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Ruijsenaars' commuting difference system from Belavin's elliptic R-matrix

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0.1 Introduction

As the master family among the integrable n-body systems on a line, or as the generalized spherical function equation on a simple Lie group, much attention has been paid for the Macdonald system [M1][M2] and the Ruijsenaars' elliptic system [R] these days. In both cases it is remarkable that the commutativity of the operators of the system is nontrivial. As is observed by Ruijsenaars, a direct computation of the commutator will lead to the Cauchy type identity of the function appeared in the coefficient of the operators. On the other hand Macdonald showed in the trigonometric case that the system is explicitly diagonalizable, which implies the commutativity of the system as a corollary. Many authors have made efforts to elucidate the nature of the commutativity, and at least two ways of understanding for Macdonald's commuting operators have been established. One is due to Cherednik [C92], who uses the double affine Hecke algebra, its representation via difference operators (Dunkl operators), and the center of the algebra. Restricting ourselves to the type A case, the other is the work by Etingof and Kirillov [EK], who obtained these operators as the image of central elements of the quantum universal enveloping algebra. These two works have related the system to the two natural, universal, commuting, but different objects. (There should be a question that these two approaches can be related or not by any means, which seems to be not yet clarified so far.) ¿From the conformal field theory point of view, the former approach can be regarded as the reformulation of the theory of the Knizhnik-Zamolodchikov equation for the n-pointed conformal block, while the latter can be considered as the analogue of the Bernard equation for the one-pointed conformal block on the torus at the critical level [FW][F].

In both above approaches, the Yang-Baxter equation can be regarded as the important background of the theory. Interesting enough, in Baxter's study of solvable lattice statistical models, the Yang-Baxter equation arise as the condition to provide sufficiently many commuting operators. This is done by taking the traces of the so-called L-operators, the operators which satisfy the "$RLL = LLR$ relation" (0.1), which is nothing but a variant of the Yang-Baxter equation. In [H3] we considered a realization of the L-operator for Belavin's elliptic R-matrix [Be] in terms of difference operators, and studied the commuting operators arising from the traces thereof. It turns out that one can recover Ruijsenaars' elliptic commuting difference operators in this way. In other word, the arguement of "commuting transfer matrices" in statistical lattice model theory provides the way to make the commutativity apparent in the elliptic case. In this brief note we will summarize the result of [H3] and will report some further facts.

Our difference L-operator can be identified with the representation of Felder's elliptic algebra [F] after some suitable similarity transformation and reformulation. This suggests that our L-operator would allow a natural formulation in a q-deformed conformal field theory in a way that our approach could be regarded as an elliptic generalization of the above mentioned work of Etingof and Kirillov.

0.2 The difference operators

For $n > 1$ let $V = \oplus_{k \in \mathbf{Z}/n\mathbf{Z}} \mathbf{C} e^k$ $(e^k = e^{k+n})$ and let $g, h \in \mathrm{GL}(V)$ to be

$$g e^k := e^k \exp\frac{2\pi i k}{n}, \quad h e^k := e^{k+1}.$$

We have $gh = hg \exp\frac{2\pi i}{n}$. Let $\hbar, \tau \in \mathbf{C}, \hbar \neq 0, \mathrm{Im}\tau > 0$.

Belavin's R-matrix $R(u) = R_\hbar(u)$ is characterized as the unique solution of the following five conditions.

- $R_\hbar(u)$ is a holomorphic $\mathrm{End}(\mathbf{C}^n \otimes \mathbf{C}^n)$ -valued function in u,
- $R_\hbar(u) = (x \otimes x) R_\hbar(u) (x \otimes x)^{-1}$ for $x = g, h$,
- $R_\hbar(u+1) = (g \otimes 1)^{-1} R_\hbar(u) (g \otimes 1) \times (-1)$,
- $R_\hbar(u+\tau) = (h \otimes 1) R_\hbar(u) (h \otimes 1)^{-1} \times (-\exp 2\pi i [u + \frac{\hbar}{n} + \frac{\tau}{2}])^{-1}$,
- $R_\hbar(0) = P : x \otimes y \mapsto y \otimes x$.

It holds that 1) there is a unique solution to the above conditions and 2) the solution satisfies the Yang-Baxter equation [Be].

In the trigonometric limit, up to a certain simple "gauge transformation" [Re] , this R-matrix degenerates to the image of the universal R-matrix for the quantum affine enveloping algebra $U_q(A^{(1)}_{n-1})$ ([J],[Dr]) in the vector representation.

