $\varpi(\mathbb{X}_{11}(z)) = X_{11}^{-1}\mathbb{X}_{11}(z)X_{1,n+1} = T_{z,s}X_{1,n+1}^{-1}T_{z,s}^{-1}\mathbb{X}_{11}(z)X_{1,n+1}$.

- $U_k = s_{k-1}\cdots s_2 s_1 \varpi s_{n-1}\cdots s_{k+1}s_k$.

  The action of $\langle U_1, U_2, \ldots, U_n\rangle \cong \mathbb{Z}^n$

  $\longrightarrow$ Quantum q-difference isomonodromic dynamical system with n time variables

- The action of $\widetilde{W}(A_{m-1}^{(1)}) \longrightarrow$ Symmetry of the dynamical system

Example $\mathbf{((m,n)=(3,2))}$ • $x_{i+3,k} = r x_{ik}$, $x_{i,k+2} = s x_{ik}$.

- $x_{11}x_{11} = x_{11}x_{11}$, $x_{21}x_{11} = q^{-2}x_{11}x_{21}$, $x_{31}x_{11} = q^{2}x_{11}x_{31}$,
  $x_{12}x_{11} = x_{11}x_{12}$, $x_{22}x_{11} = q^{2}x_{11}x_{22}$, $x_{32}x_{11} = q^{-2}x_{11}x_{32}$.
- $P_{ik} = x_{i+1,k+1} + x_{ik}$,
  $Q_{ik} = x_{i+2,k+1}x_{i+1,k+1} + x_{i+2,k+1}x_{ik} + x_{i+1,k}x_{ik}$.
- $r_1(x_{11}) = s(x_{22}+x_{11})x_{21}(x_{13}+x_{12})^{-1}$,
  $r_1(x_{21}) = s^{-1}(x_{22}+x_{11})^{-1}x_{21}(x_{13}+x_{12})$,
  $\omega(x_{ik}) = x_{i+1,k}$.
- $s_1(x_{11}) = r(x_{42}x_{32}+x_{42}x_{21}+x_{31}x_{21})^{-1}x_{12}(x_{32}x_{22}+x_{32}x_{11}+x_{21}x_{11})$,
  $s_1(x_{12}) = r^{-1}(x_{42}x_{32}+x_{42}x_{21}+x_{31}x_{21})x_{11}(x_{32}x_{22}+x_{32}x_{11}+x_{21}x_{11})^{-1}$,
  $\varpi(x_{ik}) = x_{i,k+1}$. $(U_1 = \varpi r_1,\ U_2 = r_1\varpi)$
  $U_1(x_{11}) = r(x_{43}x_{33}+x_{43}x_{22}+x_{32}x_{22})^{-1}x_{13}(x_{33}x_{23}+x_{33}x_{12}+x_{22}x_{12})$.
- U_1 generates quantum $q\mathrm{P}_{\mathrm{IV}}$ (q-difference Panlevé IV system).
  The action of $\widetilde{W}(A_2^{(1)})$ is symmetry of quantum $q\mathrm{P}_{\mathrm{IV}}$.

Action of $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ on $\mathcal{K}_{m,n}$ as alg. autom.

Theorem. For any mutually prime integers $m, n \geqq 2$, the action of $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ on $\mathcal{K}_{m,n} = Q(\mathcal{A}_{m,n})$ as algebra automorphisms is constructed. This is a quantization of the KNY action of $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ on $\mathbb{C}(x_{ik} | 1 \leqq i \leqq m, 1 \leqq k \leqq n)$.

Easy Part. Lax representations $\implies$ braid relations of r_i and s_k.

Difficult Part. To show that
r_i and s_k act on $\mathcal{K}_{m,n} = Q(\mathcal{A}_{m,n})$ as algebra automorphisms.

Sketch of proof. Let φ_i be appropriately corrected Chevalley generators in $\mathcal{B}_{m,n}$ and put $\rho_i := \varphi_i^{\alpha_i^\vee} \tilde{r}_i$. Then $\mathrm{Ad}(\rho_i) x_{jl} = \rho_i x_{jl} \rho_i^{-1} = r_i(x_{jl})$. Therefore r_i acts on $\mathcal{K}_{m,n}$ as algebra automorphisms. The duality leads to that s_k also acts on $\mathcal{K}_{m,n}$ as algebra automorphisms.