By an L-operator we mean the matrix $L(u) = [L(u)^i_j]_{i,j=1,\cdots n}$ of operators (noncommutative letters) that satisfy

$$\check{R}(u-v) L(u) \otimes L(v) = L(v) \otimes L(u) \check{R}(u-v), \tag{0.1}$$

where $\check{R}(u) := PR(u)$.

For Belavin's R-matrix we shall construct such an L-operator in the following way. Let $\mathbf{h}^*$ be the weight space for $sl_n(\mathbf{C})$ and realize $\mathbf{h}^*$ in $\mathbf{C}^n = \oplus_{i=1,\cdots,n} \mathbf{C}\epsilon_i$, $\langle\, \epsilon_i,\ \epsilon_j \rangle = \delta_{i,j}$, as the orthogonal complement to $\sum_{i=1,\cdots,n} \epsilon_i$. We denote the orthogonal projection of ϵ_i by $\bar{\epsilon}_i$. Then the root lattice is $Q = \sum_{i=1}^{n-1} \mathbf{Z}(\bar{\epsilon}_i - \bar{\epsilon}_{i+1})$ and the weight lattice is $P = \sum_{i=1}^n \mathbf{Z}\bar{\epsilon}_i$.

For each $\lambda, \mu \in \mathbf{h}^*$ and $j = 1, \cdots, n$ we put

$$\phi(u)^\mu_{\lambda j} := \begin{cases} \theta_j(\frac{u}{n} - < \lambda, \bar{\epsilon}_k >) & : \mu - \lambda = \hbar \bar{\epsilon}_k \quad \text{for some } k = 1, \cdots, n, \\ 0 & : \text{otherwise} \end{cases} \tag{0.2}$$

where

$$\theta_j(u) := \sum_{\mu \in \frac{n}{2}-j+n\mathbf{Z}} \exp 2\pi i \left[\mu(u+\frac{1}{2}) + \frac{\mu^2}{2n}\tau\right].$$

Also we define $\bar{\phi}(u)_\mu^{\mu+\hbar\bar{\epsilon}_k,j}$ to be the (k,j)-entry in the inverse matrix of $[\phi(u)_\mu^{\mu+\hbar\bar{\epsilon}_k}{}_j]_{j,k=1,\cdots,n}$, namely

$$\sum_{j=1}^{n} \bar{\phi}(u)_\mu^{\mu+\hbar\bar{\epsilon}_k j}\phi(u)_\mu^{\mu+\hbar\bar{\epsilon}_{k'}}{}_j = \delta_{k,k'}, \qquad \sum_{k=1}^{n} \phi(u)_\mu^{\mu+\hbar\bar{\epsilon_k}}{}_j\bar{\phi}(u)_\mu^{\mu+\hbar\bar{\epsilon_k}j'} = \delta_{j,j'}. \tag{0.3}$$

Then generalizing a result in the celebrated paper [S] we have

Theorem 1 ([H1], [H2]) *For a function f on $\mathbf{h}^*$, put*

$$(L(c|u)^i_j f)(\mu) := \sum_{k=1,\cdots,n} \phi(u+c\hbar)_\mu^{\mu+\hbar\bar{\epsilon_k}}{}_j\bar{\phi}(u)_\mu^{\mu+\hbar\bar{\epsilon}_k,i} f(\mu+\hbar\bar{\epsilon_k}). \tag{0.4}$$

Then for any $c \in \mathbf{C}$, the collection of difference operaors

$$L(c|u) = [L(c|u)^i_j]_{i,j=1\cdots,n}$$

satisfies the desired relation (0.1). i.e., $L(c|u)$ gives a one parameter family (in c) of L-operators.

Remark. The objects ϕ and $\bar{\phi}$ are called the "intertwining vectors" [JMO]. The trigonometric version of this construction of L operator for the standard (non-dynamical) R matrix is rather obscure and not trivial from the above [AHZ].

Recall $V = \oplus_{j=1,\cdots,n}\mathbf{C}e^j = \mathbf{C}^n$ and let $\mathcal{M}(\mathbf{h}^*)$ be the field of meromorphic functions on $\mathbf{h}^*$. Then the above L-operator is an endomorphism on the space $V \otimes \mathcal{M}(\mathbf{h}^*)$,

$$L(c|u) \in \mathrm{End}(V \otimes \mathcal{M}(\mathbf{h}^*)).$$

Here the first space $V = \mathbf{C}^n$ can be regarded as the space of defining comodule (vector "co"representation) for the bialgebra $A(R)$ defined by the relation (0.1). We can consider more complicated comodules for this bialgebra as well: for each Young diagram Y we can construct an $A(R)$-comodule $V(Y)$ whose dimension is just the same as for the GL_n-module that corresponds to Y. This is an early result known as the fusion technique [KRS][C] and done by taking the appropriate sub/quotient of the tensor comodule of $V(\square) = V = \mathbf{C}^n$. It follows that we get a collection (a matrix) of difference operators

$$L^Y(c|u) \in \mathrm{End}(V(Y) \otimes \mathcal{M}(\mathbf{h}^*))$$

for each Y and they satisfy the relation

$$\check{R}^{Y,Y'}(u-v)L^Y(c|u) \otimes L^{Y'}(c|v) = L^{Y'}(c|v) \otimes L^Y(c|u)\check{R}^{Y,Y'}(u-v), \tag{0.5}$$

where $\check{R}^{Y,Y'}$ stands for the "fused R-matrices" which gives the isomorphism between the $A(R)$-comodules $V(Y)\otimes V(Y')\to V(Y')\otimes V(Y)$.

These structures are of course now well understood for the trigonometric R-matrix case, where we have the quantised enveloping algebra and its universal R-matrix as the origin of these fused R-matrix.

Now we will state the main result, concerning with the trace of the operator $L^Y(c|u)$. We shall consider the case $Y=1^k$, the vertical k boxes case. Then L^{1^k} is a matrix of size $\dim \wedge^k \mathbf{C}^n$ whose matrix element is a difference operator. We denote the Jacobi theta function by

$$\theta(u)=\sqrt{-1}\,p^{1/8}(z^{1/2}-z^{-1/2})\prod_{m=1}^{\infty}(1-zp^m)(1-z^{-1}p^m)(1-p^m)$$

with $p^{1/8}=\exp\frac{2\pi i\tau}{8}$ and $z^{1/2}=\exp\frac{2\pi i u}{2}$.

Theorem 2 *i). Let $M^{(k)}(c|u):=\mathrm{Trace}_{V(1^k)}L^{1^k}(c|u)$, $k=1,\cdots,n$. Then we have*

$$M^{(k)}(c|u)=\frac{\theta(u+\frac{kc\hbar}{n})}{\theta(u)}\sum_{I\subset\{1,\cdots,n\},|I|=k}\left(\prod_{s\notin I,t\in I}\frac{\theta(\lambda_{s,t}+\frac{c\hbar}{n})}{\theta(\lambda_{s,t})}\right)T_I^{\hbar}, \quad (0.6)$$

where $\lambda_{s,t}:=\langle\lambda,\bar{\epsilon}_s-\bar{\epsilon}_t\rangle$ and $T_I^{\hbar}$ stands for the $\hbar$-shift operator:

$$(T_i^{\hbar}f)(\lambda):=f(\lambda+\hbar\bar{\epsilon}_i),\quad T_I^{\hbar}:=\prod_{i\in I}T_i^{\hbar}.$$

ii). For any $j,k=1,\cdots,n$, we have

$$[M^{(j)}(c|u),M^{(k)}(c|u)]=0.$$

iii).(cf.[D]) Suppose $q:=\exp 2\pi i\hbar$ satisfies $|q|<1$ and put $g:=c/n$. Let

$$d^+(z):=\prod_{k=0}^{\infty}\prod_{m=0}^{\infty}\frac{1-zq^{m+1}p^k}{1-zq^{m+g+1}p^k}\,\frac{1-z^{-1}q^{m-g}p^{k+1}}{1-z^{-1}q^mp^{k+1}}, \quad (0.7)$$

and define a function on $\mathbf{h}^*$ by $\Phi(\lambda):=\prod_{k\neq k'}d^+(z_k/z_{k'})$, where $p=\exp 2\pi i\tau$ and $z_j:=\exp 2\pi i\langle\lambda,\bar{\epsilon}_j\rangle$. Then the conjugation by the square root $\Phi^{1/2}$ yields the Ruijenaars' [R] commuting operators:

$$\begin{aligned}&\left(\frac{\theta(u+\frac{kc\hbar}{n})}{\theta(u)}\right)^{-1}\cdot\Phi^{-1/2}M^{(k)}(c|u)\,\Phi^{1/2}\\ =&\sum_{|I|=k}\left(\prod_{s\notin I,t\in I}\sqrt{\frac{\theta(c\hbar+\lambda_{s,t})}{\theta(\lambda_{s,t})}}\right)T_I^{\hbar}\left(\prod_{s\notin I,t\in I}\sqrt{\frac{\theta(c\hbar+\lambda_{t,s})}{\theta(\lambda_{t,s})}}\right).\end{aligned}$$