Chevalley generators F_i

Monodromy matrix: $\mathbb{L}(z) := L_1(r^{n-1}z)L_2(r^{n-2}z)\cdots L_{n-1}(rz)L_n(z)$.

($\mathbb{L}(z)$ is the product of the L-operators of the minimal representations.)

$$\mathbb{L}(z) = \begin{bmatrix} A_1 & B_1 & \ddots & \ddots \\ & A_2 & \ddots & \ddots \\ & & \ddots & B_{m-1} \\ 0 & & & A_m \end{bmatrix} + z \begin{bmatrix} \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ B_m & \ddots & \ddots & \ddots \end{bmatrix} + \cdots$$

- $R(z/w)\mathbb{L}(z)^1\mathbb{L}(w)^2 = \mathbb{L}(w)^2\mathbb{L}(z)^1 R(z/w)$

$\Longrightarrow$ $F_i := A_i^{-1}B_i$ satisfy the q-Serre relations.

- $R_i := F_i^{\alpha_i^\vee}\tilde{r}_i$ generate the Weyl group action on the skew field generated by A_i, B_i, and parameters $a_{\varepsilon_i^\vee} = q^{\varepsilon_i^\vee}$.

- But the action of $\mathrm{Ad}(R_i)$ does not preserve the skew field generated by $x_{ik} = a_{ik}(b_{ik}b_{i+1,k+1}\cdots b_{i+\tilde{m}m-1})^{-1}$ and parameters $a_{\varepsilon_i^\vee} = q^{\varepsilon_i^\vee}$.

Correction factors for F_i

- $\mathcal{K}_{m,n} = Q(\mathcal{A}_{m,n}) \subset Q(\mathcal{B}_{m,n})$.
- $x \simeq y \iff \exists\, c \in$ (the center of $Q(\mathcal{B}_{m,n}))^\times$ s.t. $cx = y$.
- $\tilde{n} :=$ mod-m inverse of n $(\tilde{n}n \equiv 1 \pmod m,\ \tilde{n} = 1, 2, \dots, m-1)$.
- $v_{ik} := b_{ik} b_{i+1,k+1} \cdots b_{i+\tilde{n}n-1,k+\tilde{n}n-1}$ (v_{1k} are correction factors).
  (cf. $x_{ik} = a_{ik}(b_{ik} b_{i+1,k+1} \cdots b_{i+\tilde{m}m-1})^{-1}$, $\tilde{m} =$ mod-n inverse of m)
- $c_{i1}^{-1} P_{i1} \simeq v_{i1}^{-1} F_i = v_{i1}^{-1} A_i^{-1} B_i$ (motivation to find v_{ik}).
- $\varphi_i := v_{i1} F_i = v_{i1} A_i^{-1} B_i \simeq v_{i1}^2 c_{i1}^{-1} P_{i1}$ (corrected F_i).
- Using φ_i instead of F_i, we can construct the action of the affine Weyl group $W(A_{m-1}^{(1)})$ on $\mathcal{K}_{m,n} = Q(A_{m,n})$ as algebra automorphisms.

Generators of the $W(A^{(1)}_{m-1})$-action on $\mathcal{K}_{m,n} = Q(\mathcal{A}_{m,n})$

- $\mathcal{H}_m := \mathbb{F}[q^{\pm 2\varepsilon_1^\vee}, \dots, q^{\pm 2\varepsilon_m^\vee}]$, $\varepsilon_i^\vee := E_{ii} \in \mathfrak{h}$, $\alpha_i^\vee := \varepsilon_i^\vee - \varepsilon_{i+1}^\vee$.
- $I :=$ the two-sided ideal of $\mathcal{A}_{m,n} \otimes \mathcal{H}_m$ generated by $c_{ii} \otimes 1 - 1 \otimes q^{-2\varepsilon_i^\vee}$.