In the operator $M^{(k)}(c|u)$ (0.6), the spectral parameter u appears only in the overall factor. It is easy to see that, in the trigonometric limit $p \to 0$ the commuting system (0.6) falls into the Macdonald's one with the parameters $q = \exp 2\pi i\hbar$ and $t = q^{-c/n}$. An important remark is that these operators obviously commute, as is mentioned in the introduction. This is because the extended "RLL=LLR" relation (0.5) can be rewritten as

$$\check{R}^{Y,Y'}(u-v)L^{Y}(c|u) \otimes L^{Y'}(c|v)\check{R}^{Y,Y'}(u-v)^{-1} = L^{Y'}(c|v) \otimes L^{Y}(c|u),$$

and then taking the trace simply gives

$$M^{Y}(c|u)M^{Y'}(c|v) = M^{Y'}(c|v)M^{Y}(c|u)$$

where $M^{Y}(c|u) := \mathrm{Trace}_{V(Y)}L^{Y}(c|u)$. This simple argument and the resulting operators ("commuting transfer matrices") were effectively used in Baxter's analysis of the spin chain models [B71][B]. ¿From the viewpoint of spin chain models, a solution of the Yang-Baxter equation is believed to define a solvable lattice model. Since our L operator solves the Yang-Baxter equation, we may regard the function space on which the operators $\{L(c|u)^{i}_{j}\}$ will act as the space of freedom for the site variables. Actually we can consider the tensor products ("chain") of the function spaces consisting of arbitrary number (say N) of components, on which the N-th tensor product of the L-operator will acts (using the coproduct of the algebra $A(R)$). That is, the operator $L(c|u)$ gives a function space realizaion (a Schrödinger picture) of the generalized Heisenberg finite spin chain (with the periodic boundary condition). Thus we may state the ideology:

commuting transfer matrices = commuting difference system.

Functions in $\lambda \in \mathbf{h}^*$ of course form an infinite dimensional vector space, but as we will see in the next section there exists a certain finite dimensional subspace which can be regarded as a function space realization of the symmetrically fused generalized XYZ spin chain.

Among the statements of the theorem, the computation of the trace i) is rather nontrivial and uses the formula below, which is interesting itself.

Lemma 1 *Let $Y_{r<s} := 1$ if $r < s$ holds, $Y_{r<s} := 0$ otherwise. The following formula holds:*

$$\det\left[\prod_{r=1}^{d}\theta\left(\mu_r - \lambda_{s'} + \hbar Y_{r<s} + \delta_{r,s}(u-(s-1)\hbar)\right)\right]_{s,s'=1,\cdots,d} \tag{0.8}$$
$$= \prod_{s=1}^{d-1}\theta(u-s\hbar)\prod_{1\le s<s'\le d}\theta(\lambda_{s'}-\lambda_s)\theta(\hbar+\mu_s-\mu_{s'}).$$

Proof of this formula can be done by using the standard complex analysis and the induction on k. The $\hbar = 0$ case of (0.8) is easily transformed into the Cauchy type determinant formula due to Frobenius [Frob]:

$$\det\left[\frac{\theta(\mu_s - \lambda_{s'} + u)}{\theta(\mu_s - \lambda_{s'})\theta(u)}\right]_{s,s'=1,\cdots,d} = \frac{\prod_{1\le s<s'\le d}\theta(\mu_s - \mu_{s'})\theta(\lambda_{s'} - \lambda_s)}{\prod_{s,s'=1,\cdots,d}\theta(\mu_s - \lambda_{s'})}.$$

The formula (0.8) also turns out to be useful to derive the complete generators of commuting differential operators in the limit $\hbar \to 0$.

Rescently I learned from V. Tarasov that (0.8) can be obtained as a special case of the series of determinant formulas shown in [TV], Appendix B.

0.3 An invariant subspace spanned by symmetric theta functions

Fix a nonnegative integer $l \in \mathbf{Z}_{>0}$. Let $Th_l^{S_n}$ be the vector space consisting of level l integrable $A_{n-1}^{(1)}$- characters [Kac]. This space is of dimension $\frac{(l+n-1)!}{l!(n-1)!}$ and spanned by the theta series on $\mathbf{h}^*$ that are invariant under the action of the symmetric group S_n. Then we have

Theorem 3 **([H2])** *For any $i, j = 1, \cdots, n$ we have $L(l|u)^i_j Th_l^{S_n} \subset Th_l^{S_n}$, hence $M^{(k)}(l|u)Th_l^{S_n} \subset Th_l^{S_n}$. As an $A(R)$-module, $Th_l^{S_n}$ is isomorphic to the representation $V(\square \overset{l}{\cdots} \square)$, the module corresponding to the Young diagram of l horizontal boxes.*

As an elliptic analogue of Macdonald's theory, we may define a family of orthogonal polynomials as the simultaneous eigenfunctions for our operators $M^{(k)}(c|u)$. In the trigonometric case, this diagonalization problem can be solved as one observes the triangularity of the action of the Macdonald operators with respect to the so-called orbitsum symmetric polynomials. In the elliptic case, we have the analogue of the orbitsum polynomials, namely the symmetric theta series for the affine Weyl group of type $A_{n-1}^{(1)}$. One can calculate the action of the Ruijsenaars' operators for this basis as follows, but it turns out that the triangularity property fails unfortunately.

Recall that Q and P stands for the root lattice and the weight lattice, respectively. For $\mu \in \mathbf{h}^*$, $l \in \mathbf{Z}_{>0}$, put

$$\Theta_{\mu,l}(\lambda) := \sum_{\nu\in\mu+lQ} \exp 2\pi\sqrt{-1}\left[\langle\nu,\lambda\rangle + \frac{\langle\nu,\nu\rangle}{2l}\tau\right]$$

and define the affine orbitsum function, the affine Schur function respectively by

$$m_{\mu,l}(\lambda) := \sum_{w\in S_n}\Theta_{w(\mu),l}(\lambda), \quad S_{\mu,l}(\lambda) := A_{\mu+\rho,l+n}(\lambda)/A_{\rho,n}(\lambda)$$

where

$$A_{\mu,l}(\lambda) := \sum_{w\in S_n} sgn(w)\Theta_{w(\mu),l}(\lambda)$$

and $\rho := \sum_{i=1}^{n-1}(n-i)\bar{\epsilon}_i$ stands for the half sum of positive roots for A_{n-1}. It is easy to see that the space $Th_l^{S_n}$ have the basis $\{m_{\mu,l}|\mu \in P^+(l)\}$, where $P^+(l)$ denotes the set of dominant integral weights $\mu \in P^+$ that satisfy $\langle \mu, \rho\rangle \leq l$.

Proposition 1 *For $\mu \in P^+(l)$, $k = 1, \cdots, n$, we have*

$$\begin{aligned} & M^{(k)}(l|u)m_{\mu,l} \\ = \quad & \frac{\theta(u+\frac{lk\hbar}{n})}{\theta(u)} \sum_{w\in S_n,\alpha} \left(\sum_{|K|=k} T_K^{\hbar/n}\Theta_{l\rho-nw(\mu+l\alpha),nl(n+l)}(0)\right) S_{w(\mu+l\alpha),l} \end{aligned}$$

where α runs over $Q/(l+n)Q$, K runs over the subsets of $\{1,\cdots,n\}$ consisting of k elements.

One can show this in the same manner as the calculation performed in [M], pp315-316. However, the transition matrix between the orbitsums $m_{\nu,l}$ and the Schur functions $S_{\nu,l}$ is not triangular any more. (Since the entries of the transition matrix are nothing but the so-called string functions [Kac], the above formula provides some information on the string functions.) The summation over α also reflects the affine nature of the problem; it comes from the quotient of the Cartesian product of two affine Weyl groups by a certain subgroup. According to these two facts the naive triangularity fails for the elliptic Ruijsenaars' operators. This difficulty is not an accident but can be regarded as another form of the "absense of the vacuum" in the XYZ spin chain. This model was solved by Takhtajan and Faddeev [TF] in the face formulation, which is applied to the two body ($n = 2$) case of Ruijsenaars' system by Felder and Varchenko [FV].

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References

[AHZ] A. Antonov, K. Hasegawa and A. Zabrodin, "On trigonometric intertwining vectors and non-dynamical R- matrix for the Ruijsenaars model" preprint hep-th/9704074.

[B] R. J. Baxter, *Exactly solved models in statistical mechanics,* Academic, London (1972).

[B71] R. J. Baxter, "Partition function of the eight vertex lattice model", Ann. Phys.**70** (1972) 193-228.

[Be] A. A. Belavin, "Dynamical symmetry of integrable quantum systems", Nucl. Phys. B180 [FS2] 189-200 (1981).