Then $(\mathcal{A}_{m,n} \otimes \mathcal{H}_m)/I \cong \mathcal{A}_{m,n}$. $(\omega(q^{-2\varepsilon_i^\vee}) := s^{-1} q^{-2\varepsilon_{i+1}^\vee})$

- $\tilde{r}_i \varepsilon_i^\vee \tilde{r}_i^{-1} = \varepsilon_{i+1}^\vee$, $\tilde{r}_i \varepsilon_{i+1}^\vee \tilde{r}_i^{-1} = \varepsilon_i^\vee$, $\tilde{r}_i \varepsilon_j^\vee \tilde{r}_i^{-1} = \varepsilon_j^\vee$ $(j \neq i, i+1)$.
- $\rho_i := \varphi_i^{\alpha_i^\vee} \tilde{r}_i$. (generators of the $W(A^{(1)}_{m-1})$-action on $\mathcal{K}_{m,n}$)
- $\mathrm{Ad}(\rho_i)$'s generate the action of $W(A^{(1)}_{m-1})$ on $Q(\mathcal{A}_{m,n} \otimes \mathcal{H}_m)$.
- **The actions of $\mathrm{Ad}(\rho_i)$'s on $Q(\mathcal{A}_{m,n} \otimes \mathcal{H}_m)$ induce the actions of $r_i \in W(A^{(1)}_{m-1})$ on $\mathcal{K}_{m,n} = Q(\mathcal{A}_{m,n})$:**

$$\mathrm{Ad}(\rho_i) x_{il} = r_i(x_{il}) = s P_{il} x_{i+1,l} P_{i,l+1}^{-1},$$

$$\mathrm{Ad}(\rho_i) x_{i+1,l} = r_i(x_{i+1,l}) = s^{-1} P_{il}^{-1} x_{il} P_{i,l+1},$$

$$\mathrm{Ad}(\rho_i) x_{jl} = r_i(x_{jl}) = x_{jl} \quad (j \not\equiv i, i+1 \pmod{m}).$$

Summary of Results

§2. (for any symmetrizable GCM $A = [a_{ij}]$)

- Ad-action of complex powers of Chevalley generators f_i in $U_q(\mathfrak{g})$

$\Longrightarrow$ the action of the Weyl group on $Q(U_q(\mathfrak{n}) \otimes U_q(\mathfrak{h}))$
(quantum q-difference version of the NY math.QA/0012028 action)

$\Longrightarrow$ Reconstruction of the Hasegawa math.QA/0703036 action

§3. (for any mutually prime integers $m, n \geqq 2$)

- $\mathcal{B}_{m,n} :=$ the minimal representation of $U_q(\mathfrak{b})^{\otimes n} \subset U_q(\widehat{\mathrm{gl}}_m)^{\otimes n}$.
- $\mathcal{K}_{m,n} := Q($the gauge invariant subalgebra $\mathcal{A}_{m,n}$ of $\mathcal{B}_{m,n})$

$\Longrightarrow$ $\mathcal{K}_{m,n} =$ Quantization of $\mathbb{C}(x_{ik} | 1 \leqq i \leqq m, 1 \leqq k \leqq n)$.

- Complex powers of the **corrected** Chevalley generators in $\mathcal{B}_{m,n}$

$\Longrightarrow$ $\widetilde{W}(A^{(1)}_{m-1})$-action on $\mathcal{K}_{m,n}$

$\Longrightarrow$ $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$-action on $\mathcal{K}_{m,n}$ (by the $m \leftrightarrow n$ duality)

Other Problems

Problem. Construct commuting Hamiltonians in $U_q(\mathfrak{n}) \otimes U_q(\mathfrak{h})$ with Weyl group symmetry.

Hint. Commuting transfer matrices for "$AL^1BL^2 = CL^2DL^1$" algebras. $(F = q^{-\sum H_i \otimes H^i}$, $A = P(F)^{-1}RF$, $B = F$, $C = P(F)$, $D = R)$

Problem. Construct commuting Hamiltonians in $\mathcal{A}_{m,n}$ with $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ symmetry.

Classical Case. $\det(\mathbb{X}^{(m,n)}_{11}(z) - (-1)^n w) = \det(\mathbb{X}^{(n,m)}_{11}(w) - (-1)^m z)$ generates the invariants of birational $\widetilde{W}(A^{(1)}_{m-1}) \times \widetilde{W}(A^{(1)}_{n-1})$ action.

Problem. Construct solutions of quantum (q-)isomonodromic systems.

Conjecture. Schrödinger equation of any quantum (q-)isomonodromic system has (non-confluent or confluent) (q-)hypergeometric solutions.