[C] I. V. Cherednik, "On the properties of factorized S matrices in elliptic functions", Sov.J.Nucl.Phys.(Engl.transl.)**36**(2), 320-324 (1982) ; "Some Finite dimensional representation of generalized Sklyanin algebras", Funct. Annal. and Appl. (Engl. transl.) **19**, 77-79 (1985) ; "On "quantum" deformations of irreducible finite-dimensional representations of gl_N", Sov. Math. Dokl. (Engl. transl.) **33**, 507-510 (1986)

[C92] I. V. Cherednik, " Double affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald's operators", International Mathematics Research Notices (Duke Math.), 1992 No.9, pp171-180.

[D] J. F. van Diejen, "Commuting difference operators with polynomial eigenfunctions", Compositio Math. **95** (1995), 183-233.

[Dr] V. G. Drinfeld, "Quantum groups", In *Proc. ICM 1986* (AMS., Providence 1987), 798-820.

[EK] P. I. Etingof and A. A. Kirillov Jr., "Macdonald's polynomials and representations of quantum groups", Math. Res. Lett. **1**(1994) 3, 279-296.

[F] G. Felder, "The KZB equations on Riemann surfaces", hep-th/9609153.

[FV] G. Felder and A. Varchenko, "Algebraic Bethe Ansatz for the elliptic quantum group $E_{\tau,\eta}(sl_2)$, q-alg/9605024.

[FW] G. Felder and C. Wieczerkowski, "Conformal blocks on elliptic curves and the Knizhnik–Zamolodchikov–Bernard equations", Comm. Math. Phys. **176**(1996) 133-162.

[Frob] G. Frobenius, "Ueber die elliptischen Functionen zweiter Art", J. für die reine und angew. Math. 93 (1882) 53-68.

[H1] K. Hasegawa, "On the crossing symmetry of the elliptic solution of the Yang-Baxter equation and a new L operator for Belavin's solution", J.Phys. A: Math. Gen. 26 (1993) 3211-3228.

[H2] K. Hasegawa, " L-operator for Belavin's R-matrix acting on the space of theta functions", J.Math.Phys. **35**(11), 6158-6171 (1994).

[H3] K. Hasegawa, "Ruijsenaars' commuting difference operators as commuting transfer matrices", to appear in Comm. Math. Phys.

[J] M. Jimbo, "A q-difference analog of $U(g)$ and the Yang-Baxter equation", Lett. math. Phys.**10**, 63-69 (1985).

[JMO] M.Jimbo, T. Miwa and M. Okado, "Local state provabilities of solvable lattice models: An $A^{(1)}_{n-1}$ family", Nucl. Phys. **B300**[FS22] 74-108 (1988).

[Kac] V. G. Kac, *Infinite dimensional Lie algebras, 3rd ed.* Cambridge Univ. Press (1990)

[KRS] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, "Yang-Baxter equation and representation theory I", Lett. Math. Phys. **5**, 393-403(1981).

[M] I. G. Macdonald, *Symmetric functions and Hall polynomials, 2nd ed.* Oxford Univ. Press 1995.

[M1] I. G. Macdonald, "A new class of symmetric functions", in *Actes Seminaire Lotharringen*, Publ. Inst. Rech. Math. adv., Strasbourg (1988), 131-171.

[M2] I. G. Macdonald, "Orthogonal polynomials associated with root systems", in *Orthogonal polynomials*, 311-318, P.Nevai(ed.), Kluwer Academic Publishers (1990).

[Re] N. Reshetikhin, "Multiparameter quantum groups and twisted quasitriangular Hopf algebras", Lett. Math. Phys. **20** 331-335 (1990).

[R] S. N. M. Ruijsenaars, "Complete integrability of relativistic Calogero - Moser systems and elliptic function identities", Comm. Math. Phys. **110**(1987), pp191-213.

[S] E. K. Sklyanin, “Some algebraic structure connected with the Yang-Baxter equation”, Funct. Anal. and Appl. (Engl. transl.) **16**, 27-34 (1982); **17**, 273-284 (1983).

[TF] L. A. Takhtadzhan and L. D. Faddeev, “The quantum method of the inverse problem and the Heisenberg XYZ model” (English translation), Russian math. surveys 34:5 (1979) 11-68.

[TV] V. O. Tarasov and A. N. Varchenko, “Geometry of q- hypergeometric functions, quantum affine algebras and elliptic quantum groups”, preprint q-alg/9703044.